FYSH300, fall 2013

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Part 3: Scattering theory

Basic idea

Particle physics experiment:
Know incoming particle(s) — measure outgoing particle properties:

- Momenta (absolute value and direction)
- Charges (electric, other conserved), masses, spins/polarizations
  $\Longrightarrow$ identify particle type
- Repeat experiment and measure number/frequency of events
  $\Longrightarrow$ collect **statistics**, form spectra (jakauma) $\mathrm{d}N/\mathrm{d}^3\mathbf{p}$, correlations
- Compare to theory to test/learn about fundamental interactions.

Terminology I

What goes in

Beam

- Consists of **projectile** (ammus) particles a (e.g. $e^{\pm}, \mu^{\pm}, \nu, p, \pi, \ldots$).
- $\sim$ monoenergetic & collimated ($|\Delta p|/|p| \ll 1$).
- Sufficient ($10^{11} \ldots 10^{13}$ particles/s) but not too high **intensity**: want many scatterings, but no interactions between beam particles
- May be polarized or not
- Consists of **bunches** (LHC: 7.4cm long, 7.5m $=$ 25ns apart)

Target Macroscopic sample at rest in lab (kohtio)

- *Large* number of particle b.
- Distance between scattering centers $d_b \gg \lambda_a = h/p_a$: independent scatterings
- Thin, so that each a scatters most once.

Terminology II

Where it collides

Collider experiment: **two beams collide**; LHC, Tevatron, LEP,

+ Achieve higher energy/\$ ($\sqrt{s} = E_a^* + E_b^*$, \$ $\sim E_{\text{beam}} \sim \sqrt{s}$)
- Hard to align beams: less events

(Not always symmetric: e.g. HERA 30GeV $e^- + 920$GeV p)

Fixed target experiment: **beam+target**

+ Can make target big enough so every beam particle scatters $\Longrightarrow$ better statistics
- Lower energy/\$ ($\sqrt{s} \approx \sqrt{2m_b E_a^{\text{TRF}}}$, \$ $\sim E_{\text{beam}} \sim s$)

An experimentalist cannot boost the lab to $v = 0.9999$ but a theorist can, so we can do the following derivations in the TRF (= fixed target lab frame).

Terminology III

What comes out

Exclusive experiment: measure and identify **all** final state particles. Examples

- $\pi^- + p \longrightarrow \pi^+ + \pi^- + n$
- $e^+ + e^- \longrightarrow \mu^+ + \mu^-$

Inclusive experiment: only care about/measure certain particles (rest denoted as "X"). Completely random examples:

- $p + \bar{p} \longrightarrow X$
- $\pi^- + p \longrightarrow \pi^+ + X$
- $p + \bar{p} \longrightarrow \mu^+ + \mu^- + X$

Elastic scattering: $a + b \longrightarrow a + b$ (initial and final particles same).

Inelastic scattering: final state particles $\neq$ initial state particles. Examples:

- $e^+ + e^- \longrightarrow e^+ + e^-$, elastic
- $e^+ + e^- \longrightarrow \mu^+ + \mu^-$, inelastic
- $e^+ + e^- \longrightarrow \mu^+ + \mu^- + \gamma$, inelastic
- $p + \bar{p} \longrightarrow X$, with $X \neq p + \bar{p}$ inelastic (total inelastic cross section)

Luminosity and cross section

Number of events in scattering $a + b \longrightarrow X$ depends on

- Experimental setup: Intensity of incoming projectile particles a + number of target particles b encountered
- Properties of fundamental interaction

Number of events per unit time

$$\frac{\mathrm{d}N_{\mathrm{ev}}^{a+b\to X}}{\mathrm{d}t} = \sigma^{a+b\to X} \overbrace{\Phi_a N_b}^{\mathcal{L}}, \text{ where}$$

Φ_a is the **flux** of particle a (particles/(time*area))

N_b is the number of target particles b "under" the beam

$\mathcal{L}$ is the **luminosity**

σ is the **cross section**: this encodes everything about the particle interaction.

Dimensions: $\left[\frac{\mathrm{d}N}{\mathrm{d}t}\right] = \frac{1}{s} = m^2 \frac{1}{sm^2}$ $\Longrightarrow$ Cross section is an **area**.

(Remember we chose to work in TRF, but you can always boost.)

Some interpretation

Cross section

σ can be thought of as size of particle; but "size" depends on interaction ...
σ of billiard ball scattering is $\pi(2R)^2$, (scatter if centers less than $2R$ apart.)
— but σ of neutrino-billiard ball scattering is much much less.

Loosely cross section = area $\times$ probability to scatter

Probability is dimensionless — Billiard balls have $P = 1$ for distance $< 2R$.

Cross section is Lorentz-invariant

It does not change in boost along the beam axis (or any other) direction
(unlike luminosity, because of time dilation!)

$$\text{Differential } \sigma \quad : \quad \frac{\mathrm{d}N_{\text{ev}}^{a+b\to c+X}}{\mathrm{d}t\,\mathrm{d}^3\mathbf{p}_c}(\mathbf{p}_c) = \frac{\mathrm{d}\sigma^{a+b\to c+X}}{\mathrm{d}^3\mathbf{p}_c}(\mathbf{p}_c)\,\overbrace{\Phi_a N_b}^{\mathcal{L}}$$

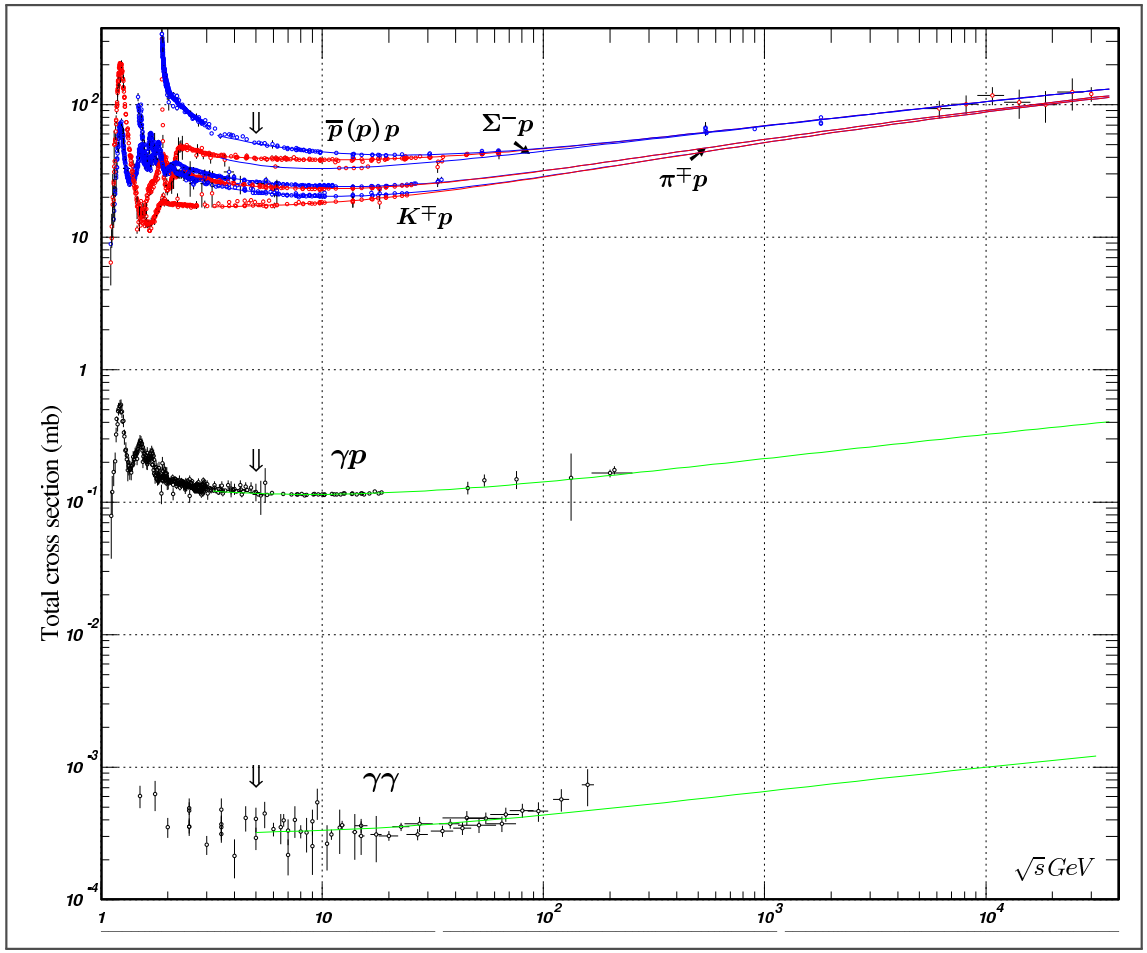
Number of scattering events where final state has particle c in momentum space cube between $\mathbf{p}_c = (p_x, p_y, p_z)$ and $(p_x + \Delta p_x, p_y + \Delta p_y, p_z + \Delta p_z)$
— divided by time interval and $\Delta p_x \Delta p_y \Delta p_z$.

Some cross sections

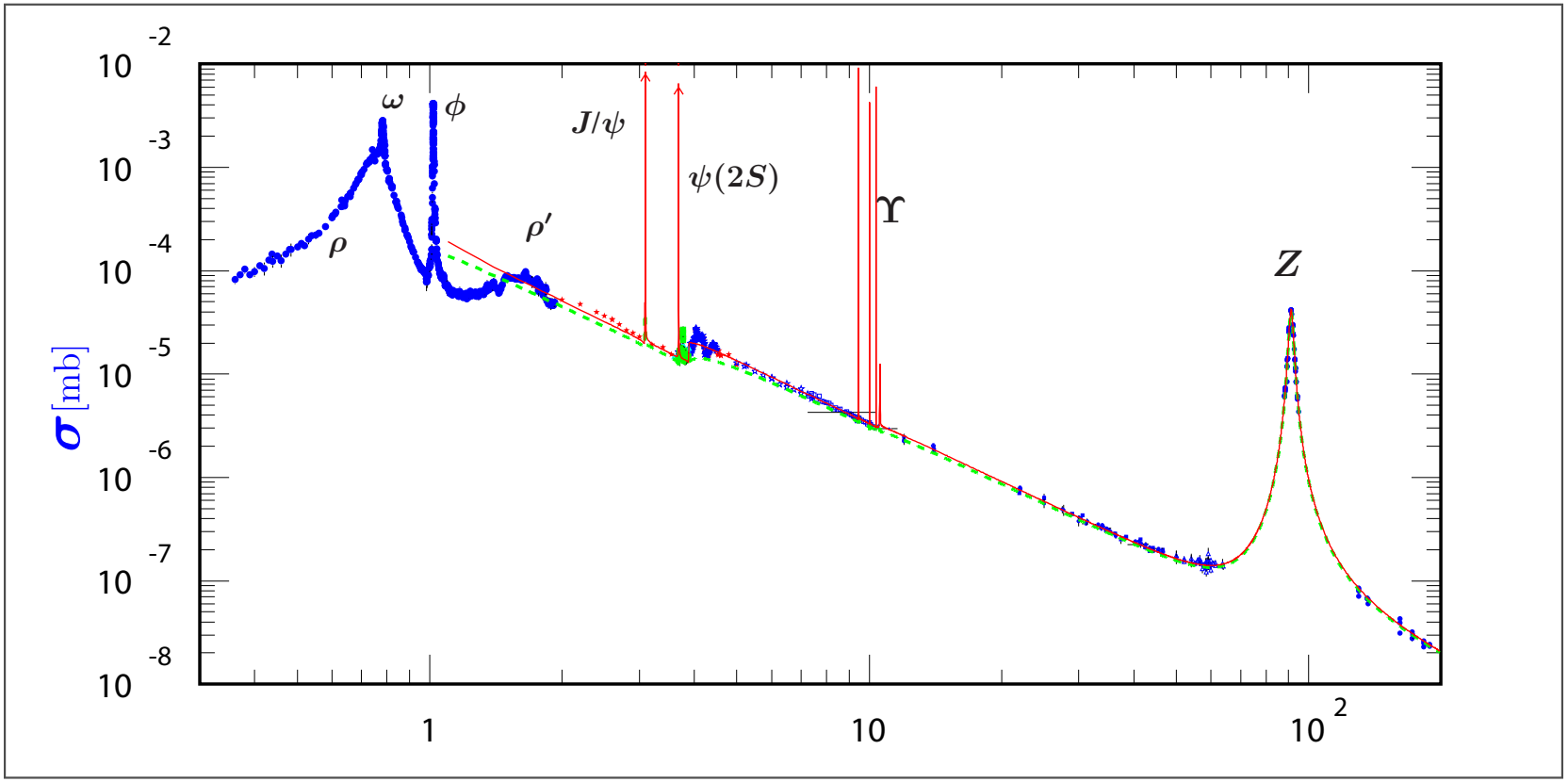
Unit of cross section
1barn $= 10^{-28} m^2$

E.g. proton-proton total inelastic σ at $\sqrt{s} = 200\text{GeV}$ is $\sim 40\text{mb} = (2\text{fm})^2$.

Named by Enrico Fermi:
(The area corresponding to a barn is the approximate area of a typical atomic nucleus which has a size of 10^{-12}cm. For most sub-nuclear processes, this is a very large cross-section, so Fermi suggested "it was as big as a barn".)

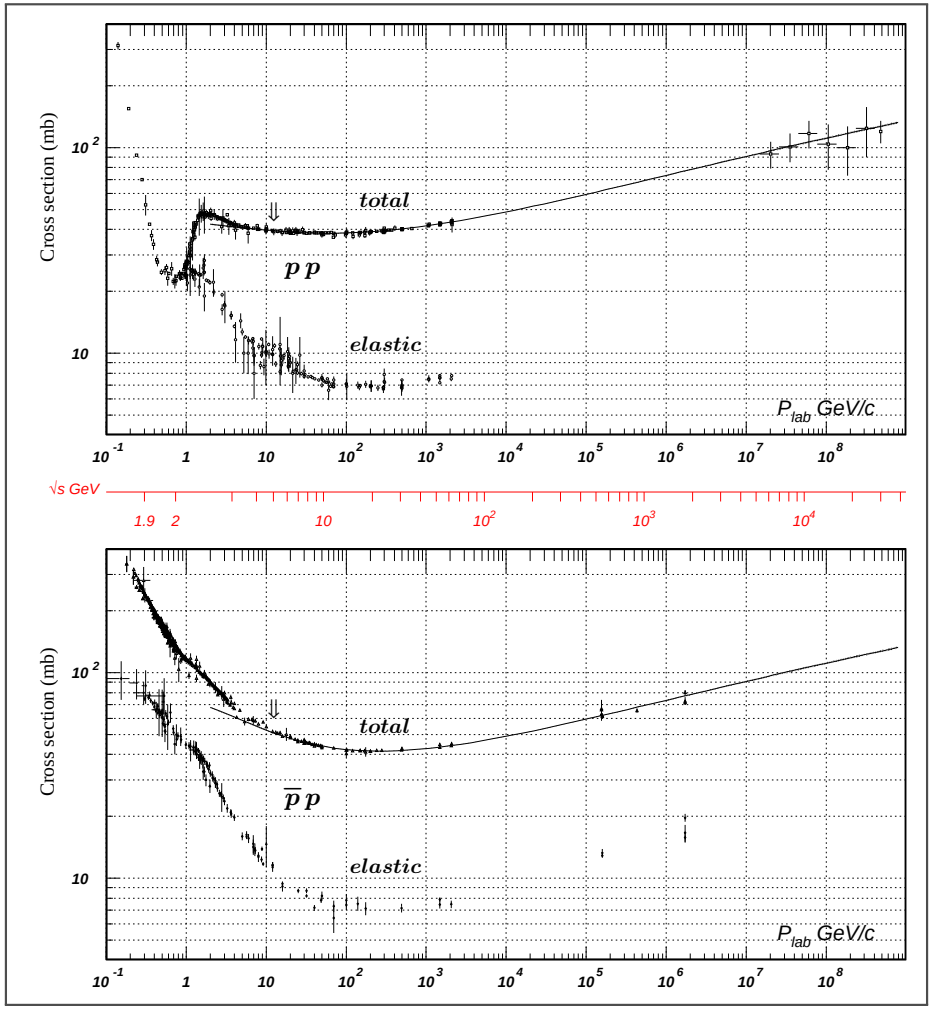


More cross sections I



$\sigma(e^+e^- \to \text{hadrons})$ vs. $\sqrt{s}/\text{GeV}$

More cross sections II



pp and $p\bar{p}$ total and elastic cross sections

Collider/experiment luminosity

- The luminosity is the "figure of merit" of a particle accelerator/collider.
- Bigger luminosity is better, because you can see rarer events.
- LHC design luminosity $\sim 10^{34}\mathrm{cm}^{-2}\mathrm{s}^{-1}$ is very big for a collider.

Often quoted is

Integrated luminosity

$$\int \mathrm{d}t\mathcal{L}$$

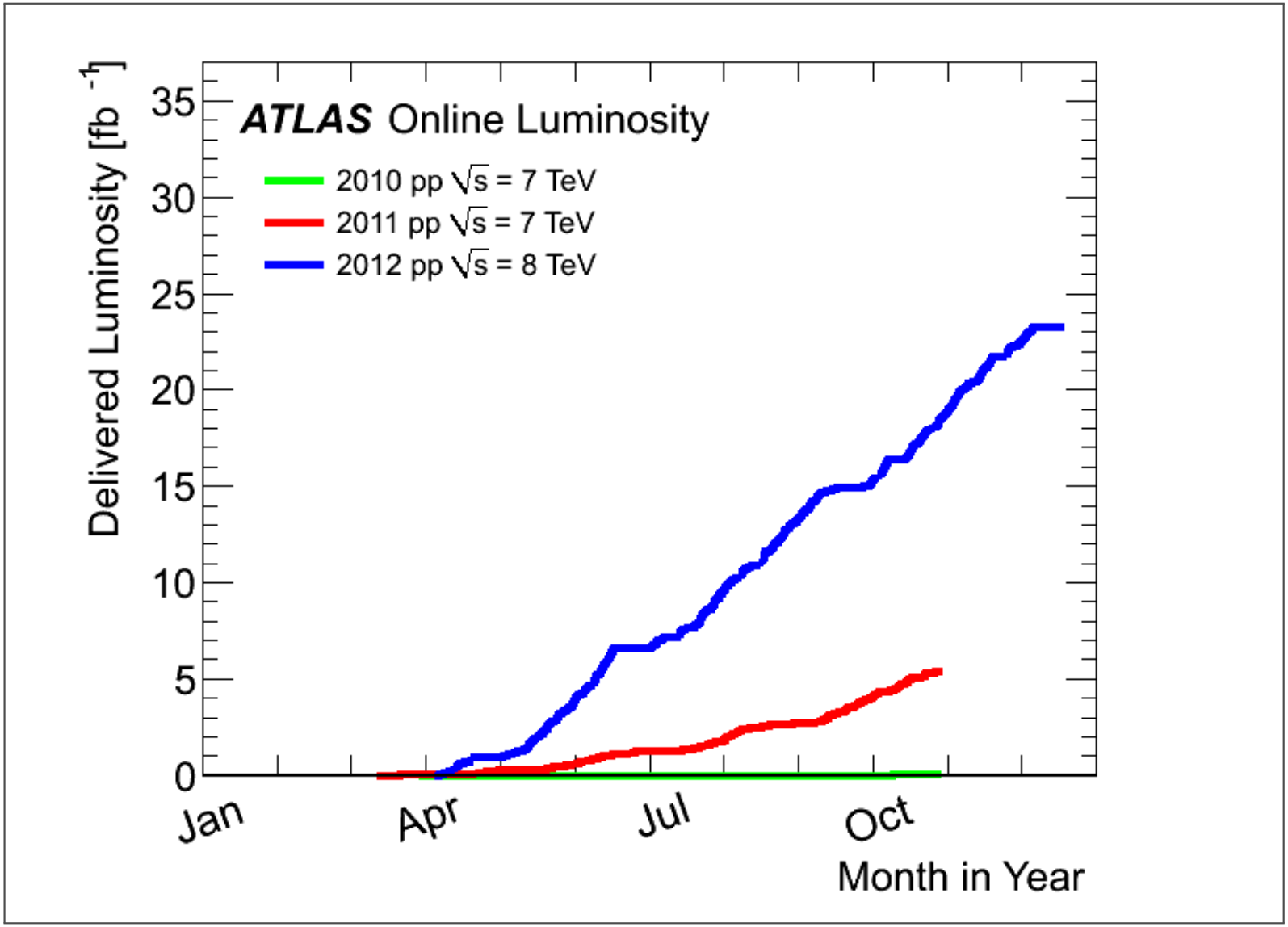
Has units $1/\mathrm{m}^2$: inverse of cross section.

Interpretation:

When LHC has delivered $1\mathrm{fb}^{-1}$ of integrated lumi, a process with cross section $\sigma = 1\mathrm{fb}$ should have happened one time ($\pm$ statistical fluctuations).

(LHC design luminosity would give $\sim 300\mathrm{fb}^{-1}$ in full year of nonstop running.)

ATLAS measured integrated luminosity at CERN/LHC



Quantum scattering theory: the S-matrix

Cross section related to quantum mechanical **matrix element**, computable from the Hamiltonian of a given theory.
Consider a scattering process $a + b \to 1 + 2 + \cdots + n$

Initial state $|i\rangle$ is system of two free particles $(a + b)$ at $t \to -\infty$.
Final state $|f\rangle$ is system of n free particles at $t \to \infty$

Time development from $|i\rangle$ to $|f\rangle$ described by S-matrix

All information is in matrix elements of $\hat{S}$, **transition amplitudes**

$$S_{fi} \equiv \langle f|\hat{S}|i\rangle, \text{ from which } P_{fi} = |S_{fi}|^2 = \langle i|\hat{S}^\dagger|f\rangle\langle f|\hat{S}|i\rangle$$

is the **transition probability**.

Probability is conserved, sum over all final states is 1 ($\mathbb{I}$=identity operator)

$$1 = \sum_f P_{fi} = \langle i|\hat{S}^\dagger \overbrace{\sum_f |f\rangle\langle f|}^{=\mathbb{I},\text{complete set}} \hat{S}|i\rangle = \langle i|\hat{S}^\dagger\hat{S}|i\rangle = 1 \forall |i\rangle, \quad \text{i.e. } \hat{S} \textbf{ unitary}$$

Subtracting trivial $|i\rangle = |f\rangle$ we get: "T-matrix" (T like transition) :

$$\hat{S} = 1 + i\hat{T}, \text{ i.e. for components } S_{fi} = \delta_{fi} + iT_{fi},$$

Quantum mechanical scattering theory

- We need the **state vectors $|i\rangle$ and $|f\rangle$** corresponding to the initial and final particle content.
- These are many particle states created from the vacuum $|0\rangle$ by particle creation operators that should be familiar if you have taken QMII.
- We will, however, not go into the second quantization formalism here; you need it to derive the Feynman rules, but we will in this course not do that.

Relativistic plane waves

- Contrary to many QM courses we want to have a completely relativistic normalization (So some things may differ from nonrelativistic treatment QM I/II, conventions in atomic physics, quantum optics etc.)
- The in and out-states are described by **a complete set of orthonormal Fock (=multiparticle) states $\{|n\rangle\}$**.

We now need to state our normalization of the wave functions (plane wave states) that correspond to these multiparticle states.

Klein-Gordon equation and plane waves

- Remember from nonrelativistic QM: $E \to i\partial_0$, $\mathbf{p} \to -i\nabla$.
- Nonrelativistic **dispersion relation** is

$$E = \frac{1}{2m}\mathbf{p}^2 \quad \Longrightarrow \quad \text{Schrödinger } i\partial_0\psi(t,\mathbf{x}) = -\frac{1}{2m}\nabla^2\psi(t,\mathbf{x})$$

- Relativistic dispersion relation

$$E^2 = \mathbf{p}^2 + m^2 \quad \Longrightarrow \quad \textbf{Klein-Gordon} \text{ eq } (i\partial_0)^2\varphi(t,\mathbf{x}) = \left[(-i\nabla)^2 + m^2\right]\varphi(t,\mathbf{x})$$

- In covariant notation (recall $\Box = \partial_\mu\partial^\mu = \partial_0^2 - \nabla^2$) relativistic plane wave

$$(\Box + m^2)\varphi(x) = 0, \text{ solution } \mathcal{N}e^{-ip\cdot x} = \mathcal{N}e^{-iE_{\mathbf{p}}t + i\mathbf{p}\cdot\mathbf{x}}$$

(E^2, not E, because $\sqrt{}$ of ∇ awkward. Physically: antiparticles, $E < 0$ solutions.)

- **Note:** $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, **the** energy corresponding to momentum $\mathbf{p}$.

Normalization in a box

Problem: want to work with plane waves (**exact**) momentum eigenstates.
Heisenberg uncertainty relation $\Longrightarrow$ **completely delocalized** in space and **unnormalizable**.

Solution: assume we are in a large box of size $V = L^3$, periodic boundary cond's.

Normalization choice for plane waves $\mathcal{N} e^{-ip\cdot x}$

$$\mathcal{N} = \frac{1}{\sqrt{V}}$$

So that

$$\int_V \mathrm{d}^3\mathbf{x} \left| \mathcal{N} e^{-ip\cdot x} \right|^2 = 1$$

Box density of states

Periodic boundary allows us to count discrete states:

$$\begin{aligned}
\varphi(x,y,z) &= & \varphi(x+L,y,z) &= & \varphi(x,y+L,z) &= & \varphi(x,y,z+L) \\
e^{i(p^1x+p^2y+p^3z)} &= & e^{i(p^1(x+L)+p^2y+p^3z)} &= & e^{i(p^1x+p^2(y+L)+p^3z)} &= & e^{i(p^1x+p^2y+p^3(z+L))} \\
1 &= & e^{ip^1L} &= & e^{ip^2L} &= & e^{ip^3L}
\end{aligned}$$

only true if $(p^1,p^2,p^3) = \frac{2\pi}{L}(n^1,n^2,n^3)$; $n^i \in \mathbb{Z}$.

$$\text{Now} \sum_{\text{states}} = \sum_{n^1,n^2,n^3} \underset{V\to\infty}{\approx} \int \mathrm{d}^3\mathbf{n} = \int \frac{V}{(2\pi)^3}\,\mathrm{d}^3\mathbf{p}$$

($\sum_n f(n) \approx \int \mathrm{d}n f(n)$, when $f(n)$ varies slowly. $\int_{n-1/2}^{n+1/2} \mathrm{d}x = 1$.
"Slowly varying" requires $V \to \infty$ because $\frac{2\pi}{L}(n^1,n^2,n^3)$ close to $\frac{2\pi}{L}(n^1+1,n^2,n^3)$.)

Now single particle states are orthonormal

$$\mathcal{N} = \frac{1}{\sqrt{V}} \implies \langle \mathbf{p}|\mathbf{q}\rangle = \int \mathrm{d}^3\mathbf{x}(\mathcal{N}e^{-ip\cdot x})^*(\mathcal{N}e^{-iq\cdot x}) = \delta_{\mathbf{p},\mathbf{q}},$$

(Discrete Kronecker delta for discrete momenta $\implies$ Now well defined matrices S and T.)

Conserved particle number current, nonrelativistic

In nonrelativistic theory:

- Particle density $\rho_{\text{NR}} = \psi^*(t, \mathbf{x})\psi(t, \mathbf{x})$,
  current $\mathbf{j}_{\text{NR}} = -\frac{i}{2m}(\psi^* \nabla\psi - \psi\nabla\psi^*)$.
- Probability conservation = continuity equation: $\partial_t \rho_{\text{NR}} + \nabla \cdot \mathbf{j}_{\text{NR}} = 0$.
- For a plane wave $\psi(t, \mathbf{x}) = \mathcal{N} e^{-iE_{\mathbf{p}}t + i\mathbf{p}\cdot\mathbf{x}}$ these are:
  $\rho_{\text{NR}} = |\mathcal{N}|^2 \quad ; \quad \mathbf{j}_{\text{NR}} = |\mathcal{N}|^2 \mathbf{p}/m$.

This does not directly work in relativistic case

- A relativistic current is $j^\mu = (\rho, \mathbf{j})$
- It should also have a continuity equation $\partial_\mu j^\mu = 0$.
- But it should be a proper 4-vector;
  thus for a plane wave we should have $j^\mu \sim p^\mu$
- In particular: **Density of free particles in a box should be proportional to the energy** in a Lorentz-covariant formulation.

(Thought experiment: Start from fixed box V, 1 particle in rest, energy m, density $1/V$.
Boost to another frame, $\gamma = 1/\sqrt{1 - v^2}$; box size contracted to V/γ,
particle energy now $m\gamma$, density γ/V.)

Conserved current, relativistic case

Definition

$$j^\mu \equiv i(\varphi^* \partial^\mu \varphi - \varphi \partial^\mu \varphi^*), \text{ for plane wave } \varphi = \mathcal{N} e^{-ip\cdot x} \implies j^\mu = 2p^\mu |\mathcal{N}|^2$$

In particular, for our choice $\mathcal{N} = \frac{1}{\sqrt{V}}$ the particle density is $\rho \equiv j^0 = \frac{2E}{V}$

- Funny dimensions (of energy density, not number density) , related to normalization of the (unnormalizable) plane waves.
- Could replace $E \implies E/m$ (some books do this) , but then $m = 0$ difficult.
- **In the end the normalization cancels along with V**

Density of states: now "$2E$" particles in V

The number of states **per particle** is

$$\sum_{\text{states}} \frac{1}{N_{\text{part/state}}} = \int \frac{V}{2E_{\mathbf{p}}(2\pi)^3} \mathrm{d}^3\mathbf{p} = V \int \mathrm{d}^4 p \frac{\delta(p^2 - m^2)}{(2\pi)^3} \theta(p^0)$$

This is **the Lorentz-invariant phase space measure**.

Back to cross section

Cross section is Lorentz-invariant, but we will derive it in TRF.

$$\text{Recall} \quad \sigma_{a+b\to 1+\cdots+n} = \frac{N_s}{\Delta t \Phi_a N_b}, \qquad b \text{ at rest}$$

- $N_b = n_b V = 2E_b^{\mathrm{TRF}} = 2m_b$, number of target particles
- $\Phi_a = n_a|\mathbf{v}_a^{\mathrm{TRF}}| = 2E_a^{\mathrm{TRF}}|\mathbf{v}_a^{\mathrm{TRF}}|/V$ flux of particle a (flux = density $\times$ velocity)
  - $2E_a^{\mathrm{TRF}}|\mathbf{v}_a^{\mathrm{TRF}}|/V = 2|\mathbf{p}_a^{\mathrm{TRF}}|/V = \sqrt{\lambda(s, m_a^2, m_b^2)}/(m_b V)$
  - Thus $(\Phi_a N_b)^{\mathrm{TRF}} = 2\sqrt{\lambda(s, m_a^2, m_b^2)}/V \implies$ Back to Lorentz-invariant form, up to volume factor V.
- $\frac{N_s}{\Delta t}$=scattering events per time = probability to go to state $|f|$; summed over states, divided by Δt

$$\frac{N_s}{\Delta t} = \frac{N(ab \to 1 \ldots n)}{\Delta t} = \frac{|T_{fi}|^2}{\Delta t} \times (\text{number of possible final states}),$$

$$= \int \prod_{m=1}^{n} \frac{V}{2E_{\mathbf{p}_.}(2\pi)^3} \, \mathrm{d}^3\mathbf{p}_m \times \frac{|T_{fi}|^2}{\Delta t}$$

T-matrix element

The last, and most important, thing we need

Number of scatterings, T-matrix

The number of scatterings is given by the T-matrix element between final and initial states. We separate out some factors:

$$T_{fi} = -i\mathcal{N}_a\mathcal{N}_b\mathcal{N}_1\ldots\mathcal{N}_n(2\pi)^4\delta^{(4)}\left(p_a + p_b - \sum_{i=1}^{n} p_i\right)\mathcal{M}_{fi},$$

Why the factors?

- Energy, momentum always conserved in **amplitude**: useful to extract $\delta^{(4)}$ to explicitly treat $(\delta^{(4)})^2$ in $|T_{fi}|^2$
- Plane wave normalizations: conventions for $\mathcal{N}$ may vary; for $\mathcal{M}_{fi}$ not.
  (Justification: T_{fi} is matrix element between our box-normalized plane waves

  $$T_{fi} \sim \langle f|\hat{T}|i\rangle \sim (\mathcal{N}_a e^{-ip_a\cdot x}\mathcal{N}_b e^{-ip_b\cdot x})^* \hat{T}(\mathcal{N}_1 e^{-ip_1\cdot x}\ldots\mathcal{N}_n e^{-ip_n\cdot x})$$

  The "invariant amplitude" $\mathcal{M}_{fi}$, o.t.o.h. is defined using unnormalized plane waves

  $$\mathcal{M}_{fi} \sim (e^{-ip_a\cdot x}e^{-ip_b\cdot x})^* \hat{T}(e^{-ip_1\cdot x}\ldots e^{-ip_n\cdot x})$$

  Why this complication? needed the box to understand delta functions, count particles...)

Putting things together

What is then $(\delta(k))^2$? Integral representation of δ:

$$\int \mathrm{d}x\, e^{ikx} = (2\pi)\delta(k), \text{ thus } \delta(k=0) = \frac{1}{2\pi}\int \mathrm{d}x = \frac{L}{2\pi} \text{ and } (2\pi)^4\delta^{(4)}(0) = \Delta t V$$

Now we have

$$\sigma = \frac{1}{\Phi_a N_b}\frac{N_s}{\Delta t} = \frac{V}{2\sqrt{\lambda(s, m_a^2, m_b^2)}} \int \prod_{m=1}^{n} \left[\frac{V}{2E_{\mathbf{p}_.}(2\pi)^3}\mathrm{d}^3\mathbf{p}_m\right]$$

$$\frac{|-i\mathcal{N}_a\mathcal{N}_b\mathcal{N}_1\ldots\mathcal{N}_n|^2\left[(2\pi)^4\delta^{(4)}(p_a+p_b-\sum_{i=1}^n p_i)\right]^2|\mathcal{M}_{fi}|^2}{\Delta t}$$

$$= \frac{V}{2\sqrt{\lambda(s, m_a^2, m_b^2)}} \int \prod_{m=1}^{n} \left[\frac{V}{2E_{\mathbf{p}_.}(2\pi)^3}\mathrm{d}^3\mathbf{p}_m\right]$$

$$\frac{V^{-n-2}(2\pi)^4\delta^{(4)}(p_a+p_b-\sum_{i=1}^n p_i)[\Delta t V]|\mathcal{M}_{fi}|^2}{\Delta t}$$

Now volume cancels — time interval cancels

Final master formula for cross section

The final formulae

$$\text{Total } \sigma(ab \to 1 \ldots n) = \frac{1}{2\sqrt{\lambda(s, m_a^2, m_b^2)}} \left[\int \prod_{m=1}^{n} \frac{\mathrm{d}^3\mathbf{p}_m}{(2\pi)^3 2E_m} \times (2\pi)^4 \delta^{(4)} \left(p_a + p_b - \sum_{i=1}^{n} p_i \right) \right] |\mathcal{M}_{ab\to 1\ldots n}|^2,$$

$$\text{Differential } \frac{\mathrm{d}\sigma_{ab\to 1\ldots n}}{\mathrm{d}^3\mathbf{p}_1 \ldots \mathrm{d}^3\mathbf{p}_n} = \frac{1}{(2(2\pi)^3 E_1) \cdots (2(2\pi)^3 E_n)} \frac{1}{2\sqrt{\lambda(s, m_a^2, m_b^2)}} \times (2\pi)^4 \delta^{(4)} \left(p_a + p_b - \sum_{i=1}^{n} p_i \right) |\mathcal{M}_{ab\to 1\ldots n}|^2$$

- Everything is Lorentz-invariant (Derived in TRF, but now works in any frame.)
- The invariant amplitude $\mathcal{M}(ab \to 1 \ldots n)$ from Feynman rules.
- Part in $[\cdot]$ is Lorentz-invariant ***n*-particle phase space**.

Special case: 2 → 2 scattering

First work in general frame, start from the general formula

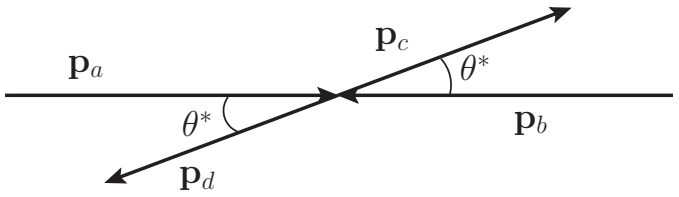
$$\sigma_{ab\to cd} = \frac{1}{2\sqrt{\lambda(s, m_a^2, m_b^2)}} \int \frac{\mathrm{d}^3\mathbf{p}_c}{2E_c(2\pi)^3} \frac{\mathrm{d}^3\mathbf{p}_d}{2E_d(2\pi)^3} (2\pi)^4 \delta^{(4)}(p_a+p_b-p_c-p_d)|\mathcal{M}|^2.$$

- Now $\delta^{(4)}(p) = \delta(E)\delta^{(3)}(\mathbf{p}) \implies$ integrate $\mathrm{d}^3\mathbf{p}_d$.
- Other variable $\mathrm{d}^3\mathbf{p}_c = |\mathbf{p}_c|^2\,\mathrm{d}|\mathbf{p}_c|\,\mathrm{d}\Omega_c$

$$\sigma_{ab\to cd} = \frac{(2\pi)^{-2}}{8\sqrt{\lambda(s, m_a^2, m_b^2)}} \int \mathrm{d}\Omega_c \int \frac{|\mathbf{p}_c|^2\,\mathrm{d}|\mathbf{p}_c|}{E_c E_d} \delta(E_a + E_b - E_c - E_d)|\mathcal{M}|^2$$

Get rid of $\delta(E_a + E_b - E_c - E_d)$, using $\int \mathrm{d}|\mathbf{p}_c|$.
Remember, now $E_c = \sqrt{|\mathbf{p}_c^2| + m_c^2}$, $\quad E_d = \sqrt{|\mathbf{p}_a + \mathbf{p}_b - \mathbf{p}_c|^2 + m_d^2}$.
$\implies \delta$-function has $|\mathbf{p}_c|$-dependence in both E_c and E_d.
Complicated in general frame, specialize to CMS.

Special case: $2 \to 2$ scattering in CMS



In CMS $|\mathbf{p}_c| = |\mathbf{p}_d| \equiv p_f^*$, $E_a^* + E_b^* = \sqrt{s}$
Get rid of $\delta(E_c + E_d - \sqrt{s})$, using $\int \mathrm{d}|\mathbf{p}_c|$.

$$\text{Remember} \quad \delta(f(x)) = \frac{\delta(x - x_0)}{|f'(x_0)|}, \text{ where } f(x_0) = 0$$

$$\text{Now} \quad \frac{\mathrm{d}}{\mathrm{d}p} E(p) = \frac{\mathrm{d}}{\mathrm{d}p} \sqrt{p^2 + m^2} = \frac{p}{\sqrt{p^2 + m^2}} = \frac{p}{E(p)}$$

$$\text{So } \delta(E_c^*(p_f^*) + E_d^*(p_f^*) - \sqrt{s}) = \frac{\delta(p_f^* - \dots)}{\frac{p_f^*}{E_c^*} + \frac{p_f^*}{E_d^*}} = \frac{E_c^* E_d^*}{p_f^* \sqrt{s}} \delta(p_f^* - \dots)$$

$$\text{and } \sigma_{ab \to cd} = \frac{(2\pi)^{-2}}{8\sqrt{\lambda(s, m_a^2, m_b^2)}} \int \mathrm{d}\Omega_c \int \frac{(p_f^*)^2 \, \mathrm{d}p_f^*}{E_c^* E_d^*} \frac{E_c^* E_d^*}{p_f^* \sqrt{s}} \delta(p_f^* - \dots) |\mathcal{M}|^2$$

$$= \frac{(2\pi)^{-2}}{8\sqrt{\lambda(s, m_a^2, m_b^2)}} \frac{p_f^*}{\sqrt{s}} \int \mathrm{d}\Omega_c^* |\mathcal{M}|^2.$$

$p_f^* = \frac{\sqrt{\lambda(s, m_c^2, m_d^2)}}{(2\sqrt{s})}$ (Remember: $\sqrt{\lambda(s, m_a^2, m_b^2)}$ came from p_i^*.)

2 → 2 scattering in CMS, final result I

Not integrating over the angle gives differential σ

$$\frac{\mathrm{d}\sigma_{ab\to cd}}{\mathrm{d}\Omega_c^*} = \frac{|\mathcal{M}_{ab\to cd}|^2}{64\pi^2 s}\sqrt{\frac{\lambda(s, m_c^2, m_d^2)}{\lambda(s, m_a^2, m_b^2)}} = \frac{|\mathcal{M}_{ab\to cd}|^2}{64\pi^2 s}\frac{p_f^*}{p_i^*}.$$

(Differential: scattering events with final particle in angle Ω_c^*, divided by angle $\mathrm{d}\Omega_c^*$)

Special case ($m_a = m_c$ and $m_b = m_d$) or ($m_a = m_d$ and $m_b = m_c$)

Leads to $p_i^* = p_f^* \implies$

$$\frac{\mathrm{d}\sigma_{ab\to cd}}{\mathrm{d}\Omega_c^*} = \frac{|\mathcal{M}_{ab\to cd}|^2}{64\pi^2 s}$$

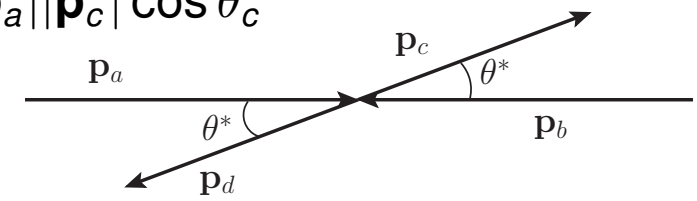
Remarks

- Energy-momentum conservation constraints now all implicitly in $|\mathcal{M}|$

$2 \to 2$ scattering, invariant $\mathrm{d}\sigma/\mathrm{d}t$

Convenient to also express $\mathrm{d}\Omega_c^*$ in a Lorentz-invariant way, via t.
In general $\mathrm{d}\Omega_c = \mathrm{d}\phi_c\,\mathrm{d}(\cos\theta_c)$. If scattering is unpolarized (summed, averaged over spins) , cross sections do not depend on $\phi_c \implies \frac{\mathrm{d}\sigma}{\mathrm{d}(\cos\theta_c)} = 2\pi\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_c}$.

$$
\begin{aligned}
t = (p_a - p_c)^2 &= m_a^2 + m_c^2 - 2E_a^* E_c^* + 2|\mathbf{p}_a^*||\mathbf{p}_c^*|\cos\theta_c^* \\
\implies \mathrm{d}t &= 2p_i^* p_f^*\,\mathrm{d}(\cos\theta_c^*) \\
\implies \frac{\mathrm{d}\sigma}{\mathrm{d}t} &= \frac{2\pi}{2p_i^* p_f^*}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_c^*}
\end{aligned}
$$



Using the expressions from the previous slide:
$\frac{\mathrm{d}\sigma_{ab\to cd}}{\mathrm{d}\Omega_c^*} = \frac{|\mathcal{M}_{ab\to cd}|^2}{64\pi^2 s}\frac{p_f^*}{p_i^*}$ and $p_i^* = \sqrt{\lambda(s, m_a^2, m_a^2)}/(2\sqrt{s})$, this leads to

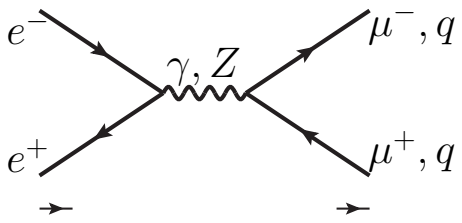
Differential cross section in manifestly Lorentz-invariant form

$$\frac{\mathrm{d}\sigma_{ab\to cd}}{\mathrm{d}t} = \frac{|\mathcal{M}_{ab\to cd}|^2}{16\pi\lambda(s, m_a^2, m_b^2)}$$

If $s \gg m_a^2, m_b^2$: $\lambda(s, m_a^2, m_b^2) = s^2 + m_a^4 + m_b^4 - 2sm_a^2 - 2m_a^2 m_b^2 - 2m_b^2 s \approx s^2$
$\implies$ this is the easiest version to remember.

Example: unpolarized $e^+e^- \to \mu^+\mu^-$ scattering

A little flavor of things to come



- Assume $\sqrt{s}$ small enough so we can neglect Z_0-boson $\Longrightarrow$ only one Feynman diagram at LO (=leading order)
- Use Feynman rules for Quantum ElectroDynamics= QED $\Longrightarrow$ get expression for $|\mathcal{M}_{e^+e^-\to\mu^+\mu^-}|^2$.
- **unpolarized** = summed over spins of $\mu^\pm$; averaged over the spins of $e^\pm$, denote $\overline{|\mathcal{M}|^2}$
- Assuming $\sqrt{s} \gg m_e, m_\mu$

$$\overline{|\mathcal{M}|^2} \equiv \frac{1}{2}\frac{1}{2}\sum_{\text{spins}} |\mathcal{M}|^2 = 2e^4\frac{t^2+u^2}{s^2}.$$

- Origin of Mandelstams: $\mathcal{M}$ contains dot products of p_a, p_b, p_c and p_d $\Longrightarrow$ written in terms of s, t and u.

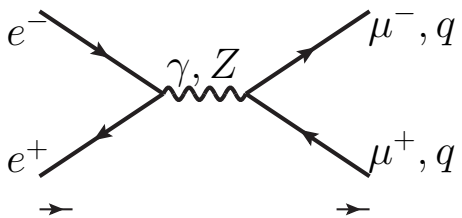
Example: $e^+e^- \to \mu^+\mu^-$ continued

The differential cross section for the above process becomes ($p_i^* \approx p_f^*$ in $\sqrt{s} \gg m_e, m_\mu$ limit)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_c^*} = \frac{\alpha^2}{4s}(1 + \cos^2\theta_\mu^*).$$

From this, the total cross section is

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}.$$



- Two interactions vertices $\sim e$, so $\mathcal{M}(e^+e^- \to \mu^+\mu^-) \propto e^2$, and $|\mathcal{M}|^2 \propto e^4 \propto \alpha^2$; $\alpha = e^2/(4\pi) \approx 1/137$.
- Check dimensions: $[s] = \text{GeV}^2$

$e^+e^- \to \mu^+\mu^-$, comments

Angular distribution depends on spin structure

- Angular distribution $1 + \cos^2\theta^*$ characteristic of spin-1 particle (virtual photon γ^*) decaying into two spin 1/2 particles.
- For spin interpretation, there are two spin states with amplitudes: $\sim (1 \pm \cos\theta)$. These are different final states $\Longrightarrow$ they do not interfere, each amplitude is squared separately, then added.
- Cross section for $\gamma^* \to 2$ spin-0 particles would be $\sim \sin^2\theta \sim |Y_1^1(\theta,\phi)|^2$; spin has to go into $m = 1, \ell = 1$ angular momentum of the pair.

See Halzen, Martin, sec 6.6.

Squaring and averaging

- Sum of diagrams=amplitude with one initial and final state (incl. spin)
- Then square each amplitude separately
- Only then sum/average $|\mathcal{M}|^2$'s from different spin states $\Longrightarrow$ this is denoted by $\overline{|\mathcal{M}|^2}$

Particle decay

Initial state $|i\rangle$ = unstable particle a, decaying to n particle final state $|f\rangle$.

- Only natural frame is CMS = rest frame of a.
- Momentum conservation $p_a^2 = (p_1 + \cdots + p_n)^2$.
  In CMS: $m_a^2 = (E_1 + \cdots + E_n)^2 \geq (m_1 + \ldots m_n)^2$
  $\Longrightarrow$ **threshold** $m_a \geq m_1 + \cdots + m_n$.

  (Thus a has $m_a > 0$, there is always a rest frame.)

Terminology

- In general different possible final states $|f\rangle$: **decay channels**.
- Define the **decay width** Γ_f for channel f as

$$\Gamma_f \equiv \frac{N_f}{N_a \Delta T} \text{ in } a \text{ rest frame.}$$

N_f is the number of decays to state f
N_a is the number of decaying particles a
ΔT is the time interval required for N_f decays.

All experimentally accessible, like def. of σ.

Now want to tie this definition with theoretically calculable quantities, skipping details that are similar to the derivation with cross sections.

Decay width and amplitude

- Recall: for cross section we had $\frac{N_s}{\Delta t} = \frac{N(ab \to 1 \ldots n)}{\Delta t} = \frac{|T_{fi}|^2}{\Delta t} \times$ (number of possible final states),
- Also now

$$\frac{N_f}{\Delta t} = \frac{|T_{fi}|^2}{\Delta t} \times \text{(number of possible final states)}.$$

- Again transition amplitude T_{fi} and the invariant amplitude $\mathcal{M}_{fi}$:

$$T_{fi} = -i\mathcal{N}_a\mathcal{N}_1 \ldots \mathcal{N}_n (2\pi)^4 \delta^{(4)} \left(p_a - \sum\nolimits_{j=1}^{n} p_j\right) \mathcal{M}_{fi}(a \to 1 \ldots n).$$

- $|T_{fi}|^2$ has a $\delta^{(4)}$ squared $\Longrightarrow$ treated similarly as before.
- Again the number of final states per particle is $\frac{1}{2E}\frac{V}{(2\pi)^3}\,\mathrm{d}^3\mathbf{p}$

Result: decay width

$$\Gamma_f = \frac{1}{2E_a} \int \prod_{m=1}^{n} \frac{\mathrm{d}^3\mathbf{p}_m}{(2\pi)^3 2E_m} (2\pi)^4 \delta^{(4)} \left(p_a - \sum\nolimits_{i=1}^{n} \mathbf{p}_i\right) |\mathcal{M}(a \to 1 \ldots n)|^2.$$

Γ_a and E_a depend on frame; in CMS $E_a = m_a$. Rest is Lorentz-invariant.

Branching ratios

- One unstable particle; many possible finals tates $|f\rangle$, each has width Γ_f. Define **total decay width**

$$\Gamma \equiv \sum_f \Gamma_f,$$

- Fraction of each final state is **branching ratio**

$$B_f \equiv \frac{\Gamma_f}{\Gamma}; \qquad \text{often in \%.}$$

- **mean lifetime**

$$\tau \equiv \frac{1}{\Gamma}.$$

Branching ratio example

Example

For the Z_0 boson we have the following branching ratios:

$$\begin{array}{lll} Z_0 \to & e^+e^- & 3.363\% \\ & \mu^+\mu^- & 3.366\% \\ & \tau^+\tau^- & 3.370\% \end{array}$$

In addition:

- Z_0 can decay "invisibly" to a pair of any of the three neutrino species: total branching ratio 20.00%.
- Hadronic decay modes: remaining 69.91% of the total width

$$\Gamma(Z_0) = 2.4952 \pm 0.0023\text{GeV} \qquad \tau \simeq 0.079\text{fm} \approx 3 \cdot 10^{-25}\ \text{s}.$$

Hence the Z_0 boson is very short-lived.
In particle physics "long-lived particles" have lifetime $\sim 10^{-16}$s or more
$\Longrightarrow$ can be actually seen directly in experiment.

Some decay times

- Out of gauge bosons γ is absolutely stable, $W^{\pm}$ and Z^0 extremely unbstable, $\Gamma \sim 2\text{GeV}$ and $\tau \approx 3 \cdot 10^{-25}$s.
- gluon is confined, stability/unstability not well defined
- Neutrinos, $e^{\pm}$ stable
- $\tau^{\pm}, \mu^{\pm}$ decay via weak interaction: $\tau_\mu \approx 2 \cdot 10^{-6}$s and $\tau_\tau \approx 3 \cdot 10^{-13}$ s.
- Quarks u, d, c, s and b form first a hadron which then decays. Out of all hadrons only the proton is absolutely stable.
- Heaviest t quark decays before it has formed a hadron, $\tau_t \approx 0.02$fm ; It would take a time of the order of 1 fm to make a hadron.

$1 \to 2$ decay

Analogous to $\sigma(2 \to 2) \implies$ we'll be brief. Frame is CMS, omit $*$.

$$\Gamma_{a\to cd} = \frac{1}{2m_a}\int \frac{\mathrm{d}^3\mathbf{p}_c}{(2\pi)^3 2E_c}\frac{\mathrm{d}^3\mathbf{p}_d}{(2\pi)^3 2E_d}(2\pi)^4\delta(m_a - E_c - E_d)\delta^{(3)}(\mathbf{p}_c + \mathbf{p}_d)|\mathcal{M}|^2.$$

- Use $\delta^{(3)}(\mathbf{p}_c + \mathbf{p}_d)$ to do $\int \mathrm{d}^3\mathbf{p}_d$; sets $|\mathbf{p}_c| = |\mathbf{p}_d| \equiv p_f$
- Write $\mathrm{d}^3\mathbf{p}_c = \mathrm{d}\Omega p_f^2\,\mathrm{d}p_f$, want to use $\delta(m_a - E_c - E_d)$ to do $\int \mathrm{d}p_f$
- Energy conservation

$$\delta(m_a - E_c - E_d) = \frac{E_c E_d}{p_0 m_a}|_{p_f = p_0}\delta(p_f - p_0),$$

where p_0 is obtained by solving $m_a = E_c + E_d$, i.e. explicitly

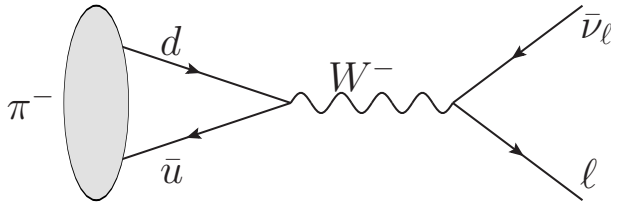
$$p_0 = \frac{\sqrt{\lambda(s, m_c^2, m_d^2)}}{2\sqrt{s}} = \frac{\sqrt{\lambda(m_a^2, m_c^2, m_d^2)}}{2m_a}.$$

Final result $\Gamma(a \to cd) = \dfrac{\sqrt{\lambda(m_a^2, m_c^2, m_d^2)}}{64\pi^2 m_a^3}\displaystyle\int \mathrm{d}\Omega|\mathcal{M}_{fi}(a \to cd)|^2.$

Example: π^- decay

Two possible final states, $\mu^-\bar{\nu}_\mu$ and $e^-\bar{\nu}_e$ (threshold,conservation laws).

Experiments: $$\frac{\Gamma(\pi^- \to e^-\bar{\nu}_e)}{\Gamma(\pi^- \to \mu^-\bar{\nu}_\mu)} \approx 1.23 \cdot 10^{-4} \quad \text{very small, why?}$$



- Annihilate $d\bar{u} \Longrightarrow$ charged weak current
- One W-boson, 2 weak vertices: $\mathcal{M} \sim \frac{g_W^2}{m_W^2} \sim G_F$ ("Fermi constant")
- $\pi^- \leftrightarrow d\bar{u}$: "pion decay constant" $\mathcal{M} \sim f_\pi$.

Evaluation of the Feynman diagram gives:

(Actually Feynman rule for $W\ell\bar{\nu}_\ell$ vertex + information of pion wavefunction in f_π)

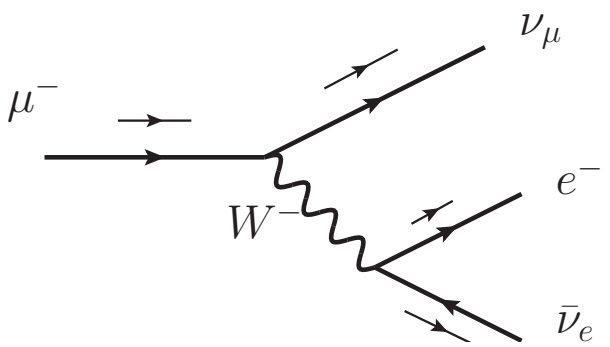
$$\overline{|\mathcal{M}(\pi^- \to \ell^-\bar{\nu}_\ell)|^2} = 4G_F^2 f_\pi^2 m_\ell^2 (p_\ell \cdot p_{\bar{\nu}}) \quad = \quad 2G_F^2 f_\pi^2 m_\ell^2 m_\pi^2 \left(1 - \frac{m_\ell^2}{m_\pi^2}\right),$$

Then (exercise) the decay width is $$\Gamma(\pi^- \to \ell^-\bar{\nu}_\ell) \quad = \quad \frac{1}{8\pi} G_F^2 f_\pi^2 m_\pi m_\ell^2 \left(1 - \frac{m_\ell^2}{m_\pi^2}\right)^2,$$

which implies the ratio $$\frac{\Gamma(\pi^- \to e^-\bar{\nu}_e)}{\Gamma(\pi^- \to \mu^-\bar{\nu}_\mu)} = \left(\frac{m_e}{m_\mu}\right)^2 \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 \approx 1.28 \cdot 10^{-4}.$$

Example 2: Muon decay $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$, estimates I

$1 \longrightarrow 3$ decay



1. Conservation laws determine final particles:
   1.1 Charge conservation: e^- (only negative particle with $m < m_\mu$).
   1.2 μ number conservation: $L_\mu = 1 \implies \nu_\mu$
   1.3 e number: $L_e = 0$: $\bar{\nu}_e$ to compensate e^-
2. Only e^- detected. 2 others $\implies$ continuous distribution of E_e.

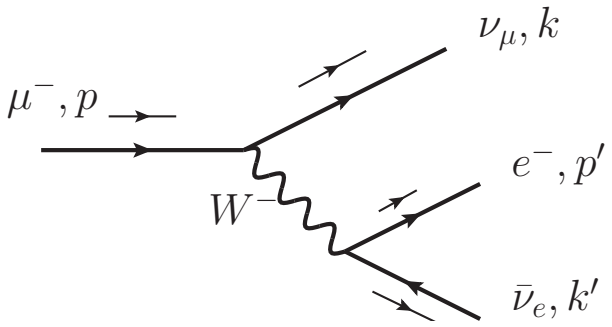
As before, we can estimate the decay width as

- One W propagator, 2 W-fermion vertices: $\mathcal{M} \sim g_W^2/m_W^2 \sim G_F$
- $m_\mu \gg m_e, m_{\nu_\ell} \implies$ dimensionally must have $\Gamma \sim G_F^2 m_\mu^5$

Example 2: Muon decay $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$, calculation

The actual invariant amplitude, from S.M. Feynman rules, is

$$\overline{|\mathcal{M}(\mu \to e^- \bar{\nu}_e \nu_\mu)|^2} = 64 G_F^2 (k \cdot p')(k' \cdot p).$$



Denote
$$\begin{aligned} p &= (m_\mu, \mathbf{0}) \\ p' &= (E', \mathbf{p}') \\ k &= (\omega, \mathbf{k}) \\ k' &= (\omega', \mathbf{k}') \end{aligned}$$

Differential width is $\quad \mathrm{d}\Gamma = \dfrac{1}{2m_\mu} \overline{\mathcal{M}(\mu \to e^- \bar{\nu}_e \nu_\mu)|^2}\, \mathrm{d}Q,$

with $\quad \mathrm{d}Q = \dfrac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3 2E'} \dfrac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3 2\omega'} \dfrac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2\omega} (2\pi)^4 \delta^{(4)}\left(p - p' - k - k'\right).$

This can be simplified using $\quad \displaystyle\int \frac{\mathrm{d}^3\mathbf{k}}{2\omega} = \int \overbrace{\mathrm{d}^4 k}^{\text{Eats } \delta^{(4)}}\ \theta(\omega) \overbrace{\delta(k^2)}^{\text{1d}}.$

Example 2: Muon decay $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$, result

$$\text{We get} \quad \mathrm{d}Q = \frac{1}{(2\pi)^5} \frac{\mathrm{d}^3\mathbf{p}'}{2E'} \frac{\mathrm{d}^3\mathbf{k}'}{2\omega'} \theta(m_\mu - E' - \omega')\delta((p - p' - k')^2)$$

After intermediate steps (exercise) we get

$$\mathrm{d}\Gamma = \frac{G_F^2}{2\pi^3} \mathrm{d}E' \, \mathrm{d}\omega' \, m_\mu \omega' (m_\mu - 2\omega'),$$

$$\text{with restrictions} \qquad \begin{aligned} \frac{1}{2}m_\mu - E' \quad &\leq \omega' \leq \quad \frac{1}{2}m_\mu \\ 0 \quad &\leq E' \leq \quad \frac{1}{2}m_\mu \end{aligned}$$

$$\text{Integrating, final result is (exercise)} \qquad \Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

- Estimate $\sim G_F^2 m_\mu^5$ true
- But constant of proportionality is very small!

Details: Halzen-Martin, sec. 12.5. Also in KJE's notes, after page 87.