FYSH300, fall 2013

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Part 5: Spacetime symmetries

Reminder of quantum mechanics

- Dynamics (time development) is in the **Hamiltonian** operator $\hat{H}$
  $\Longrightarrow$ Schrödinger $i\partial_t\psi = \hat{H}\psi$
- Observable $\Longrightarrow$ Hermitian operator $\hat{A} = \hat{A}^\dagger$
- A is **conserved quantity** (liikevakio) , iff (if and only if)

$$[\hat{A}, \hat{H}] = 0 \Longleftrightarrow \frac{d}{dt}\langle\hat{A}\rangle = 0$$

note: Commutation relation $[\hat{A}, \hat{H}] = 0 \Longrightarrow$ state can be eigenstate of $\hat{A}$ and $\hat{H}$ at the same time

- Conserved quantity associated with **symmetry**; Hamiltonian invariant under transformations generated by $\hat{A}$

(What does it mean to “generate transformations”? We’ll get to examples in a moment.)

Translations and momentum

- Define the operation of "translation" $D_{\mathbf{a}}$ as: $\mathbf{x} \xrightarrow{D_{\mathbf{a}}} \mathbf{x}' = \mathbf{x} + \mathbf{a}$
- A system is **translationally invariant** iff $\hat{H}(\mathbf{x}') \equiv \hat{H}(\mathbf{x} + \mathbf{a}) = \hat{H}(\mathbf{x})$
  (In particle physics systems usually are; nothing changes if you move all particles by fixed **a**.)
- How does this operation affect Hilbert space states $|\psi\rangle$? Define corresponding operator $\hat{D}_{\mathbf{a}}$ (Now with hat, this is linear operator in Hilbert space!) :

$$\hat{D}_{\mathbf{a}}|\psi(\mathbf{x})\rangle = |\psi(\mathbf{x} + \mathbf{a})\rangle$$

- Infinitesimal translation, **a** small, Taylor series

$$|\psi(\mathbf{x}+\mathbf{a})\rangle \approx (1 + i(-i\mathbf{a}\cdot\nabla) + \frac{1}{2}(i(-i\mathbf{a}\cdot\nabla))^2 + \ldots)|\psi(\mathbf{x})\rangle \qquad \hat{p} = -i\nabla$$

$$= (1 + i\mathbf{a}\cdot\hat{p} + \frac{1}{2}(i\mathbf{a}\cdot\hat{p}) + \ldots)|\psi(\mathbf{x})\rangle = e^{i\mathbf{a}\cdot\hat{p}}|\psi(\mathbf{x})\rangle$$

Translations generated by momentum operator $\hat{p}$

- $|\psi(\mathbf{x} + \mathbf{a})\rangle = e^{i\mathbf{a}\cdot\hat{p}}|\psi(\mathbf{x})\rangle$
- Translational invariance $\Longrightarrow$ momentum conservation; $[\hat{p}, \hat{H}] = 0$
  (Note: rules out external potential $V(\mathbf{x})$!)

Rotations, angular momentum

Rotation around z axis:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x\cos\varphi - y\sin\varphi \\ x\sin\varphi + y\cos\varphi \\ z \end{pmatrix} \underset{\varphi\to 0}{\approx} \begin{pmatrix} x - y\varphi \\ y + x\varphi \\ z \end{pmatrix}$$

Wavefunction (scalar) rotated in infinitesimal rotation as

$$\psi(\mathbf{x}) \to \psi(\mathbf{x}') \approx \psi(\mathbf{x}) + i\delta\varphi(-i)\left(x\partial_y - y\partial_x\right)\psi(\mathbf{x}) = \left(1 + i\delta\varphi\hat{\mathbf{L}}_z\right)\psi(\mathbf{x})$$

Rotation around general axis $\boldsymbol{\theta}$

Direction: $\boldsymbol{\theta}/|\boldsymbol{\theta}|$, angle $\theta = |\boldsymbol{\theta}|$.

$$\psi(\mathbf{x}) \longrightarrow e^{i\boldsymbol{\theta}\cdot\hat{\mathbf{L}}}\psi(\mathbf{x})$$

Rotations generated by (orbital) **angular momentum** (pyörimismäärä) operator $\hat{\mathbf{L}}$.

Rotations as part of Lorentz group

$\hat{\mathbf{L}}$ rotates coordinate-dependence the wavefunction.
You know that particles also have **spin**=intrinsic angular momentum.

Total angular momentum is $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$

- $\hat{\mathbf{J}}$ is one of the generators of the Lorentz group $\Longrightarrow$ conserved in Lorentz-invariant theories
- $\hat{\mathbf{L}}$,$\hat{\mathbf{S}}$ not separately conserved
- For $S \neq 0$ particles also spin rotates in coordinate transformation: $\psi(\mathbf{x}) \longrightarrow e^{i\boldsymbol{\theta}\cdot\hat{\mathbf{J}}}\psi(\mathbf{x})$
- Spectroscopic notation ${}^{2S+1}L_J$
  L often denoted by letter: $L = 0 : S; \quad L = 1 : P; \quad L = 2 : D \quad \ldots$

Spin S of composite particle = total J of constituents: examples

$\Delta^{++} = uuu$ has $S = 3/2$: S-wave ($L = 0$), quark spins ${}^4S_{3/2}\ |\uparrow\rangle|\uparrow\rangle|\uparrow\rangle$

$\chi_c = c\bar{c}$ has $S_{\chi_c} = 0$, resulting from P-wave orbital state $L = 1$ and quark spins $|\uparrow\rangle|\uparrow\rangle$ for total $S = 1$, but $\mathbf{S} \upharpoonleft\downharpoonright \mathbf{L} \Longrightarrow J = 0.\ {}^3P_0$

QM of angular momentum

Recall from QMI

- Commutation relations: $[\hat{J}_i, \hat{J}_j] = i\varepsilon_{ijk}\hat{J}_k$, $[\hat{J}^2, \hat{J}_i] = 0$.
- Eigenstates of $\hat{J}_3$ & $\hat{J}^2$ are denoted $|j, m\rangle$ with

$$\hat{J}_3|j, m\rangle = m|j, m\rangle \qquad \hat{J}^2|j, m\rangle = j(j+1)|j, m\rangle$$

- Permitted range $m = -j, -j+1, \ldots j$
- Ladder operators $\hat{J}_\pm = \hat{J}_1 \pm i\hat{J}_2$ raise and lower the z-component of the spin:

$$\begin{aligned} \hat{J}_+|j, m\rangle &= \sqrt{j(j+1) - m(m+1)}|j, m+1\rangle \implies 0, \text{ if } m = j \\ \hat{J}_-|j, m\rangle &= \sqrt{j(j+1) - m(m-1)}|j, m-1\rangle \implies 0, \text{ if } m = -j \end{aligned}$$

Coupling of angular momentum

Recall from QMI

- How to add two angular momenta $\hat{J} = \hat{J}_1 + \hat{J}_2$?
- State (tensor) product $|j_1, m_1\rangle \otimes |j_2, m_2\rangle \equiv |j_1, m_1\rangle|j_2, m_2\rangle \equiv |j_1, m_1, j_2, m_2\rangle$.
- Want to express $|j_1, j_2, J, M = m_1 + m_2\rangle$ in terms of $|j_1, m_1\rangle|j_2, m_2\rangle$.
- This is done via Clebsch-Gordan coefficients:

$$
\begin{aligned}
|j_1, j_2, J, M\rangle &= \left[\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1, m_1\rangle|j_2, m_2\rangle\langle j_1, m_1, j_2, m_2|\right] |j_1, j_2, J, M\rangle \\
&= \sum_{m_1, m_2} \overbrace{\langle j_1, m_1, j_2, m_2|\, j_1, j_2, J, M\rangle}^{CG,\text{from tables}} |j_1, m_1\rangle\, |j_2, m_2\rangle
\end{aligned}
$$

- Possible values $J = |j_1 - j_2|, \ldots, j_1 + j_2$. CG$\neq 0$ only if $m_1 + m_2 = M$

Not only for angular momentum

Other quantum numbers have similar mathematics: SU(N) group

Isospin SU(2) (like angular momentum);

Color SU(3) (slightly more complicated).

Properties of Clebsch-Gordan coefficients

Recall from QMI

One can also go the other way

$$|j_1, m_1\rangle|j_2, m_2\rangle = \sum_{M,J} \langle j_1, j_2, J, M|j_1, m_1, j_2, m_2\rangle \quad |j_1, j_2, J, M\rangle$$

Transformation between orthonormal bases is always unitary
+ Clebsch's are real

$\Longrightarrow$ matrix of Clebsch's actually orthogonal.

$$\overbrace{\langle j_1, j_2, J, M|j_1, m_1, j_2, m_2\rangle}^{\text{elem } JM, j_1 j_2 \text{ in uncoupled} \to \text{coupled}} = \overbrace{\langle j_1, m_1, j_2, m_2|j_1, j_2, J, M\rangle}^{\text{elem } j_1, m_1, j_2, m_2 \text{ in coupled} \to \text{uncoupled}}$$

Consequence: probability to find coupled state in uncoupled is the same as probability to find uncoupled in coupled:

$$|\langle j_1, m_1, j_2, m_2|j_1, j_2, J, M\rangle|^2$$

Clebsch tables

36. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONIC AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M$$

1/2×1/2

m_1	m_2	J=1, M=+1		
+1/2	+1/2	1		
		J=1, M=0	**J=0, M=0**	
+1/2	−1/2	1/2	1/2	
−1/2	+1/2	1/2	−1/2	
		J=1, M=−1		
−1/2	−1/2	1		

1×1/2

m_1	m_2		
		3/2, +3/2	
+1	+1/2	1	
		3/2, +1/2	**1/2, +1/2**
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3
		3/2, −1/2	**1/2, −1/2**
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3
		3/2, −3/2	
−1	−1/2	1	

2×1

m_1	m_2			
		3, +3		
+2	+1	1		
		3, +2	**2, +2**	
+2	0	1/3	2/3	
+1	+1	2/3	−1/3	
		3, +1	**2, +1**	**1, +1**
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10
		3, 0	**2, 0**	**1, 0**
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10
		3, −1	**2, −1**	**1, −1**
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5
		3, −2	**2, −2**	
−1	−1	2/3	1/3	
−2	0	1/3	−2/3	
		3, −3		
−2	−1	1		

1×1

m_1	m_2			
		2, +2		
+1	+1	1		
		2, +1	**1, +1**	
+1	0	1/2	1/2	
0	+1	1/2	−1/2	
		2, 0	**1, 0**	**0, 0**
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3
		2, −1	**1, −1**	
0	−1	1/2	1/2	
−1	0	1/2	−1/2	
		2, −2		
−1	−1	1		

3/2×1

m_1	m_2			
		5/2, +5/2		
+3/2	+1	1		
		5/2, +3/2	**3/2, +3/2**	
+3/2	0	2/5	3/5	
+1/2	+1	3/5	−2/5	
		5/2, +1/2	**3/2, +1/2**	**1/2, +1/2**
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6
		5/2, −1/2	**3/2, −1/2**	**1/2, −1/2**
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2
		5/2, −3/2	**3/2, −3/2**	
−1/2	−1	3/5	2/5	
−3/2	0	2/5	−3/5	
		5/2, −5/2		
−3/2	−1	1		

2×1/2

m_1	m_2		
		5/2, +5/2	
+2	+1/2	1	
		5/2, +3/2	**3/2, +3/2**
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5
		5/2, +1/2	**3/2, +1/2**
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5
		5/2, −1/2	**3/2, −1/2**
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5
		5/2, −3/2	**3/2, −3/2**
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5
		5/2, −5/2	
−2	−1/2	1	

3/2×1/2

m_1	m_2		
		2, +2	
+3/2	+1/2	1	
		2, +1	**1, +1**
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4
		2, 0	**1, 0**
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2
		2, −1	**1, −1**
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4
		2, −2	
−3/2	−1/2	1	

Example

Short notation for representations according to dimension:

1 Singlet, $j = 0 = m$
2 Doublet $j = \frac{1}{2}, \quad m = \pm\frac{1}{2}$
3 Triplet $j = 1, \quad m = -1, 0, 1 \quad \ldots$

Couple two spin-1/2 particles $2 \otimes 2 = 3 \oplus 1$

$$\text{triplet} \begin{cases} \left|\frac{1}{2}, \frac{1}{2}, 1, 1\right\rangle & = \quad 1 \left|\frac{1}{2}, \frac{1}{2}\right\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle \\ \left|\frac{1}{2}, \frac{1}{2}, 1, 0\right\rangle & = \quad \frac{1}{\sqrt{2}} \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle + \frac{1}{\sqrt{2}} \left|\frac{1}{2}, \frac{1}{2}\right\rangle \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \\ \left|\frac{1}{2}, \frac{1}{2}, 1, -1\right\rangle & = \quad 1 \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \end{cases}$$

$$\text{singlet} \left\{ \left|\frac{1}{2}, \frac{1}{2}, 0, 0\right\rangle \quad = \quad \frac{1}{\sqrt{2}} \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle - \frac{1}{\sqrt{2}} \left|\frac{1}{2}, \frac{1}{2}\right\rangle \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \right.$$

- Interpretation: prepare two particles in $|\frac{1}{2}, -\frac{1}{2}\rangle|\frac{1}{2}, \frac{1}{2}\rangle \implies$ measure $J = 0, 1$ with probability $|1/\sqrt{2}|^2 = \frac{1}{2}$
- Triplet symmetric in $1 \leftrightarrow 2$, singlet antisymmetric in $1 \leftrightarrow 2$
  (Why singlet $|J = 0, M = 0\rangle$ has $-$ and triplet $|J = 1, M = 0\rangle$ $+$? Try operating with $J_\pm$ ...)
- More particles: repeat. E.g. $2 \otimes 2 \otimes 2 = 2 \otimes (3 \oplus 1) = 4 \oplus 2 \oplus 2 \implies 3$ spin $1/2$ particles can form quadruplet ($j = 3/2$) or 2 doublets.

Parity transformation

- Flip the signs of all **space**-components of normal (=polar) four-vectors:

$$\mathbf{x} \xrightarrow{P} -\mathbf{x}$$
$$\mathbf{p} \xrightarrow{P} -\mathbf{p}$$

- Cross-products $(\mathbf{a} \times \mathbf{b})^k = \varepsilon^{ijk} a^i b^k$ are not real vectors, but rank 2 P-invariant antisymmetric tensors, i.e. 3 component-**pseudovectors**. E.g. angular momentum

$$\mathbf{L} = \mathbf{x} \times \mathbf{p} \xrightarrow{P} (-\mathbf{x}) \times (-\mathbf{p}) = \mathbf{x} \times \mathbf{p} = \mathbf{L}$$

scalar: Stays same in both rotations and parity transformations: e.g. $\mathbf{x}^2 \rightarrow \mathbf{x}^2$, mass m, charge e etc.

pseudoscalar: Stays same in rotations, but changes sign under parity, e.g. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \varepsilon^{ijk} a^i b^j c^k$

(If you already know the Dirac equation: matrix elements of γ^μ are vectors, those of $\gamma^5\gamma^\mu$ pseudovectors. $\mathrm{Tr}\, \gamma^5\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta = -4i\varepsilon^{\alpha\beta\gamma\delta}$.)

Intrinsic parity

In QM, transformation = operator in Hilbert space. Parity operator $\hat{\mathrm{P}}$.

Parity conservation

Strong and electromagnetic interactions conserve parity, i.e. $[\hat{\mathrm{P}}, \hat{\mathrm{H}}] = 0$
Particles that are (1) stable or (2) decay via weak interaction only
$\Longrightarrow$ are eigenstates of strong & e.m. Hamiltonian
$\Longrightarrow$ particles are (can be chosen as (*)) eigenstates of $\hat{\mathrm{P}}$.
Notation: J^P. E.g. pion $\pi^{\pm}$: $J^P = 0^-$; proton $J^P = \frac{1}{2}^+$

Whole transformation on single particle (a) parity eigenstate

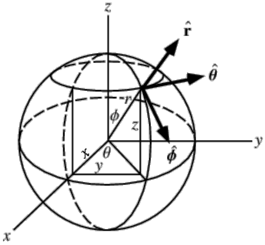
$$\hat{\mathrm{P}}\psi_a(t, \mathbf{x}) \longrightarrow \overbrace{P_a}^{\text{eigenvalue}} \psi_a(t, -\mathbf{x}).$$

$$\hat{\mathrm{P}}^2 = 1 \Longrightarrow P_a = \pm 1$$

P_a is **intrinsic parity** of particle a, can be ± 1.

(* Suppose $|\psi\rangle$ is eigenstate of $\hat{\mathrm{H}}$: $\hat{\mathrm{H}}|\psi\rangle = E|\psi\rangle$. Because $[\hat{\mathrm{P}}, \hat{\mathrm{H}}] = 0$ we have $\hat{\mathrm{H}}(\hat{\mathrm{P}}|\psi\rangle) = \hat{\mathrm{P}}(\hat{\mathrm{H}}|\psi\rangle) = E(\hat{\mathrm{P}}|\psi\rangle)$, so $\hat{\mathrm{P}}|\psi\rangle$ is also eigenstate. Form new linear combinations $|\psi\pm\rangle \equiv (1/\sqrt{2})(|\psi\rangle \pm \hat{\mathrm{P}}|\psi\rangle)$ that are eigenstates of both $\hat{\mathrm{H}}$ and $\hat{\mathrm{P}}$: $\hat{\mathrm{P}}|\psi\pm\rangle = \pm|\psi\pm\rangle$.)

Parity of bound state



Recall from QMI: bound state Schrödinger eq. in rotationally invariant potential $V(r)$

$\Longrightarrow$ Solutions eigenfunctions of $\hat{\mathbf{L}}^2 \& \hat{L}_z$,
These are **spherical harmonics** $Y_l^m(\theta, \varphi)$.
Under parity transformation $\mathbf{x} \to -\mathbf{x}$:

$$\theta \to \pi - \theta \quad \Longrightarrow \quad z = r\cos\theta \to -z$$

$$\varphi \to \varphi + \pi \quad \Longrightarrow \quad (x, y) = r\sin\theta(\cos\varphi, \sin\varphi) \to -(x, y)$$

Spatial wavefunction under parity

$$Y_\ell^m(\theta, \varphi) \xrightarrow{P} Y_\ell^m(\pi - \theta, \varphi + \pi) = (-1)^\ell Y_\ell^m(\theta, \varphi)$$

So a (bound) state of particles a and b in an ℓ wave state transforms as

$$|a, b, \ell\rangle \xrightarrow{P} P_a P_b (-1)^\ell |a, b, \ell\rangle$$

Intrinsic parity of elementary particles

Spin $1/2$ particles f (fermion) and $\bar{f}$ (corresponding antifermion)

Obey the Dirac equation (details later) $\Longrightarrow$ one can show

$$P_f P_{\bar{f}} = -1 \quad \Longrightarrow \quad \text{convention } P_\ell = P_q = +1, \quad P_{\bar{\ell}} = P_{\bar{q}} = -1$$

Photon γ (and gluon) parity

- Photon: e.m. field A^μ. Electric field $\mathbf{E} = -\nabla A^0 - \partial_t \mathbf{A}$ satisfies Gauss' law $\nabla \cdot \mathbf{E} = -\nabla^2 A^0 - \partial_t \nabla \cdot \mathbf{A} = \rho$.
- Vacuum $\rho = 0$ and one can choose (gauge symmetry, see later) $\nabla \cdot \mathbf{A} = 0$ $\Longrightarrow A^0 = 0$.
- Because $\mathbf{A} \xrightarrow{P} -\mathbf{A}$, the parity of a photon is $P_\gamma = -1$.

Consequence: positronium decay $e^+e^- \to \gamma\gamma$

- Para-positronium, $S_{ee} = 0$, $L_{ee} = 0 \Longrightarrow J = 0$. $P = -1 \Longrightarrow L_{\gamma\gamma} = 1$,
  (Can deduce $S_{\gamma\gamma} = 1$, because $(L_{\gamma\gamma} = 1) \otimes (S_{\gamma\gamma} = 0/2)$ do not include $J = 0$.) .
- Ortho-positronium, $S_{ee} = 1$, $L_{ee} = 0 \Longrightarrow J = 1$.
  (Decay to $\gamma\gamma$ not allowed because of charge conjugation, as we will see later.)

Intrinsic parity of hadrons

Mesons

Meson is bound state of $q\bar{q}$: parity $P_M = P_q P_{\bar{q}} (-1)^L = -(-1)^\ell = (-1)^{\ell+1}$
Ground state is $\ell = 0 \implies$ lightest mesons usually $P = -1$.

Baryon

Baryon is bound state of qqq: parity

$$P_B = P_q P_q P_q (-1)^{L_{12}+L_3} = 1^3 (-1)^{L_{12}+L_3} = (-1)^{L_{12}+L_3}$$

Antibaryon

$$P_{\bar{B}} = P_{\bar{q}} P_{\bar{q}} P_{\bar{q}} = (-1)^3 (-1)^{L_{12}+L_3} = -(-1)^{L_{12}+L_3} = -P_B$$

$\implies$ just like should be for spin $1/2$ (Dirac) particle.
Ground states are $L_{12} = L_3 = 0$.

(L_{12}: orbital angular momentum of quarks 1&2. L_3: quark 3 w.r.t. the other two.)

Meson spin and parity states, examples

Example: pseudoscalar meson

E.g. in π^0 (pion) $u\bar{u}, d\bar{d}$ have spin $\sim |q\uparrow\rangle|\bar{q}\downarrow\rangle - |q\downarrow\rangle|\bar{q}\uparrow\rangle$

$\implies$ singlet $S = 0$.

Ground state $L = 0$; Spectroscopic notation $^{2S+1}L_J = {}^1S_0$.

Example: vector mesons

- Example $J/\Psi = c\bar{c}$. Spin state $|c\uparrow\rangle|\bar{c}\uparrow\rangle$, ground state $\ell = 0$ $\implies J = S + L = 1 + 0 = 1$.
- Parity $P_{J/\Psi} = (-1)^{\ell=0} P_c P_{\bar{c}} = 1 \cdot 1 \cdot (-1) = -1$.
- Vector field A^μ; corresponding particle photon has $J = 1$, $P = -1$.
- Particles with these quantum numbers are called "vectors". In particular **vector meson**.
- Practical consequence: very clean decay modes $J/\Psi \to \gamma^* \to e^+e^-$

Charge conjugation

Charge conjugation Ĉ: change every particle to its antiparticle

Strong and electromagnetic interaction remain invariant, weak interaction not.

Antiparticles have opposite charges.
Electric charge: $Q \xrightarrow{C} -Q$; strangeness $S \xrightarrow{C} -S$ ($s \xrightarrow{C} \bar{s}, \bar{s} \xrightarrow{C} s$).

***C*-parity**

Some particles are their own antiparticles, i.e. eigenstates of Ĉ.
$\hat{C}|\psi\rangle = C_\psi|\psi\rangle$, $C_\psi = \pm 1$.
Examples: photon, neutral mesons ($u\bar{u} \xrightarrow{C} \bar{u}u \sim u\bar{u}$)

Photon *C*-parity is -1

Question: e^- in magnetic field **B**; trajectory bends in one direction.
C: $e^- \rightarrow e^+$, does e^+ bend in other direction, e.m. still invariant?
No, because charge conjugation also changes direction of **B**: $\mathbf{A} \xrightarrow{C} -\mathbf{A}$

$$\Longrightarrow \quad C_\gamma = -1.$$

Two particle bound state C-parity

Bound state of particle and its antiparticle is a C eigenstate.

Orbital $|\Psi(\mathbf{x})\bar{\Psi}(-\mathbf{x})\rangle \sim Y_\ell^m(\theta,\varphi) \xrightarrow{C} Y_\ell^m(\pi-\theta,\varphi+\pi) = (-1)^\ell Y_\ell^m(\theta,\varphi)$

(bound state C.M. frame) $\Longrightarrow$ Action of C on orbital wavefunction of particle-antiparticle state is $(-1)^L$, just like parity.

Spin Recall $1/2 \otimes 1/2$ coupling:

singlet $S = 0 \sim |\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle$ antisymmetric,
triplet $S = 1 \sim |\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle$ symmetric.

For fermions: **spin odd is symmetric, even antisymmetric**

(See CG table, exercise) factor $(-1)^{S+1}$ in C-parity.

Bosons: even S is symmetric: $(-1)^S$

Exchange If particle/antiparticle are **fermions**; additional factor (-1) for exchanging them back after C. (Fermion/antifermion creation/annihilation operators anticommute: $b^\dagger d^\dagger = -d^\dagger b^\dagger$, multiparticle wavefunction is antisymmetric w.r.t. particle exchange. Remember Pauli rule.)

All together

- Fermion-fermion (bound) state (e.g. meson) $C = (-1)^{S+L}$
- Boson-antiboson (bound) state (e.g. $\pi^+\pi^-$ "atom") $C = (-1)^{S+L}$

Pion decay to photons

- Neutral pion π^0 is its own antiparticle ($u\bar{u} - d\bar{d}$, mysterious - sign later)
- It mostly decays via $\pi^0 \to \gamma\gamma$.
- Photon final state, lifetime $c\tau = 25.1$nm $\implies$ looks like e.m. decay.
- Let us check P and C conservation in the quark model.
  - Pion is $L = S = 0$ state (pseudoscalar)
  - $P_{\pi^0} = P_q P_{\bar{q}}(-1)^L = -1; \quad C_{\pi^0} = (-1)^{L+S} = 1$
  - $\gamma\gamma$: $\quad P_{\gamma\gamma} = (P_\gamma)^2(-1)^{L_{\gamma\gamma}} = (-1)^{L_{\gamma\gamma}} = -1$. $C_{\gamma\gamma} = (-1)^{L_{\gamma\gamma}+S_{\gamma\gamma}} = 1$.
  - This is possible with $L_{\gamma\gamma} = 1$ and $S_{\gamma\gamma} = 1$ coupled into $J = 0$ $\implies$ works.
- Is $\pi^0 \to \gamma\gamma\gamma$ possible? $C_{\gamma\gamma\gamma} = -1 \neq C_{\pi^0}$.
  - If allowed, expect suppression by $\alpha_{\text{e.m.}} \approx 1/137$.
  - Experiment: $B(\pi^0 \to \gamma\gamma\gamma) < 3.1 \times 10^{-8} B(\pi^0 \to \gamma\gamma)$
    Much bigger suppression $\implies$ decay conserves C.

Similarly η meson ($J^{PC} = 0^{-+}$) decays:

- $\eta \to \gamma\gamma$ ($B = 39\%$)
- $\eta \to \pi^0 + \pi^0 + \pi^0$ ($B = 33\%$) — Check P,C
- $\eta \to \pi^0 + \pi^+ + \pi^-$ ($B = 23\%$) — Exercise: Why no $\eta \to \pi^0\pi^0$?

Back to positronium decay, charge conjugation

Recall positronium decay $e^+e^- \to n \times \gamma$ ($n \geq 2$)

Para-positronium $S = 0, L = 0 \Longrightarrow J = 0.$

Ortho-positronium $S = 1, L = 0 \Longrightarrow J = 1$

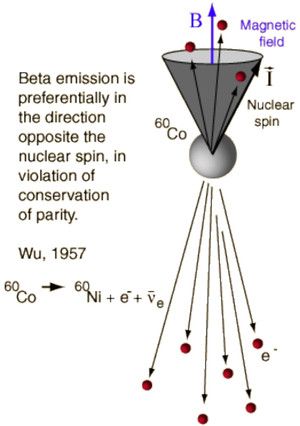
Parity $P = P_{e^+} P_{e^-} (-1)^{L_{e^+e^-}} = -1$
Can we decay to $n = 2$ photons? $P = -1 = (-1)^{L_{\gamma\gamma}} \Longrightarrow L_{\gamma\gamma} = 1.$

- Para, $J = 0$: $C = (-1)^{S_{\gamma\gamma}+L_{\gamma\gamma}} = 1.$
  Can get $J = 0$, $L_{\gamma\gamma} = 1$ with $S_{\gamma\gamma} = 1$ $\Longrightarrow$ Decay to $\gamma\gamma$ possible
- Ortho $C = (-1)^{S_{\gamma\gamma}+L_{\gamma\gamma}} = -(-1)^{S_{\gamma\gamma}} = -1.$
  Combined C and P would require $S_{\gamma\gamma} = 0$ or 2
  But cannot get total $J = 1$ from $S_{\gamma\gamma} = 0$ or 2 and $L_{\gamma\gamma} = 1$
  ($L_{\gamma\gamma} = 1$; $S_{\gamma\gamma}$ even is antisymmetric, bosons) $\Longrightarrow$ $\gamma\gamma$ not possible

Ortho-positronium can only decay to $\gamma\gamma\gamma$.
Lifetimes $S = 0$ $\tau = 1.244 \times 10^{-10}$s — $S = 1$ $\tau = 1.386 \times 10^{-7}$s,
$\Longrightarrow$ consistent with suppression by $\alpha_{\text{e.m.}} \approx 1/137$

C, P violation in weak interaction, experimental evidence

C,P violated "maximally" in weak interactions



$\mathbf{B}, \mathbf{S} \xrightarrow{P} \mathbf{B}, \mathbf{S}$; $\mathbf{p} \xrightarrow{P} -\mathbf{p}$

$\mathbf{p} \sim \mathbf{B}$ violates P.

Also C transformation $\mathbf{B} \to -\mathbf{B}$

$\implies$ Wu exp does not violate CP

SM also has a **very small** violation of CP

Decay of C-conjugate pair

$K^0 = d\bar{s}, \bar{K}^0 = \bar{d}s$.

Can form linear combinations:

- 2 CP eigenstates: decay into $\pi\pi$ ($CP = 1$) or $\pi\pi\pi$ ($CP = -1$)
- 2 weak int. eigenstates: K_S^0 K_L^0 (short, long: different lifetimes).
- If weak interactions respect CP, these are the same.

Cronin, Fitch 1964: K_L^0 decays mostly to $\pi\pi\pi$, but also to $\pi\pi$ $\implies$ weak and CP eigenstates not same.