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Part 6: Hadrons: quantum numbers and excited states

Introductory remarks

- Properties of hadrons can be understood (if not always calculated) from quark model
- Free quarks never seen. Independently of the existence of quarks, some of the observed systematics can also be analyzed based on observed symmetries: parity, charge conjugation, masses, spins etc.
  - Setting (by convention) the parity of 6 hadrons one can deduce others from conservation laws
- Note: in this section we will neglect electroweak interaction and consider e.m. or weakly decaying partices as stable.

Observation: groups of similar mass hadrons, "multiplets"

All within 5MeV , similar quantum numbers except electric charge

- Pions (pseudoscalar) $\pi^{\pm}$(139.57MeV), π^{0}(134.98MeV)
- Kaons ("strange" pseudoscalar) $K^{\pm}$(494MeV), $K^{0}, \bar{K}^{0}$(497MeV) $(K^{+}, K^{0} : S = 1, K^{-}, \bar{K}^{0} : S = -1)$
- ρ mesons (vector) $\rho^{0,\pm}$(775MeV)
- Nucleons, p(938.27MeV),n(939.57MeV), $\bar{p}, \bar{n}$
- Δ baryons $\Delta^{++,+,0,-}$(1232MeV)
- $\Sigma^{+,0,-}$ baryons (1189,1192,1197MeV) $(S = -1)$
- Some more detached "loners": pseudoscalars $\eta(548), \eta'(958)$, vectors ω(782MeV), ϕ(1020MeV), baryon $\Lambda(1116)$

In the quark model these are "easily" explained: u and d have same mass, changing u to d gives these "multiplets".
Before quarks were discovered the equivalent systematics was described purely in terms of **symmetries** and **charges**.

Hypercharge and isospin

Define **isospin** I_3 (really "3rd component of isospin") and **hypercharge** Y.

$$Y \equiv B + S + C + \widetilde{B} + T \qquad I_3 = Q - \frac{Y}{2}$$

Historically: no $C + \widetilde{B} + T$ known:

- Fixed strangeness S, trade B, Q for Y, I_3
- Fixed baryon number B, trade S, Q for Y, I_3

Groups have same B, S, Y

Isospin looks like angular spin, $I_3 = -I, -I+1, \ldots, I$

(Originally "isotopic spin")

Particles in multiplet = different isospin states of same particle.

Why the funny assignment?
Check B, S, Y, I_3 for hadron multiplets:

	B	S	Y	I_3
$\pi^{\pm,0}$	0	0	0	$-1, 0, 1$
K^+, K^0	0	1	1	$\frac{1}{2}, -\frac{1}{2}$
$K^-, \bar{K}^0$	0	-1	-1	$-\frac{1}{2}, \frac{1}{2}$
$\rho^{0,\pm}$	0	0	0	$-1, 0, 1$
p, n	1	0	1	$\frac{1}{2}, -\frac{1}{2}$
$\bar{p}, \bar{n}$	-1	0	-1	$-\frac{1}{2}, \frac{1}{2}$
$\Delta^{++,+,0,-}$	1	0	1	$\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$
$\Sigma^{+,0,-}$	1	-1	0	$1, 0, -1$
$\eta, \eta', \omega, \phi$	0	0	0	0
Λ	1	-1	0	0

Hypercharge and isospin of quarks

$$
\begin{aligned}
Y &\equiv B + S + C + \widetilde{B} + T \\
I_3 &\equiv Q - \frac{Y}{2}
\end{aligned}
$$

Interpretation in quark model

	Y	I_3
u	$\frac{1}{3}$	$\frac{1}{2}$
d	$\frac{1}{3}$	$-\frac{1}{2}$
s	$-\frac{2}{3}$	0
c	$\frac{4}{3}$	0
b	$-\frac{2}{3}$	0
t	$\frac{4}{3}$	0

Isospin measures $N_u - N_{\bar{u}} - N_d + N_{\bar{d}}$

$I_3 = 0$ for other quarks.

Hadron mass independent of I_3 because:

- $m_u \approx m_d$
- Strong interaction treats u and d equally $\Longrightarrow$ **isospin symmetry**

From the assignments of isospin numbers we can guess that the mathematics of isospin is just like that of spin. i.e. SU(2).

One can show from the Lagrangian that this is indeed the case. With strangeness this becomes SU(3) $\Longrightarrow$ come back to this later.

SU(2) structure of isospin

Isospin operator vector $\hat{\mathbf{I}}$ $\quad [\hat{I}_i, \hat{I}_j] = i\epsilon_{ijk}\hat{I}_k \qquad \left[\hat{\mathbf{I}}, \hat{H}_{\text{strong}}\right] = 0$

All hadrons are eigenstates of strong interaction Hamiltonian

$\Longrightarrow$ can be chosen as isospin eigenstates:

Hadron $|I, I_3\rangle$ $\quad \hat{I}_3|I, I_3\rangle = I_3|I, I_3\rangle \quad \hat{\mathbf{I}}^2|I, I_3\rangle = I(I+1)|I, I_3\rangle, \quad I_3 = -I, -I+1, \ldots I$

Isospin assignments, examples

$$-|\pi^+\rangle, |\pi^0\rangle, |\pi^-\rangle = |1, 1\rangle, |1, 0\rangle, |1, -1\rangle \qquad \text{Note sign in } \pi^+$$

$$|p\rangle, |n\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle, \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$$

$$|\Delta^{++}\rangle, |\Delta^+\rangle, |\Delta^0\rangle, |\Delta^-\rangle = \left|\frac{3}{2}, \frac{3}{2}\right\rangle, \left|\frac{3}{2}, \frac{1}{2}\right\rangle, \left|\frac{3}{2}, -\frac{1}{2}\right\rangle, \left|\frac{3}{2}, -\frac{3}{2}\right\rangle,$$

(Note: $\left[\hat{\mathbf{I}}^2, \hat{I}_i\right] = 0$, so there can be a term $\sim \hat{\mathbf{I}}^2$ in $\hat{H}_{\text{strong}}$ $\Longrightarrow$ This leads to different masses for Δ's ($I = 3/2$) than p, n ($I = 1/2$), even if other quantum numbers are the same.)

If only I_3 changes, same mass; if I different, mass different.

Strong interactions conserve isospin

$$\left[\hat{\mathbf{I}}, \hat{H}_{\text{strong}}\right] = 0$$

- Initial and final states of a strong interaction process have same I, I_3
- Scattering amplitude (derived from Hamiltonian) independent of I_3 (but can depend on I)

Practical procedure

1. Initial/final state with several particles: tensor in isospin space.
2. Use Clebsch-Gordans to decompose initial state into **total** I, I_3 states.
3. Scattering amplitude can only depend on total I (in strong interaction process)
4. Again use CG's to find amplitudes for going from each total I, I_3 to a particular many-particle final state

- If initial state is just one unstable particle, step 2 is trivial
- Also note that coupling a singlet $I = 0$ to something is trivial . . .

Example 1, $\Delta \to N\pi$ decay ($\pi = \pi^{\pm 0}$ $N = p, n$)

$\Delta^+(1232)$ (uud) is $I = 3/2$, $I_3 = 1/2$, decays via $\Delta^+ \to \left\{ \begin{array}{l} p\pi^0 \\ n\pi^+ \end{array} \right.$

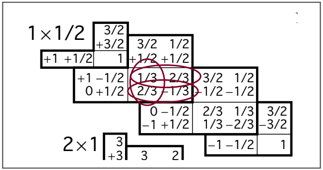
$$\text{Now } \Gamma(\Delta^+ \to N\pi) = K \int d\Omega |T(\Delta^+ \to N\pi)|^2,$$

- K identical for $N = p, n$ since the masses $\approx$ same.
- T only depends on I. In terms of isospin states

$$\begin{aligned} T(\Delta^+ \to p\pi^0) &= T(I=3/2) \left(\overbrace{\left\langle \frac{1}{2}, +\frac{1}{2} \right|}^{p} \otimes \overbrace{\left\langle 1, 0 \right|}^{\pi^0} \right) \overbrace{\left| \frac{3}{2}, +\frac{1}{2} \right\rangle}^{\Delta^+} \\ T(\Delta^+ \to n\pi^+) &= -T(I=3/2) \left(\underbrace{\left\langle \frac{1}{2}, -\frac{1}{2} \right|}_{n} \otimes \underbrace{\left\langle 1, +1 \right|}_{-\pi^+} \right) \underbrace{\left| \frac{3}{2}, +\frac{1}{2} \right\rangle}_{\Delta^+} \end{aligned}$$

(We'll come back to - sign in π^+ later)

Example 1 continued, $\Delta \to N\pi$ decay ($N = p, n$)



Using Clebsch tables

$$\left|\frac{1}{2},+\frac{1}{2}\right\rangle\left|1,0\right\rangle = -\frac{1}{\sqrt{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{3}{2},\frac{1}{2}\right\rangle,$$

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle\left|1,1\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle + \frac{1}{\sqrt{3}}\left|\frac{3}{2},\frac{1}{2}\right\rangle$$

Project out the Δ^+ state from these:

$$T(\Delta^+ \to p\pi^0) = \sqrt{\frac{2}{3}}T(I = 3/2)$$

$$T(\Delta^+ \to n\pi^+) = -\frac{1}{\sqrt{3}}T(I = 3/2) \qquad (\text{'–' from } |\pi^+\rangle = -|1,1\rangle)$$

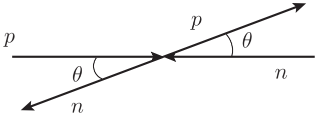
And we get $$\frac{\Gamma(\Delta^+ \to p\pi^0)}{\Gamma(\Delta^+ \to n\pi^+)} = \frac{|T(\Delta^+ \to p\pi^0)|^2}{|T(\Delta^+ \to n\pi^+)|^2} = \frac{2}{1} = 2$$

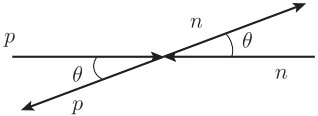
You can also directly read the table the other way:

$$\left|\frac{3}{2},\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle\left|1,1\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2},+\frac{1}{2}\right\rangle\left|1,0\right\rangle$$

Example 2: NN scattering

Consider scattering processes $pp \to pp, pn \to pn, nn \to nn$
Initial, final states are $2 \otimes 2 = 3 \oplus 1$: only 2 independent amplitudes
$T_0(\cos\theta) \equiv T(I=0, \cos\theta)$ and $T_1(\cos\theta) \equiv T(I=1, \cos\theta)$.
Note: "$pn \to pn$": outgoing p to angle θ "$pn \to np$": outgoing n to angle θ.





Masses $\Longrightarrow$ kinematic factors are same in all cases:

$$\frac{d\sigma(N_1 N_2 \to N_1' N_2')}{d\cos\theta^*} = K\,\overline{|\mathcal{M}(N_1 N_2 \to N_1' N_2')|^2}$$

Example 2: NN scattering, isospin amplitudes

$$\begin{aligned}
\mathcal{M}(pp \to pp) &\sim \left\langle \frac{1}{2},\frac{1}{2} \right| \left\langle \frac{1}{2},\frac{1}{2} \middle| 1,1 \right\rangle T_1 \left\langle 1,1 \middle| \frac{1}{2},\frac{1}{2} \right\rangle \left| \frac{1}{2},\frac{1}{2} \right\rangle \\
\mathcal{M}(pn \to pn) &\sim \left\langle \frac{1}{2},\frac{1}{2} \right| \left\langle \frac{1}{2},-\frac{1}{2} \middle| 1,0 \right\rangle T_1 \left\langle 1,0 \middle| \frac{1}{2},\frac{1}{2} \right\rangle \left| \frac{1}{2},-\frac{1}{2} \right\rangle \\
&\quad + \left\langle \frac{1}{2},\frac{1}{2} \right| \left\langle \frac{1}{2},-\frac{1}{2} \middle| 0,0 \right\rangle T_0 \left\langle 0,0 \middle| \frac{1}{2},\frac{1}{2} \right\rangle \left| \frac{1}{2},-\frac{1}{2} \right\rangle \\
\mathcal{M}(pn \to np) &\sim \left\langle \frac{1}{2},-\frac{1}{2} \right| \left\langle \frac{1}{2},\frac{1}{2} \middle| 1,0 \right\rangle T_1 \left\langle 1,0 \middle| \frac{1}{2},\frac{1}{2} \right\rangle \left| \frac{1}{2},-\frac{1}{2} \right\rangle \\
&\quad + \left\langle \frac{1}{2},-\frac{1}{2} \right| \left\langle \frac{1}{2},\frac{1}{2} \middle| 0,0 \right\rangle T_0 \left\langle 0,0 \middle| \frac{1}{2},\frac{1}{2} \right\rangle \left| \frac{1}{2},-\frac{1}{2} \right\rangle \\
\mathcal{M}(nn \to nn) &\sim \left\langle \frac{1}{2},-\frac{1}{2} \right| \left\langle \frac{1}{2},-\frac{1}{2} \middle| 1,-1 \right\rangle T_1 \left\langle 1,-1 \middle| \frac{1}{2},-\frac{1}{2} \right\rangle \left| \frac{1}{2},-\frac{1}{2} \right\rangle
\end{aligned}$$

Example 2: NN scattering continued

Using Clebsches these are

$$
\begin{aligned}
\mathcal{M}(pp \to pp) &\sim 1 \cdot 1 T_1 \\
\mathcal{M}(pn \to pn) &\sim \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} T_1 + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} T_0 \\
\mathcal{M}(pn \to np) &\sim \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} T_1 + \left(-\frac{1}{\sqrt{2}}\right) \cdot \frac{1}{\sqrt{2}} T_0 \\
\mathcal{M}(nn \to nn) &\sim 1 \cdot 1 T_1
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{d\sigma(pp \to pp)}{d\cos\theta*} = \frac{d\sigma(nn \to nn)}{d\cos\theta*} &= K |T_1(\cos\theta^*)|^2 \\
\frac{d\sigma(pn \to pn)}{d\cos\theta*} &= \frac{K}{4} |T_1(\cos\theta^*) + T_0(\cos\theta^*)|^2 \\
\frac{d\sigma(pn \to np)}{d\cos\theta*} &= \frac{K}{4} |T_1(\cos\theta^*) - T_0(\cos\theta^*)|^2
\end{aligned}
$$

These are nontrivial constraints.
E.g. when $\cos\theta^* = 0$ (90° scattering) *np* and *pn* are the same:

$$
T_0(0) = 0 \quad \text{and} \quad \left.\frac{d\sigma(pn \to pn)}{d\cos\theta*}\right|_{\theta*=\pi/2} = \frac{1}{4} \left.\frac{d\sigma(pp \to pp)}{d\cos\theta*}\right|_{\theta*=\pi/2}
$$

Charge conjugation and isospin doublet

Consider isospin doublet — such as $\begin{pmatrix} p \\ n \end{pmatrix}$ or $\begin{pmatrix} u \\ d \end{pmatrix}$.

The corresponding antiparticles should also form isospin doublet.

(E.g. $\bar{p}$ and $\bar{n}$ have $\sim$ same masses, different from other antibaryons.)

$$\text{Charge conjugation} \quad \begin{pmatrix} p \\ n \end{pmatrix} \xrightarrow{C} \begin{pmatrix} \bar{p} \\ \bar{n} \end{pmatrix} \qquad \begin{pmatrix} u \\ d \end{pmatrix} \xrightarrow{C} \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}$$

But these cannot be isospin doublets, since:

$I_3(\bar{p}) = I_3(\bar{u}) = -\frac{1}{2}$, $\Longrightarrow$ should be lower component of doublet

$I_3(\bar{n}) = I_3(\bar{d}) = \frac{1}{2}$ $\Longrightarrow$ should be upper component of doublet

Solution: perform isospin SU(2) rotation

$[\hat{\mathbf{I}}, \hat{H}_{\text{strong}}] = 0 \Longrightarrow$ If $|\psi\rangle$ is strong eigenstate (hadron), also $e^{i\boldsymbol{\theta}\cdot\hat{\mathbf{I}}}|\psi\rangle$ is.

Now, in this matrix notation, the representation of $\hat{\mathbf{I}}$ is $\boldsymbol{\sigma}/2$.

So we can rotate by SU(2) matrix $e^{-i\pi\sigma_2/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to get the correct

Antifermion isospin doublets

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{p} \\ \bar{n} \end{pmatrix} = \begin{pmatrix} -\bar{n} \\ \bar{p} \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix} = \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}$$

Signs of pion states in triplet

We now have

$$\begin{pmatrix} |u\rangle \\ |d\rangle \end{pmatrix} = \begin{pmatrix} \left|I=\frac{1}{2}, I_3=\frac{1}{2}\right\rangle \\ \left|I=\frac{1}{2}, I_3=-\frac{1}{2}\right\rangle \end{pmatrix} \qquad \begin{pmatrix} |\bar{d}\rangle \\ |\bar{u}\rangle \end{pmatrix} = \begin{pmatrix} -\left|I=\frac{1}{2}, I_3=\frac{1}{2}\right\rangle \\ \left|I=\frac{1}{2}, I_3=-\frac{1}{2}\right\rangle \end{pmatrix}$$

Pions are isospin triplet

$$\begin{array}{lclclcl} |1,1\rangle & = & \left|\frac{1}{2}\right\rangle\left|\frac{1}{2}\right\rangle & = & -|u\bar{d}\rangle & = & -|\pi^+\rangle \\ |1,0\rangle & = & \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\right\rangle\left|-\frac{1}{2}\right\rangle + \left|-\frac{1}{2}\right\rangle\left|\frac{1}{2}\right\rangle\right) & = & \frac{1}{\sqrt{2}}|u\bar{u}\rangle - \frac{1}{\sqrt{2}}|d\bar{d}\rangle & = & |\pi^0\rangle \\ |1,-1\rangle & = & \left|-\frac{1}{2}\right\rangle\left|-\frac{1}{2}\right\rangle & = & |d\bar{u}\rangle & = & |\pi^-\rangle \end{array}$$

This is why

- $-|\pi^+\rangle$ is isospin state
- $|\pi^0\rangle$ has minus sign in $\frac{1}{\sqrt{2}}|u\bar{u}\rangle - \frac{1}{\sqrt{2}}|d\bar{d}\rangle$

This is of course just a convention;

we could equally well choose $\begin{pmatrix} |\bar{d}\rangle \\ -|\bar{u}\rangle \end{pmatrix}$ as the antiquark doublet

(Isospin rotation by 0, π, 0 in stead of 0, $-\pi$, 0)

States $|\pi^+\rangle$ and $-|\pi^+\rangle$ are the same physical particle.

Short- and long lived hadrons

Consider 3 hadrons:

- Δ $I = 3/2$, $m = 1232$MeV, decays strongly: $\Gamma = 118$MeV, $c\tau \approx 1.7$fm
- Λ^0 $I = 0$, $S = -1$, $m = 1116$MeV, decays weakly: $c\tau = 7.89$cm, $\Gamma \approx 2.5 \times 10^{-12}$MeV
- π^0 $I = 1$, $m = 135$MeV, decays electromagnetically: $c\tau = 25.2\text{nm} = 25.2 \times 10^6$fm; $\Gamma = 7.9 \times 10^{-6}$MeV.

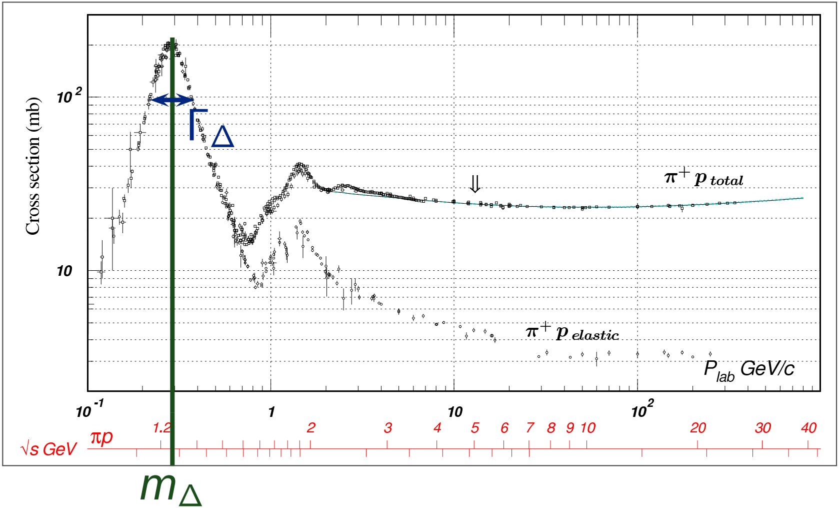
Compare lifetimes to typical hadronic distance: proton charge radius 0.88fm, or to resolution of particle detector.

- Λ^0 Weak decay, can be **seen** in detector
- π^0 E.m. decay; only decay products seen, but $c\tau \gg$ size of hadrons R_{had}
- Δ Strong decay; $c\tau \sim R_{\text{had}}$; $\Gamma \sim m$, Can decay even be separated from production?

So how do we know what the lifetime/width of the Δ is?
Answer: it is seen as a **resonance** in a $N\pi$ cross section.

$$N_{\text{ev}}(\sqrt{s}) \sim \frac{1}{(\sqrt{s} - m_\Delta)^2 + \frac{\Gamma^2}{4}}$$ **Breit-Wigner formula**

Proton-pion cross section



Time dependent perturbation theory

Why does unstable particle show up as a peak in the spectrum?
Time-dependent perturbation theory.

$$\text{Hamiltonian } \hat H = \hat H_0 + \hat V$$

Develop in $|\psi_n\rangle$; eigenstates of free Hamiltonian $\hat H_0$: i.e. $\hat H_0|\psi_n\rangle = E_n|\psi_n\rangle$.

$$|\psi(t)\rangle = \sum_n e^{-iE_n t} a_n(t)|\psi_n\rangle,$$

Initial state $|\psi_0\rangle$, i.e. initial conditions $a_0(0) = 1$, $a_n(0) = 0, n \geq 1$.

$$\begin{aligned}
\text{Schroedinger eq.} \quad i\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle &= \hat H|\psi(t)\rangle \\
\sum_m e^{-iE_m t}\left[\cancel{E_m a_m(t)} + i\dot a_m(t)\right]|\psi_m\rangle &= \sum_m e^{-iE_m t}\left[\cancel{\hat H_0} + \hat V\right] a_m(t)|\psi_m\rangle \\
\langle\psi_n|e^{iE_n t}, \langle\psi_n|\psi_m\rangle = \delta_{mn}\Big\| \quad \Longrightarrow \quad i\dot a_n(t) &= \sum_m e^{-i(E_m-E_n)t}\underbrace{\langle\psi_n|\hat V|\psi_m\rangle}_{\equiv H_{nm}} a_m(t)
\end{aligned}$$

(We can assume $H_{nn} = 0$, diagonal part in $\hat H_0$, not $\hat V$.)

Time dependence of decay states

Perturbation theory

Assume $\hat{V}$ is small.
With the initial condition $a_0(0) = 1$, $a_n(0) = 0, n \geq 1$ assume $a_{n\geq 1} \sim \hat{V}$.
Keep only $H_{n0}a_0 \sim V$, not other $H_{nm}a_m \sim V^2$.
Physically: $|\psi_0\rangle$ is decaying particle. Only keep transitions from $|\psi_0\rangle$ to decay products $|\psi_{n\geq 1}\rangle$, not transitions among $|\psi_{n\geq 1}\rangle$.

$$i\dot{a}_n(t) = e^{-i(E_0-E_n)t} H_{n0} a_0(t).$$

If $|\psi_0\rangle$ decays with rate Γ, $|a_0(t)|^2 = e^{-\Gamma t}$, then

$$\begin{aligned}
i\dot{a}_n(t) &= e^{-i(E_0-E_n-i\Gamma/2)t} H_{n0} \\
P_n(t \gg 1/\Gamma) &= \frac{|H_{n0}|^2}{|E_0-E_n-i\Gamma/2|^2} = \frac{|H_{n0}|^2}{(E_0-E_n)^2+(\Gamma/2)^2} \\
P_n(t \gg 1/\Gamma) &= \frac{2\pi}{\Gamma}|H_{n0}|^2 \underbrace{\frac{\Gamma/(2\pi)}{(E_0-E_n)^2+(\Gamma/2)^2}}_{\equiv P(E_n-E_0)}
\end{aligned}$$

Separated factors for (continuum) normalized $\int_{-\infty}^{\infty} \mathrm{d}E\, P(E-E_0) = 1.$

Add density of states

We were assuming discrete spectrum of states (box normalization. . .)
General density of final states $\rho_f(E) = \sum_{\text{states} f} \delta(E - E_f)$

In other words $\rho_f(E)\,\mathrm{d}E$ = number of final states between E and $E + \mathrm{d}E$.
Now probability to decay into state with energy E is

$$P_f(E)\,\mathrm{d}E = \frac{2\pi}{\Gamma} \overbrace{|H_{n0}|^2}^{\equiv|H_{f0}|} \overbrace{P(E - E_0)}^{\text{peak}} \rho_f(E)\,\mathrm{d}E \approx \frac{2\pi|H_{f0}|^2}{\Gamma} P(E - E_0)\rho_f(E_0)\,\mathrm{d}E$$

Breit-Wigner

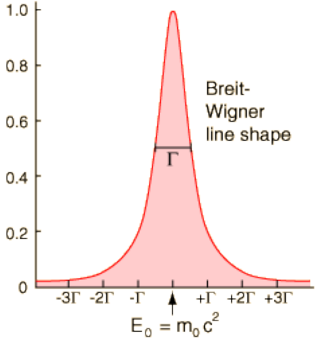
Decay into channel f with

Branching fraction

$$B_f = \frac{\Gamma_f}{\Gamma} = \frac{2\pi |H_{f0}|^2}{\Gamma} \rho_f(E_0)$$

Probability distribution of energies

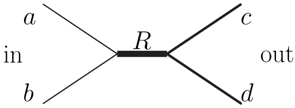
$$P(E - E_0) = \frac{\Gamma/(2\pi)}{(E - E_0)^2 + (\Gamma/2)^2}$$



- Experimentally: lifetime of unstable particle from this formula.
- Note: f labels particle content of final state; $P(E - E_0)$ the probability distribution of energy of those particles.
- Γ in peak is **total** width, not partial width to channel f.
- This is the **unrelativistic** version, similar in relativistic case.

Formation and decay of resonance

Now include **production** of resonance in same (nonrigorous) treatment.



Assume we have $|\psi(t)\rangle = \sum_n a_n(t)|\psi_n\rangle$

- Intial state (particles *ab*) $|\psi_1\rangle$
- Unstable resonance state $|\psi_0\rangle$
- Possible final states $n \geq 2$.

Initial condition $a_1(t=0) = 1, a_0 = a_{n\geq 2} = 0$

Amplitude of resonance state

$$i\dot{a}_0(t) = e^{-i(E_1-E_0)t} H_{01} \overbrace{a_1(t)}^{\mathcal{O}(1)} + \sum_{m\geq 2} e^{-i(E_m-E_0)t} H_{0m} \overbrace{a_m(t)}^{\mathcal{O}(H_{1m})} \approx e^{-i(E_1-E_0)t} H_{01}$$

This is just the creation of the resonant state. Approximate $a_1 = 1$
R also decays: add damping term by hand (instead of treating it rigorously)

$$\dot{a}_0(t) \approx -ie^{-i(E_1-E_0)t} H_{01} - \frac{\Gamma}{2} a_0(t)$$

$$P(\mathrm{R}) = P_0(t \gg 1/\Gamma) = |a_0(t)|^2 \xrightarrow{t\to\infty} \frac{|H_{01}|^2}{(E_1-E_0)^2 + \Gamma^2/4}$$

Resonance peak in cross section

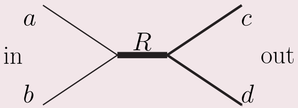
We see that the transition probability from initial state ab to R is

$$P(ab \to R) \sim \frac{|H_{R,ab}|^2}{(E_{ab} - E_0)^2 + \Gamma^2/4} \sim P(R \to ab) \sim \Gamma_{R\to ab};$$

Peak at $E_{ab} = \sqrt{s} = E_0 = m_R$, just like the (time reversed) decay $R \to ab$.

Resonance peak in cross section $ab \to cd$

For $\sqrt{s}$ close to the resonance mass, the $ab \to R$ cross section has a peak $\Longrightarrow$ this peak dominates the cross section $\sigma_{ab\to cd} \sim P(ab \to R)\Gamma_{R\to cd}$.



$$\sigma_{ab\to cd} = \frac{\pi}{(q_i^{\mathrm{TRF}})^2} \frac{\Gamma_{R\to ab}\Gamma_{R\to cd}}{(E_{ab} - m_R)^2 + \Gamma^2/4}$$

- q_i^{TRF} is momentum of a in TRF (b rest frame); See Martin&Shaw, KJE for factors.
- Isospin: CG coefficients for $R \leftrightarrow ab, R \leftrightarrow cd$ in $\Gamma_{R\to ab}, \Gamma_{R\to cd}$.
- Remember: scattering also without intermediate R $\Longrightarrow$ but no peak in σ

Resonance particle properties

- Measure hadronic reaction vs. invariant mass; $\exists$ peak $\Longrightarrow$ resonance
- Obtain resonance particle mass, lifetime from peak center, width.
- Deduce quantum numbers (isospin etc.) from decay products (or initial particles).
- Information about decay channels $\Gamma_{R\to X}$ from height of peak.

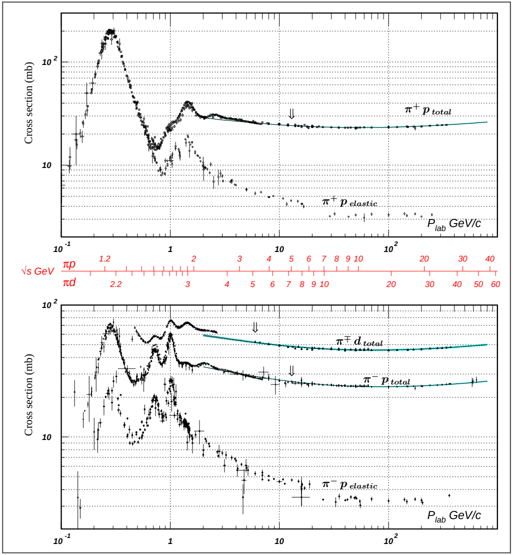
πN scattering, baryonic resonances

- $\pi^+ p$, in isospin basis $|1,+1\rangle|1/2,+1/2\rangle = |3/2,+3/2\rangle = |\Delta^{++}\rangle$
- $\pi^- p$, in isospin basis $|1,-1\rangle|1/2,+1/2\rangle = c_1|3/2,-1/2\rangle + c_2|1/2,-1/2\rangle = c_1|\Delta^0\rangle + c_2|N^0\rangle$ $\Longrightarrow$ more peaks, also N^0 resonance (quantum numbers of neutron).

Production of resonances in $a+b \to R+X \to c+d+X$ $\Longrightarrow$ other particles X involved

Formation of resonances: $a+b \to R \to c+d$ $\Longrightarrow$ cleaner signature

πN cross section plot

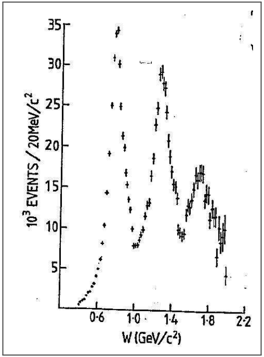


Example: mesonic resonances

Consider $\pi^- + p \to n + \pi^- + \pi^+$.

Plot cross section vs. invariant mass of $X = \pi^+\pi^-$

Note: now not $\sqrt{s}$, but m^2 of subset of final particles.



$X \to \pi^+\pi^-$ quantum numbers:

1. $J_X = J_{\pi\pi} = L$, since $J_\pi = 0$.
2. $C_X = C_{\pi\pi} = (-1)^L$
3. $P_X = P_{\pi\pi} = (-1)^L$
4. $B = 0$, $Q = 0 \Rightarrow I_3 = 0$
5. $I_\pi = 1 \Rightarrow I_{\pi\pi} = 0, 1, 2$

- $B = 0 \implies X$ meson
- $I_X = 0, 1$ or 2 and $I_{3,X} = 0$
- $\pi^+\pi^-$ angular distribution $\implies$ measure L

Peaks have $J^{PC} = 1^{--}, 2^{++}, 3^{--} \implies$ Identify

- $\rho^0(770)$, $J^{PC} = 1^{--}$, $I = 0$, $I_3 = 0$
- $f_2^0(1270)$ $J^{PC} = 2^{++}$, $I = 0$, $I_3 = 0$
- $\rho_3(1690)$ $J^{PC} = 3^{--}$, $I = 1$, $I_3 = 0$

Why ”resonance”, classical mechanics analogue

Classical Harmonic oscillator with friction and external driving force

$$\ddot{x}(t) = -\overbrace{\omega_0^2 x(t)}^{\text{spring}} - \overbrace{\gamma \dot{x}(t)}^{\text{friction}} + \overbrace{F\sin(\omega t)}^{\text{external force}}$$

If $\gamma = 0$, $F = 0$, solution $x(t) = A\sin(\omega_0 t + \varphi_0)$
Resonance: driving frequency ω = characteristic frequency ω_0
At $t \gg 1/\gamma$ modes with $\sin(\omega_o t)$ die away with friction, remains:

$$x(t) \xrightarrow[t\to\infty]{} A\sin(\omega t + \varphi) \quad \tan\varphi = -\frac{\gamma\omega}{\omega_0^2 - \omega^2} \quad A^2 = \frac{F^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

This is Breit-Wigner, analogy

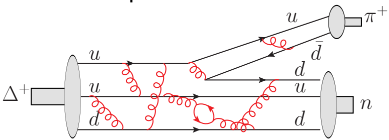
- F Driving force $\sim$ matrix elements $ab \to R$, $\quad R \to cd$
- ω^2 Driving frequency $\sim$ energy of ab (energy = frequency!)
- ω_0^2 Characteristic frequency $\sim$ mass of resonance
- γ Friction $\sim$ decay width

(Note at $\omega \approx \omega_0$ $(\omega_0^2 - \omega^2)^2 = (\omega_0 - \omega)^2(\omega_0 + \omega)^2 \approx 4\omega_0^2(\omega_0 - \omega)^2$, so
$A^2 \approx (F/2\omega_0)^2/((\omega_0 - \omega)^2 + \gamma^2/4))$)

Strong interaction process, quark diagram

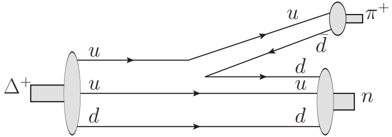
Recall: gluon interactions in hadrons not calculable in perturbation theory.

Example: $\Delta^+ \to \pi^+ + n$



Feynman diagram: cannot calculate all these gluons.

But: strong inteactions conserve quark flavor number: just draw quark lines:



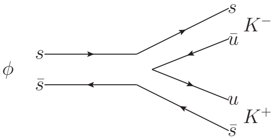
Quark diagram

Quark diagram $\neq$ Feynman diagram

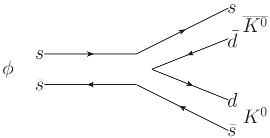
- In quark diagram: only draw quark lines
- keep track of quark flavor conservation
- Gluon can create quark-antiquark pair $\Longrightarrow$ in quark diagram can create $q\bar{q}$ pairs from nothing (same flavor!)

In Feynman diagram vertices are important part — in quark diagram the lines

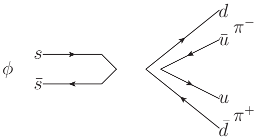
Zweig rule

- 

  $B_{\phi \to K^- K^+} \approx 49.2\%$

- 

  $B_{\phi \to K^0 \overline{K^0}} \approx 34.0\%$

- 

  $B_{\phi \to \text{pions}} \approx 15\%$

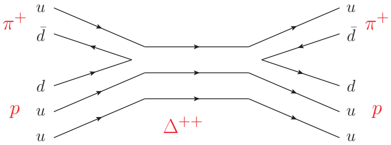
> **Zweig rule**
>
> Meson decays involving $q\bar{q}$ annihilation are suppressed.
> This is particularly important for heavy quark physics.

Baryons, mesons

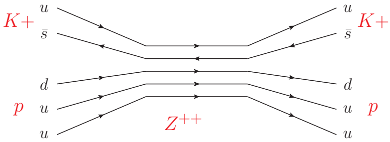
All observed resonances so far are baryons or mesons.

Some **exotic** bound quark states, such as $qqqq\bar{q}$ are possible by symmetries, but not seen in nature.

Quark diagrams are a convenient way to see whether there could be resonance peak in cross section.



Resonance peak seen in $\pi^+ p \to \pi^+ p$

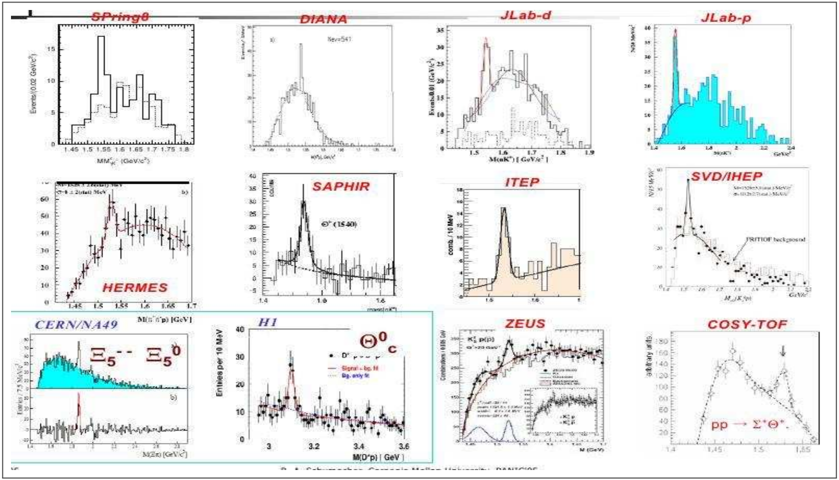


No peaks in $K^+ p \to K^+ p$

$\Longrightarrow$ No Z^{++} resonance found,

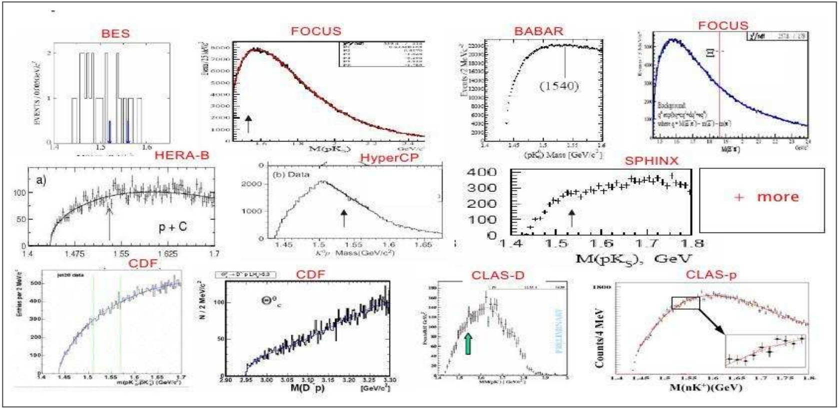
Recent history: pentaquark craze

Starting in fall of 2002, claims of evidence for **pentaquark =** $qqqq\bar{q}$.
E.g. $uudd\bar{s}$ state $\Theta^{+}(1530)$ $\Gamma = 15\text{MeV}$ resonance peak.



(From talk by Schumacher, PANIC05)

But then the evidence went away . . .



(From talk by Schumacher, PANIC05)

By 2005 the pentaquark was dead.

Lesson: finding resonances is a tricky and complicated business.

SU(2) symmetry of isospin

Two light quarks and their antiquarks form SU(2) doublets.

$$\begin{pmatrix} |u\rangle \\ |d\rangle \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} |\bar{u}\rangle \\ -|\bar{d}\rangle \end{pmatrix}$$

Isospin SU(2) invariance: we can rotate these doublets by $U \in$SU(2)

$$\begin{pmatrix} |u'\rangle \\ |d'\rangle \end{pmatrix} = U \begin{pmatrix} |u\rangle \\ |d\rangle \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} |\bar{u}'\rangle \\ -|\bar{d}'\rangle \end{pmatrix} = U \begin{pmatrix} |\bar{u}\rangle \\ -|\bar{d}\rangle \end{pmatrix},$$

and the new states $|u'\rangle, |d'\rangle, |\bar{u}'\rangle, |\bar{d}'\rangle$ behave just like the old ones
— under the strong interaction.

What about other quarks? **Strong** interaction treats **all flavors** similarly, so for N flavors we should have SU(N) symmetry.

Next step: add s quark, quark triplet $\Longrightarrow$ **flavor SU(3) symmetry**

$$\begin{pmatrix} |u\rangle \\ |d\rangle \\ |s\rangle \end{pmatrix} \longrightarrow \begin{pmatrix} |u'\rangle \\ |d'\rangle \\ |s'\rangle \end{pmatrix} = U \begin{pmatrix} |u\rangle \\ |d\rangle \\ |s\rangle \end{pmatrix}, \qquad \text{with } U^\dagger U = 1, \det U = 1.$$

Explicit form of SU(3) generators

$t_a = \lambda_a/2$

Gell-Mann matrices:

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned}$$

- An SU(2) doublet is the two eigenstates of $\sigma_3/2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ — the only diagonal generator $\Longrightarrow$ enough to know one eigenvalue ($\pm 1/2$)
- SU(3) states are characterized by the eigenvalues of the **two** diagonal generators, t_3, t_8 — isospin and hypercharge

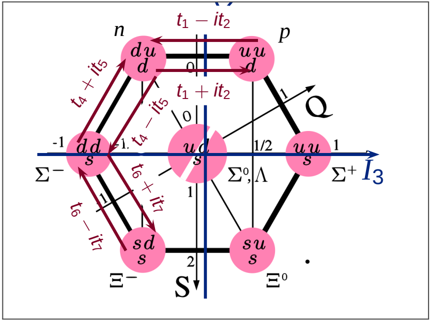
SU(2) and SU(3) ladder operators

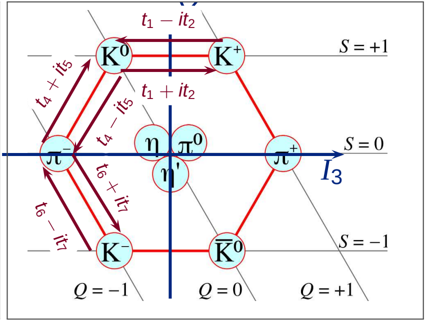
- Recall for SU(2):
  - One diagonal generator $\sigma_3/2$
  - Eigenstates $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for spin $1/2$ representation
  - from other two can form ladder operators $\sigma_+ = (\sigma_1 + i\sigma_2)/2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

    $\sigma_- = (\sigma_1 - i\sigma_2)/2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ that raise or lower eigenvalue.
- For SU(3) 2 diagonal generators: t_3 (eigenvalue isospin I_3) and t_8 (eigenvalue $\frac{\sqrt{3}}{2}Y$; Y= hypercharge).
- Eigenstates characterized by values of I_3 and Y
- Other generators combined into ladder operators $t_1 \pm it_2$; $t_4 \pm it_5$; $t_6 \pm it_7$ that raise/lower I_3, Y
- Fundametal representation (like spin $1/2$ for SU(2)) generated by t_a's: flavor u, d, s (or color of QCD quarks)
- $\exists$ higher representations (like spin 1, $3/2$ etc for SU(2)) with higher I, Y; but ladder operators raise/lower quantum numbers the same way.

SU(3) multiplets

Hadrons can indeed be organized into representations of SU(3):





These are not the simplest representation

$\Longrightarrow$ but effect of ladder operators is the same.

But different hypercharge/strangeness states do not have the same masses.

We do not go into representations of SU(3) here, subject for a full group theory course.

Mass matrix in SU(N) rotation

The mass part of the $\hat{H}$ (for quarks) is $m_u|u\rangle\langle u| + m_d|d\rangle\langle d| + m_s|s\rangle\langle s|$.
How does this transform in a $SU(3)$ rotation?

$$(|u\rangle\ |d\rangle\ |s\rangle)\begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}\begin{pmatrix} \langle u| \\ \langle d| \\ \langle s| \end{pmatrix}$$

$$\longrightarrow (|u'\rangle\ |d'\rangle\ |s'\rangle)U^*\begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}U^T\begin{pmatrix} \langle u| \\ \langle d| \\ \langle s| \end{pmatrix}$$

If masses were equal

$$\begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} = m\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Longrightarrow U^*mU^T = m(UU^\dagger)^* = m$$

Quark masses break flavor SU(N) symmetry

- SU(2) of isospin is a very good symmetry becuse $m_u \approx m_d$
- $m_s \gg m_{u,d} \Longrightarrow$ strange particles heavier, flavor SU(3) not as good.

Neutral meson mixing and mass matrix

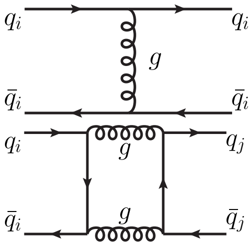
Neutral mesons states $|u\bar{u}\rangle, |d\bar{d}\rangle, |s\bar{s}\rangle$ have same charge, parity, C-parity etc.
$\Longrightarrow$ they can **mix**.

Physical mesons

- Pseudoscalar $J^{PC} = 0^{-+}$: $|\pi^0\rangle, |\eta\rangle, |\eta'\rangle$ ($L = 0$, spin singlet $S = 0$)
- Vector $J^{PC} = 1^{--}$: $|\rho^0\rangle, |\omega^0\rangle|\phi^0\rangle$ ($L = 0$, spin triplet $S = 1$)

are linear combinations of $|u\bar{u}\rangle, |d\bar{d}\rangle, |s\bar{s}\rangle$.
What linear combinations? The eigenvectors of the mass matrix.
Components of the mass matrix H_{ij}, $i, j = u\bar{u}, d\bar{d}, s\bar{s}$ have contributions from

- Quark masses $m_{q_i}\delta_{ij}$ $\Longrightarrow$ only important for s quark
- Gluon exchange between quark and antiquark (binding potential from QCD); only diagonal component $\sim \delta_{ij}$, do not change flavor
- Quark-antiquark annihilation and creation of a new pair: same for all component combinations H_{ij}



Simple model for neutral meson mass matrix

Choose basis $|u\bar{u}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ |d\bar{d}\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ |s\bar{s}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$

Simplest SU(3) symmetric mass matrix would be

$$H_m = \begin{pmatrix} m+\epsilon & \epsilon & \epsilon \\ \epsilon & m+\epsilon & \epsilon \\ \epsilon & \epsilon & m+\epsilon \end{pmatrix}.$$

Strange quark mass is heavier, add 1 parameter:

$$H_m = \begin{pmatrix} m+\epsilon & \epsilon & \epsilon \\ \epsilon & m+\epsilon & \epsilon \\ \epsilon & \epsilon & m'+\epsilon \end{pmatrix}.$$

This can be used to describe vector mesons $|\rho^0\rangle, |\omega^0\rangle, |\phi^0\rangle$ (exercise) — Pseudoscalar mesons $|\pi^0\rangle, |\eta\rangle, |\eta'\rangle$ are more complicated.

Pseudoscalar mixings

Two natural assumptions for physical particle states:

- Flavor SU(3) symmetry conserved
  - 2 SU(3) octet states $|\pi^0\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle)$, $|\eta_8\rangle = \frac{1}{\sqrt{6}}(|u\bar{u}\rangle + |d\bar{d}\rangle - 2|s\bar{s}\rangle)$,
  - SU(3) singlet state $|\eta_1\rangle = \frac{1}{\sqrt{3}}(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle)$,
- Flavor SU(3) completely broken: strange quark very heavy. Eigenstates:
  - Isospin triplet $|\pi^0\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle)$, and singlet $\frac{1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle)$,
  - Strange quark state = $|s\bar{s}\rangle$

For neutral pseudoscalar mesons neither! Physical states are a mixture

$$\begin{aligned}
|\eta(547)\rangle &= \cos\theta|\eta_8\rangle - \sin\theta|\eta_1\rangle \\
&= 0.5810(|u\bar{u}\rangle + |d\bar{d}\rangle) - 0.5698|s\bar{s}\rangle \\
|\eta'(958)\rangle &= \sin\theta|\eta_8\rangle + \cos\theta|\eta_1\rangle \\
&= 0.4029(|u\bar{u}\rangle + |d\bar{d}\rangle) + 0.82179|s\bar{s}\rangle \qquad \theta = -20^\circ
\end{aligned}$$

(Either KJE or I has a typo here in numerical values. . .)