FYSH300, fall 2011

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Part 2: Relativistic kinematics

Galilei and Einstein

Galileian coordinates:

- Time t same for all observers.
- Space $\vec{x}$ depends on observer.
(Think two events Δt apart, happening on a moving train v. For observer in train they are at same position, for observer on outside at distance $v\Delta t$)
- **Inertial** coordinate systems: $F = ma$, moving with constant velocity w.r.t. each other
- Laws of nature have **rotational** $SO(3)$-symmetry.

Special relativity:

- Speed of light c same for all observers (Maxwell, Michelson, Morley)
- **Time, space** depend on observer.
(Train example: also time interval different for observer on train and outside.)
- Inertial coordinate systems same as Galilei (General relativity: gravity is described as a kind of acceleration; on this course only inertial coordinate systems.)
- Laws of nature have larger **Lorentz-symmetry** ($\ni$ rotations) .

The mathematics of special relativity is described in terms of **4-vectors**

$$x = (ct, x^1, x^2, x^3) \equiv (x^0, x^1, x^2, x^3)$$

that live in **Minkowski space**. (Note c above; it is essential that it is same for everybody.)

4-vectors

Elements of Minkowski space are 4-vectors such as

$$x = (ct, x^1, x^2, x^3), \quad p = (E, p^1, p^2, p^3) \quad A^\mu = (\varphi(x), \mathbf{A}(x))$$

Notations:

- 3-vector has bar/boldface, 4-vector not $x = (x^0, \vec{x}) = (x^0, \mathbf{x})$
- Components x^μ, $\mu = 0, 1, 2, 3$, Greek letter $(\mu, \nu, \rho, \sigma, \alpha, \beta, \dots)$ is 4-index,

  (note: not 1 .. 4; that would be imaginary time)

  Sometimes notation x^μ refers to whole vector (should be clear from the context)
- Components of 3-vector $\vec{x}$ are x^i, $i = 1, 2, 3$, Latin $(i, j, k, \dots)$ is 3-index
- **Index up x^μ is different from index down x_μ**

  In differential geometry, x^μ are components of a vector, x_μ of a dual vector (a "form"). They are also called
  - x^μ "contravariant" vector,
  - x_μ "covariant" vector,

Minkowski metric

Scalar product of Minkowski space

We define a scalar product $x \cdot y = (x^0, \mathbf{x}) \cdot (y^0, \mathbf{y}) = x^0 y^0 - \overbrace{\mathbf{x} \cdot \mathbf{y}}^{\text{normal}\ \cdot}$

In terms of components: $\boxed{x \cdot y = x^\mu y^\nu g_{\mu\nu}}$ with the **metric tensor**

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

- This scalar product determines a (pseudo) **metric** for Minkowski space: $x^2 = x^\mu x^\nu g_{\mu\nu} = (x^0)^2 - \mathbf{x}^2$. (In mathematics a metric is positive definite, this is not.)
- The standard metric tensor is often written $g_{\mu\nu} = \mathrm{diag}(+ - - -)$. (Another convention (especially in relativity, cosmology) is $g_{\mu\nu} = \mathrm{diag}(- + + +)$.)
- Note: in some other (e.g. curved) coordinate system $g_{\mu\nu}$ is not always diagonal. However, it is always symmetric $g_{\mu\nu} = g_{\nu\mu}$.

Lowering indices

Mathematical digression (M=Minkowski space):

Covariant vectors a_μ are elements of M^* = dual of M, i.e. linear maps from M to real numbers $a_\mu \in M^* : M \to \mathbb{R}$. An element of M^* operates on a contravariant vector b^μ via $b^\mu \mapsto a_\mu b^\mu \in \mathbb{R}$. Now every contravariant vector c^μ also defines a linear map from M to $\mathbb{R}$ via the scalar product

$$b^\mu \mapsto \underbrace{c^\nu g_{\nu\mu}}\, b^\mu$$ called c_μ with same letter

Thus we can define a linear map from M to M^* via $c^\nu \mapsto c_\mu = g_{\mu\nu} c^\nu$.

This operation is called **lowering the index**, in terms of components:

if $a^\mu = (a^0, a^1, a^2, a^3)$ then $a_\mu = (a_0, a_1, a_2, a_3) = g_{\mu\nu} a^\nu = (a^0, -a^1, -a^2, -a^3)$

Lowering the index changes sign of space (123) components

Now scalar product is $a \cdot b = a^\mu g_{\mu\nu} b^\nu = a^\mu b_\mu = a_\nu b^\nu$

(note summation convention: repeated index summed, always one down, one up)

Inverse metric, raising indices

Metric $g_{\mu\nu}$ lowers the index $a_\mu = g_{\mu\nu} a^\nu$.

The inverse operation, **raising the index** is achieved by **inverse metric** $g^{\mu\nu}$
(Note convention; same letter for g and inverse g^{-1})

$$\text{Inverse is defined by } g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma = g_{\gamma\beta} g^{\beta\alpha} \qquad (\delta^\alpha_\gamma = \mathrm{diag}(1,1,1,1))$$

$$\text{Now } a^\mu = g^{\mu\nu} a_\nu = g^{\mu\nu}(g_{\nu\rho} a^\rho) = (g^{\mu\nu} g_{\nu\rho}) a^\rho = \delta^\mu_\rho a^\rho = a^\mu$$

Note identity matrix is δ^μ_ν with one index down, one up, thus $\delta^\mu_\nu a^\nu = a^\mu$, $\delta^\mu_\nu a_\mu = a_\nu \implies$ when identity matrix operates on a vector it stays in same space, the index stays up or down.

A diagonal matrix is easy to invert

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{01} & g^{02} & g^{03} \\ g^{10} & g^{11} & g^{12} & g^{13} \\ g^{20} & g^{21} & g^{22} & g^{23} \\ g^{30} & g^{31} & g^{32} & g^{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Derivatives

The spacetime coordinate x^μ has index up (contravariant).
The corresponding derivative

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

has index down (unless raised with $g^{\mu\nu}$).

Thus $\partial_\mu x^\nu = \delta^\nu_\mu$ identity, all 1's

A Lorentz invariant second derivative is the "d'Alembertian"

$$\Box \equiv g^{\mu\nu}\partial_\mu\partial_\nu = \partial_\mu\partial^\mu = \partial_0^2 - \nabla^2$$

Plane wave

Is an eigenfunction of $\Box$:

$$\Box e^{-ip_\alpha x^\alpha} = g^{\mu\nu}\partial_\mu\partial_\nu e^{-ip_\alpha x^\alpha} = g^{\mu\nu}(-ip_\mu)(-ip_\nu)e^{-ip_\alpha x^\alpha} = -p^2 e^{-ip_\alpha x^\alpha}$$

Same for tensors

Here a tensor is a linear animal with n indices that can be raised or lowered. For example: energy-momentum tensor $T^{\mu\nu}$ or field strength tensor $F^{\mu\nu}$.

$$\text{Lowering indices: } T_{\mu\nu} = g_{\mu\rho} g_{\nu\sigma} T^{\rho\sigma}$$

$$\text{Raising indices: } T^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} T_{\rho\sigma}$$

(Note: this is not the same thing as taking the trace $\mathrm{Tr}\, T = T^{\mu\nu} g_{\mu\nu}$, both indices separately)

Calculational rule

One index down, one up, summed over.

— Doesn't matter which one up/down: $(a)_\mu \cdots (b)^\mu = (a)^\mu \cdots (b)_\mu$

Never same index twice up or twice down $\cancel{(\cdot)_\mu \cdots (\cdot)_\mu}$

All of this may sound overly formal, and many people are sloppy with indices. It is only in other coordinate systems that it becomes really essential to keep track of up/down indices, but it is better to learn from the start.

Types of 4-vectors

4-vector a^μ has norm $a^2 = (a^0)^2 - \mathbf{a}^2$

$a^2 > 0$: a^μ is **timelike** vector (time $(t, \mathbf{0})$ is timelike)

$a^2 < 0$: a^μ is **spacelike** vector (distance $(0, \mathbf{x})$ is spacelike)

$a^2 = 0$: a^μ is **lightlike** vector

Consider 2 events in space and time:

1. Light ray departs at $(t_0, \mathbf{x}_0) = x_0^\mu$
2. Light ray arrives at $(t_1, \mathbf{x}_1) = x_1^\mu$

$$c = 1 = \frac{|\mathbf{x}_1 - \mathbf{x}_0|}{t_1 - t_0} \implies (t_1 - t_0)^2 = (\mathbf{x}_1 - \mathbf{x}_0)^2 \implies (x_1 - x_0)^2 = 0$$

$\implies$ this is why $a^2 = 0$ is “lightlike”

Lorentz-invariance: speed of light is same in all inertial coordinate systems.

$a^2 = 0$ in one inertial system $\implies a^2 = 0$ in all inertial systems.

Actually: all scalar products are independent of coordinate system (“frame”)

Lorentz-transformation

A rotation is a **linear** coordinate transformation that leaves the norm $\mathbf{x}^2$ invariant. Analoguously:

Lorentz transformation

Is a **linear** coordinate transformation $\Lambda^\mu_{\ \nu}$

$$x'^\mu = \Lambda^\mu_{\ \nu} x^\nu$$

that leaves the scalar product invariant $x' \cdot y' = x \cdot y$.

The definition implies

$$x'^\mu g_{\mu\nu} y'^\nu = \Lambda^\mu_{\ \rho} x^\rho g_{\mu\nu} \Lambda^\nu_{\ \sigma} y^\sigma = x^\rho g_{\rho\sigma} y^\sigma \text{ or}$$

$$g_{\rho\sigma} = \Lambda^\mu_{\ \rho} g_{\mu\nu} \Lambda^\nu_{\ \sigma} = (\Lambda^T)_\rho^{\ \mu} g_{\mu\nu} \Lambda^\nu_{\ \sigma} \qquad g = (\Lambda^T) g \Lambda$$

(“Transpose” refers to 4×4-matrix product, does not “know” about index up/down. $(\Lambda^T)_\rho^{\ \mu} = \Lambda^\mu_{\ \rho}$.)

Rotations are also Lorentz-transformations

Lorentz boost

Derivation

Mix time and space, i.e. **boost** (suom. pusku) to a different velocity.
Most general linear form that mixes t and z coordinates:

$$\begin{aligned} t' &= \gamma t - \beta\gamma z \\ x' &= x \\ y' &= y \\ z' &= \gamma' z - \beta'\gamma' t, \end{aligned}$$

Now we require that the scalar product is invariant, i.e.

$$(t')^2 - (x')^2 - (y')^2 - (z')^2 = t^2 - x^2 - y^2 - z^2$$

$$(t')^2 - (z')^2 = \overbrace{\left(\gamma^2 - (\beta')^2(\gamma')^2\right)}^{=1} t^2 - 2\overbrace{\left(\beta\gamma^2 - \beta'(\gamma')^2\right)}^{=0} tz - \overbrace{\left((\gamma')^2 - \beta^2\gamma^2\right)}^{=1} z^2$$

This is solved only by $\gamma = \gamma'$, $\beta = \beta'$ and

$$\boxed{\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \geq 1}$$

Lorentz boost

Intepretation

We now have a coordinate transformation from coordinates K to K':

$$\begin{aligned} t' &= \gamma(t - \beta z) \\ x' &= x \\ y' &= y \\ z' &= \gamma(z - \beta t), \end{aligned}$$

- Origin of new coordinates is $z' = 0 \implies z = \beta t \implies$ **New system moves with velocity** $v = \beta$
- Time interval Δt at same place $z = 0$ in K
  $\implies$ Longer time $\Delta t' = \gamma \Delta t$ in K', **time dilation**
  (But beginning and end of interval measured at different places z' in new system.)
- Unmoving stick $\Delta z = L$ in K. Length in K'? Lengths are measured at equal time: $\Delta t' = 0 \implies \Delta t = \beta L$. Now length in K' is $\Delta z' = \gamma(L - \beta(\beta L)) = L/\gamma < L \implies$ **Lorentz contraction** of lengths.
  (Measure ends of stick at same time $t' = 0$ in K', but at different times in K!)

Definitions

$$\text{Remember } g_{\rho\sigma} = \Lambda^{\mu}{}_{\rho} g_{\mu\nu} \Lambda^{\nu}{}_{\sigma} = (\Lambda^T)_{\rho}{}^{\mu} g_{\mu\nu} \Lambda^{\nu}{}_{\sigma} \qquad g = \Lambda^T g \Lambda$$

$$\text{Now } \det g = 1 \cdot (-1)^3 = \det(\Lambda^T g \Lambda) = (\det \Lambda)^2 \det g \implies \det \Lambda = \pm 1$$

$$\text{Also } 1 = g_{00} = g_{\mu\nu} \Lambda^{\mu}{}_{0} \Lambda^{\nu}{}_{0} = (\Lambda^{0}{}_{0})^2 - \overbrace{\left((\Lambda^{1}{}_{0})^2 + (\Lambda^{2}{}_{0})^2 + (\Lambda^{3}{}_{0})^2\right)}^{\geq 0}$$

$$\implies (\Lambda^{0}{}_{0})^2 \geq 1 \implies \Lambda^{0}{}_{0} \geq 1 \text{ or } \leq -1$$

Proper Lorentz group

= Lorentz transf's continuously connected to identity $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu}$, i.e.

$\Lambda^{0}{}_{0} \geq 1$ **and** $\det \Lambda = 1$.

Full Lorentz group also includes

- Time reversal $\Lambda = \mathrm{diag}(-1, 1, 1, 1)$ (T)
- Space inversion $\Lambda = \mathrm{diag}(1, -1, -1, -1)$ **parity** P

(Strong interaction also invariant under P, T, weak interaction not!)

Inverse Lorentz transformation

$$\text{Remember } g_{\rho\sigma} = \Lambda^{\mu}{}_{\rho} g_{\mu\nu} \Lambda^{\nu}{}_{\sigma} \qquad \Big| \, (\Lambda^{-1})^{\rho}{}_{\alpha} g^{\sigma\beta}$$

$$\begin{aligned} g_{\rho\sigma}(\Lambda^{-1})^{\rho}{}_{\alpha} g^{\sigma\beta} &= \delta^{\beta}_{\rho}(\Lambda^{-1})^{\rho}{}_{\alpha} \\ &= (\Lambda^{-1})^{\beta}{}_{\alpha} \\ = \Lambda^{\mu}{}_{\rho} g_{\mu\nu} \Lambda^{\nu}{}_{\sigma}(\Lambda^{-1})^{\rho}{}_{\alpha} g^{\sigma\beta} &= g_{\mu\nu}\Lambda^{\nu}{}_{\sigma} g^{\sigma\beta}\delta^{\mu}_{\alpha} \\ &= g_{\alpha\nu}\Lambda^{\nu}{}_{\sigma} g^{\sigma\beta} \end{aligned}$$

$$\left(\Lambda^{-1}\right)^{\beta}{}_{\alpha} = g_{\alpha\nu}\Lambda^{\nu}{}_{\sigma} g^{\sigma\beta} = g^{\beta\sigma}\Lambda^{\nu}{}_{\sigma} g_{\nu\alpha} = g^{\beta\sigma}\left(\Lambda^{T}\right)_{\sigma}{}^{\nu} g_{\nu\alpha} \quad \text{i.e.} \quad \Lambda^{-1} = g\Lambda^{T} g$$

Inverse Lorentz transf. by transpose + raise/lower index

- Remember what $\cdot^{T}$ means: Both Λ and Λ^{-1} take x^{μ} to y^{μ}, must have indices $\Lambda^{\bullet}{}_{\bullet}$; $(\Lambda^{-1})^{\bullet}{}_{\bullet}$, but transpose has $(\Lambda^{T})_{\bullet}{}^{\bullet}$, so must raise and lower.
- Analogy with rotations: $R^{-1} = R^{T}$, now also change some signs.

(For boost (HT) amounts to $\beta \to -\beta$)

Adding velocities

Consider two successive boosts in the same direction,
from coordinate system K to K'; then from K' to K'':

$$\begin{aligned} t' &= \gamma_1(t - v_1 z) \\ x' &= x \\ y' &= y \\ z' &= \gamma_1(z - v_1 t), \end{aligned} \qquad \begin{aligned} t'' &= \gamma_2(t' - v_2 z') \\ x'' &= x' \\ y'' &= y' \\ z'' &= \gamma_2(z' - v_2 t'), \end{aligned}$$

This is the same (exercise) as boosting directly from K to K'' with velocity

$$v = \frac{v_1 + v_2}{1 + v_1 v_2}$$

- Velocities add up only when small: $v \approx v_1 + v_2$ when $v_{1,2} \ll 1 = c$
- When $v_1 = 1 = c$ or $v_2 = 1 = c$, $v = 1 = c$ $\Longrightarrow$ if K moves with light velocity w.r.t. K', it also moves with c w.r.t. K''.

4-momentum

Particle motion is characterized by the momentum 4-vector $p = (E, \mathbf{p})$

(E is the relativistic energy of the particle, i.e. it includes energy of the mass.)

In the **rest frame** (K' here) the 4-momentum is $p' = (m, \mathbf{0})$ In K this is:

$$
\begin{aligned}
E &= \gamma(E' + vp'^3) && = \gamma m && = mu^0 \\
p^1 &= p'^1 && = 0 && = mu^1 \\
p^2 &= p'^2 && = 0 && = mu^2 \\
p^3 &= \gamma(p'^3 + vE') && = \gamma vm && = mu^3,
\end{aligned}
$$

where $u^\mu = \gamma(1, \mathbf{v}) = (1, \mathbf{v})/\sqrt{1 - v^2}$ is the **4-velocity**.

(Nonrelativistic limit: $E = m/\sqrt{1 - v^2} \approx m + \frac{1}{2}mv^2$)

The invariant mass of a particle

Is $p^2 \equiv p \cdot p = E^2 - \mathbf{p}^2 = m^2$ in any coordinate system

(This is a scalar product, it is Lorentz-invariant.)

We say that the **particle is on mass shell $p^2 = m^2$.**

(“Off-shell” particles are “virtual”, short lived quantum fluctuations.)

Rapidity

Successive boosts in one direction with velocity v become easy if we define the **rapidity** ξ as $v = \tanh \xi$. Now for successive boosts $\xi = \xi_1 + \xi_2$. (exercise)

Note: this is only simple for boosts in the same direction. Usually in particle collisions this is the **beam axis**.
Treating the **transverse** (1,2) and **longitudinal** (3) directions differently!

4-momentum is parametrized in terms of (momentum space) rapidity y;
$v^z = \tanh y$

$$E = \sqrt{m^2 + p_T^2} \cosh y = m_T^2 \cosh y \qquad \mathbf{p}_T = (p^1, p^2)$$
$$p^3 = \sqrt{m^2 + p_T^2} \sinh y = m_T^2 \sinh y \qquad p_T^2 = \mathbf{p}_T^2$$

When boosting with a boost ξ this behaves as you would expect . . . (exercise)

Pseudorapidity η is same for $m = 0$: $E = p_T \cosh \eta, \qquad p^3 = p_T \sinh \eta$

+ Just angle, easy to measure $\Longrightarrow$ experiments use this
- Does not Lorentz-transform easily for massive particles

Final remark

Note the spelling: Hendrik **Lorentz**, Dutch theoretical physicist. Nobel 1902 for explaining Zeeman effect. Developed Lorentz-transformations to understand result of Michelson-Morley experiment.

There is also the Danish physicist Ludvig **Lorenz**. The Lorenz gauge condition $\partial_\mu A^\mu = 0$ is named after him.

Trivia exercise: find out what is the “Lorentz-Lorenz equation”!

2 Basic problems in particle physics

Particle decay (hajoaminen)

Unstable particle decays:

$$a \to cd + \dots$$

Only one reasonable coordinate system: rest frame of decaying particle:
$p_a = (m_a, \mathbf{0})$

Scattering (sironta)

Two particles collide:

$$ab \to cd + \dots$$

Two convenient frames:

TRF **T**arget **R**est **F**rame: rest frame of a or b. (Also called "lab frame", confusingly even when experiment is not fixed target ...)

CMS **C**enter of **M**ass **S**ystem: total 3-momentum of all particles is zero; $\mathbf{p}_a = -\mathbf{p}_b$

Momentum conservation

Total 4-momentum is always conserved

Decay: $p_a = p_c + p_d + \dots$ Scattering: $p_a + p_b = p_c + p_d + \dots$

This means all 4 components conserved separately

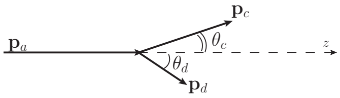
2→2 process $ab \to cd$

Count independent kinematical variables

1. $4 \times 4 = 16$ energy/momentum components
2. 4 constraints from $p_a^2 = m_a^2 \ldots$
3. 4 constraints from energy-momentum conservation
4. Can rotate whole system (all particles) around 3 axes or boost in 3 directions: 3+3=6.

$\Longrightarrow$ Leaves $16 - 4 - 4 - 6 = 2$ independent variables that scattering probability can depend on — which ones? Possible choices:

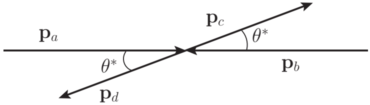
TRF



E.g. give $|\mathbf{p}_a|$ and θ_c^{TRF}

(Need to specify at least one absolute value of momentum $|p|$, only angles cannot give scale. The other variable can be an absolute value or and angle.)

CMS (denote variables by star *)



Typically give $p^* \equiv |\mathbf{p}_a| = |\mathbf{p}_b|$

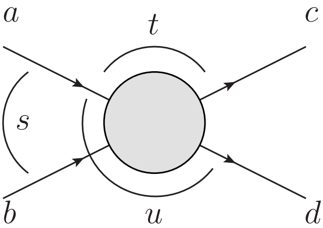
(By definition $\mathbf{p}_a = -\mathbf{p}_b$)

and $\theta^* = \theta_{ac} = \theta_{bd}$.

($\mathbf{p}_c = -\mathbf{p}_d$, but $|\mathbf{p}_a|$ is not necessarily same as $|\mathbf{p}_c|$ if masses different.)

Mandelstam variables

The two independent kinematical variables are usually expressed in terms of **Mandelstam variables** s, t, u



(Remember $p_a + p_b = p_c + p_d$)

$$\begin{aligned} s &= (p_a + p_b)^2 = (p_c + p_d)^2 \\ t &= (p_a - p_c)^2 = (p_b - p_d)^2 \\ u &= (p_a - p_d)^2 = (p_b - p_c)^2 \end{aligned}$$

Only 2 independent (exercise) :

$$s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2$$

Mandelstam variables are Lorentz-invariant

You can compute them from 4-vectors in any frame.

Total collision energy

Look at first Mandelstam invariant s

$$s = (p_a + p_b)^2 = (E_a + E_b)^2 - (\mathbf{p}_a + \mathbf{p}_b)^2.$$

CMS

$$s = (E_a^* + E_b^*)^2 - (\mathbf{p}_a^* + \mathbf{p}_b^*)^2.$$

By definition $\mathbf{p}_a^* + \mathbf{p}_b^* = \mathbf{0}$, and hence $s = (E_a^* + E_b^*)^2$ which means that $\sqrt{s} = E_a^* + E_b^* = E_{\mathrm{tot}}^*$ is the **total energy of the collision in the CMS**.

TRF

$$\begin{aligned} s &= (E_a^{\mathrm{TRF}} + E_b^{\mathrm{TRF}})^2 - (\mathbf{p}_a^{\mathrm{TRF}} + \mathbf{p}_b^{\mathrm{TRF}})^2 \\ &= m_a^2 + m_b^2 + 2E_a^{\mathrm{TRF}} m_b, \end{aligned}$$

where we used $\mathbf{p}_b^{\mathrm{TRF}} = \mathbf{0}$ and $(E_a^{\mathrm{TRF}})^2 - (\mathbf{p}_a^{\mathrm{TRF}})^2 = m_a^2$. Hence

$$E_{\mathrm{tot}}^* = \sqrt{s} = \sqrt{m_a^2 + m_b^2 + 2m_b E_a^{\mathrm{TRF}}}.$$

General results, CMS

Best left as an exercise:

CMS

$$E_a^* = \frac{s + m_a^2 - m_b^2}{2\sqrt{s}}, \quad E_b^* = \frac{s - m_a^2 + m_b^2}{2\sqrt{s}} \quad |\mathbf{p}_a^*| = |\mathbf{p}_b^*| = \frac{\sqrt{\lambda(s, m_a^2, m_b^2)}}{2\sqrt{s}}$$

$$E_c^* = \frac{s + m_c^2 - m_d^2}{2\sqrt{s}}, \quad E_d^* = \frac{s - m_c^2 + m_d^2}{2\sqrt{s}} \quad |\mathbf{p}_c^*| = |\mathbf{p}_d^*| = \frac{\sqrt{\lambda(s, m_c^2, m_d^2)}}{2\sqrt{s}}$$

Here we have defined $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$.

General results, TRF

Best left as an exercise:

TRF

$$E_a^{\mathrm{TRF}} = \frac{s - m_a^2 - m_b^2}{2m_b}, \qquad |\mathbf{p}_a^{\mathrm{TRF}}| = \frac{\sqrt{\lambda(s, m_a^2, m_b^2)}}{2m_b},$$

$$E_b^{\mathrm{TRF}} = m_b, \qquad |\mathbf{p}_b^{\mathrm{TRF}}| = 0$$

$$E_c^{\mathrm{TRF}} = -\frac{u - m_b^2 - m_c^2}{2m_b}, \qquad |\mathbf{p}_c^{\mathrm{TRF}}| = \frac{\sqrt{\lambda(u, m_b^2, m_c^2)}}{2m_b},$$

$$E_d^{\mathrm{TRF}} = -\frac{t - m_b^2 - m_d^2}{2m_b}, \qquad |\mathbf{p}_d^{\mathrm{TRF}}| = \frac{\sqrt{\lambda(t, m_b^2, m_d^2)}}{2m_b}.$$

Here we have defined $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$.

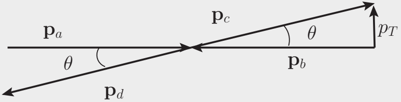
Massless/ultrarelativistic limit

Neglecting all the masses (This is a good approximation when all momenta are $\gg m_{abcd}$) one gets **(Memorize these!)**

$$
\begin{aligned}
s &\approx 4|\mathbf{p}_a^*|^2 \\
t &\approx -\frac{s}{2}(1-\cos\theta_c^*) \\
u &\approx -\frac{s}{2}(1+\cos\theta_c^*).
\end{aligned}
$$

Note that $t<0$ and $u<0$ and with $m=0$ we have $s+t+u\approx 0$.

Further approximation: small angle scattering $\theta^* \ll 1$



Now $p_c^z \approx p_a^z \gg |\mathbf{p}_c|\sin\theta^* \equiv p_T$; $\sin\theta^* \approx \theta^*$ and $\cos\theta^* \approx 1-\frac{1}{2}(\theta^*)^2$.

We get $t \approx -p_T^2 \qquad u \approx -s$

CMS/TRF quantities and invariants example I

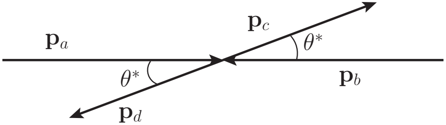
Elastic scattering $p + p \to p + p$

In CMS all four momenta are equal in magnitude, denote this by $|\mathbf{p}^*|$. Now, suppose that in CMS we have $|\mathbf{p}^*| = \sqrt{3}m_p$ and $\theta_c^* = 90^\circ$.
Question: What are θ_c^{TRF} and θ_d^{TRF}?
Idea of solution: use invariance of s, t and u.

1. Evaluate invariants in CMS
2. Solve TRF variables in terms of invariants

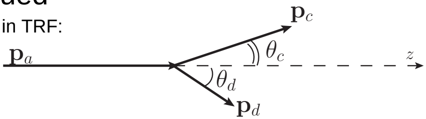
First, in CMS we have

$$\begin{aligned} t &= -2|\mathbf{p}^*|^2(1 - \cos\theta_c^*) = -2 \cdot 3m_p^2 = -6m_p^2 \\ s &= (E_a^* + E_b^*)^2 = \left(2\sqrt{m_p^2 + |\mathbf{p}^*|^2}\right)^2 = 16m_p^2 \\ u &= 4m_p^2 - s - t = -6m_p^2. \end{aligned}$$



Example continued

Now evaluate invariants in TRF:



$$
\begin{aligned}
t &= (p_a - p_c)^2 = p_a^2 + p_c^2 - 2E_aE_c + 2|\mathbf{p}_a||\mathbf{p}_c|\cos\theta_c \\
&= 2m_p^2 - 2E_a^{\mathrm{TRF}}E_c^{\mathrm{TRF}} + 2|\mathbf{p}_a^{\mathrm{TRF}}||\mathbf{p}_c^{\mathrm{TRF}}|\cos\theta_c^{\mathrm{TRF}} \\
s &= (p_a + p_b)^2 = 2m_p^2 + 2m_pE_a^{\mathrm{TRF}} \\
u &= (p_b - p_c)^2 = 2m_p^2 - 2m_pE_c^{\mathrm{TRF}}
\end{aligned}
$$

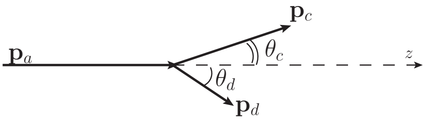
From t we obtain $\cos\theta_c^{\mathrm{TRF}}$ provided that we know E_a^{TRF}, $|\mathbf{p}_a^{\mathrm{TRF}}|$, E_c^{TRF} and $|\mathbf{p}_c^{\mathrm{TRF}}|$. These are straightforward to obtain from s and u:

$$
\begin{aligned}
E_a^{\mathrm{TRF}} &= 7m_p, \ \ |\mathbf{p}_a^{\mathrm{TRF}}| = \sqrt{(E_a^{\mathrm{TRF}})^2 - m_p^2} = \sqrt{48}m_p \\
E_c^{\mathrm{TRF}} &= 4m_p, \ \ |\mathbf{p}_c^{\mathrm{TRF}}| = \sqrt{(E_c^{\mathrm{TRF}})^2 - m_p^2} = \sqrt{15}m_p
\end{aligned}
$$

$$
\textit{Result}: \cos\theta_c^{\mathrm{TRF}} = \frac{-6m_p^2 - 2m_p^2 + 2\cdot 7m_p\cdot 4m_p}{2\sqrt{48}m_p\cdot\sqrt{15}m_p} = \frac{2}{\sqrt{5}}.
$$

Example continued I

What about the other scattering angle θ_d^{TRF}?



Consider again u in TRF, but now with p_d

$$u = (p_a - p_d)^2 = 2m_p^2 - 2E_a^{\mathrm{TRF}}E_d^{\mathrm{TRF}} + 2|\mathbf{p}_a^{\mathrm{TRF}}||\mathbf{p}_d^{\mathrm{TRF}}|\cos\theta_d^{\mathrm{TRF}}.$$

From this we can evaluate $\cos\theta_d^{\mathrm{TRF}}$, knowing the energy and momentum of d. This is done in similar matter to the cases above, by evaluating t in TRF and the results are: $E_d^{\mathrm{TRF}} = 4m_p$ and $|\mathbf{p}_d^{\mathrm{TRF}}| = \sqrt{15}m_p$. Hence we find that $\cos\theta_d^{\mathrm{TRF}} = 2/\sqrt{5}$, and finally

$$\theta_d^{\mathrm{TRF}} = \theta_c^{\mathrm{TRF}} \approx 26.6^\circ.$$

In this specific case we have equal angles (which is due to the equality of t and u; $\theta^* = 90^\circ$ in CMS and masses equal).

Invariant mass of a system of particles

For single particle on the mass shell $p^2 = p_\mu p^\mu = m^2$.
For a set of many particles, $1, \ldots n$ particles with momenta $p_1, \ldots, p_n$ **define the invariant mass as**

$$M^2 \equiv (p_1 + \cdots + p_n)^2.$$

As the name suggests, M is invariant under Lorentz transformations.
In the CMS of n particles we have

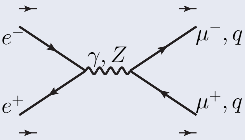
$$M^2 = (E_1^* + E_2^* + \cdots + E_n^*)^2 \geq (m_1 + m_2 + \cdots + m_n)^2.$$

Decay of an unstable particle

Consider the decay $A \to 1 + 2 + \cdots + n$.
4-momentum is conserved so $p_A^2 = (p_1 + \cdots + p_n)^2 = M^2 =$ invariant mass of the decay products.
What is the relation between the real physical mass of A, m_A^2, and the invariant mass M^2, and in particular can we have $m_A^2 \neq M^2$?

- If $M^2 = p_A^2 = m_A^2$ then we say that A is a real particle on its mass shell.
- If $s = M^2 \neq m_A^2$, A was a **virtual** particle.

Example: $e^+e^- \to \mu^+\mu^-$



Scattering can happen via photon γ or Z^0 boson.

- For $s = (p_{e^+} + p_{e^-})^2 = m_{Z^0}^2$ created a real Z^0 in the intermediate state; collision rate has a peak (like resonance of classical mechanics oscillator).
- $s = (p_{e^+} + p_{e^-})^2 > 0$ always: cannot create a real photon, $m_\gamma = 0$
- $s \neq m_{Z^0}^2$ scattering mediated by intermediate virtual Z^0 or γ.

Threshold energy for a reaction

Consider process $b + t \to 1 + 2 + \cdots + n$. (b=beam, t=target)
If the masses of the final state particles, $m_1 + m_2 + \cdots m_n > m_b + m_t$, then then process requires a certain amount of energy in order to proceed. This energy is called the threshold energy of the process. In CMS we have

$$\sqrt{s} = E_b^* + E_t^* = E_1^* + \cdots + E_n^* \geq m_1 + \cdots + m_n,$$

and using $E_b^* = \sqrt{m_b^2 + |\mathbf{p}*|^2}$ and $E_t^* = \sqrt{m_t^2 + |\mathbf{p}*|^2}$ one solves for the minimum $|\mathbf{p}^*|$ for which the above inequality is satisfied.
In the TRF on the other hand we have

$$s = (p_b + p_t)^2 = m_b^2 + m_t^2 + 2m_t E_b^{\mathrm{TRF}} \geq (m_1 + m_2 + \cdots + m_n)^2,$$

from which we obtain the threshold energy in TRF:

$$E_b^{\mathrm{TRF}} \geq \frac{(m_1 + \cdots + m_n)^2 - m_b^2 - m_t^2}{2m_t} \equiv (E_b^{\mathrm{TRF}})_{\mathrm{min}}$$

Examples

$$E_b^{\mathrm{TRF}} \geq \frac{(m_1 + \cdots + m_n)^2 - m_b^2 - m_t^2}{2m_t} \equiv (E_b^{\mathrm{TRF}})_{\mathrm{min}}$$

Antiproton production in collision $p + p \to p + p + \bar{p} + p$:

$$(E_b^{\mathrm{TRF}})_{\mathrm{min}} = \frac{(4m_p)^2 - 2m_p^2}{2m_p} = 7m_p \simeq 6.568 \ \mathrm{GeV}.$$

Pion production in $p + p$-collision:

$$\begin{aligned} p + p \to \quad & p + p + \pi^0 \\ & p + n + \pi^+. \end{aligned}$$

$$\begin{aligned} (E_b^{\mathrm{TRF}})_{\mathrm{min}}^{p+p+\pi^0} &= 1.218 \ \mathrm{GeV} \\ (E_b^{\mathrm{TRF}})_{\mathrm{min}}^{p+n+\pi^+} &= 1.231 \ \mathrm{GeV}. \end{aligned}$$