ORLICZ-SOBOLEV EXTENSIONS AND MEASURE DENSITY CONDITION

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Abstract. We study the extension properties of Orlicz-Sobolev functions both in Euclidean spaces and in metric measure spaces equipped with a doubling measure. We show that a set $E \subset \mathbb{R}^n$ satisfying a measure density condition admits a bounded linear extension operator from the trace space $W^{1,\Psi}(\mathbb{R}^n)|_E$ to $W^{1,\Psi}(\mathbb{R}^n)$. Then we show that a domain, in which the Sobolev embedding theorem or a Poincaré type inequality holds, satisfies the measure density condition. It follows that the existence of a bounded, possibly non-linear extension operator or even the surjectivity of the trace operator implies the measure density condition and hence the existence of a bounded linear extension operator.

1. Introduction

In this paper, we consider extension domains for Orlicz-Sobolev spaces both in the Euclidean space and in a metric measure space equipped with a doubling measure. Recall that, for a domain $\Omega \subset \mathbb{R}^n$ and a Young function Ψ, the Orlicz-Sobolev space $W^{1,\Psi}(\Omega)$ consists of the functions $u \in L^\Psi(\Omega)$ whose first order weak derivatives belong to the Orlicz space $L^\Psi(\Omega)$, see Section 2 for the definition of Young function and Orlicz space. The space $W^{1,\Psi}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{W^{1,\Psi}(\Omega)} = \|u\|_{L^\Psi(\Omega)} + \||\nabla u|\|_{L^\Psi(\Omega)},$$

where $\|\cdot\|_{L^\Psi(\Omega)}$ is the Luxemburg norm and ∇u is the weak gradient of u.

We say that Ω is a $W^{1,\Psi}$-extension domain if there exists a bounded linear operator $\mathcal{E} : W^{1,\Psi}(\Omega) \to W^{1,\Psi}(\mathbb{R}^n)$ such that $\mathcal{E}u|_\Omega = u$ for each $u \in W^{1,\Psi}(\Omega)$.

Extendability of the classical Sobolev functions is studied by several authors, see for example [3], [16], [21], [23], [26], [30], [32], [34]. The existence of an extension operator guarantees that $W^{k,p}(\Omega)$ inherits many properties possessed by $W^{k,p}(\mathbb{R}^n)$. For instance, Sobolev imbeddings hold for functions of $W^{k,p}(\Omega)$, not merely for functions with zero boundary values. Each $L^p(\Omega)$-function has a trivial extension to a function of $L^p(\mathbb{R}^n)$, whereas the problem in extending Sobolev functions is the extension across the boundary. According to well-known results of Calderón and Stein, every bounded Lipschitz domain is a $W^{m,p}$-extension domain, for all p. By Jones [23], (ε,δ)-domains, and hence uniform domains are extension domains. This result is best possible in the sense that each finitely connected planar $W^{1,2}$-extension domain is necessarily (ε,δ). Each (ε,δ)-domain satisfies measure density condition (1.1) which is closely related with the extension property. We say that a measurable

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set $E \subset \mathbb{R}^n$ satisfies the measure density condition if there is a constant $c > 0$ such that

$$|E \cap B(x,r)| \geq cr^n \tag{1.1}$$

for each $x \in \overline{E}$ and all $0 < r \leq 1$. By Rychkov [30], this condition implies the existence of a continuous linear extension operator from the trace space of $W^{k,p}(\mathbb{R}^n)$ to $W^{k,p}(\mathbb{R}^n)$ when $p > 1$. On the other hand, the unit disc in the plane with a radius removed shows that (1.1) cannot imply the extension property. Moreover, cusp domains, typical examples that are not $W^{m,p}$-extension domains, do not satisfy (1.1). In [16], the second author with Hajłasz and Koskela gave several characterizations of an extension domain for $W^{m,p}$, $p > 1$, and showed that extension domains satisfy (1.1). This together with a result of Shvartsman [32] shows that the surjectivity of the trace operator implies the existence of a bounded linear extension operator when $p > 1$, see [16].

Characterization of extension domains is an interesting problem also for Orlicz-Sobolev spaces but unfortunately, there are only few extension results for the classical Orlicz-Sobolev spaces. By [8], every domain that satisfies a restricted cone condition, and hence every bounded Lipschitz domain, is a $W^{1,\Psi}$-extension domain if Ψ is a sufficiently nice, doubling N-function.

We will generalize Sobolev space extension results from [15] and [16] to Orlicz-Sobolev spaces. We prove the results in the setting of metric measure spaces but our results are new even in the Euclidean spaces. Below, we denote the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$ by $|E|$ and integration with respect to the Lebesgue by dx. The norms $\|\cdot\|_{W^{1,\Psi}(\Omega;\lambda)}$, where $\lambda > 0$, are taken with the weighted measure $\lambda\, dx$ (and similarly for other norms).

Our first result gives an exact description of the space of traces of $W^{1,\Psi}(\mathbb{R}^n)$ in E, if E satisfies the measure density condition.

Theorem 1.1. *Let Ψ and its conjugate be doubling N-functions. If a measurable set $E \subset \mathbb{R}^n$ satisfies measure density condition* (1.1)*, then*

$$W^{1,\Psi}(\mathbb{R}^n)|_E = M^{1,\Psi}(E)$$

as sets and the norms are equivalent. Moreover, there is a bounded linear extension operator

$$\mathcal{E}\colon M^{1,\Psi}(E) \to M^{1,\Psi}(\mathbb{R}^n) = W^{1,\Psi}(\mathbb{R}^n).$$

Here $M^{1,\Psi}(E)$ is the Orlicz-Sobolev space defined using pointwise inequalities, see Section 2. The fact that $M^{1,\Psi}(\mathbb{R}^n)$ coincides with $W^{1,\Psi}(\mathbb{R}^n)$, if both Ψ and its conjugate are doubling, was proved in [2]. Note that the conjugate of Ψ is doubling if and only if the Hardy-Littlewood maximal operator is bounded in L^Ψ, see [10].

Our next result shows that if a suitable version of the Sobolev embedding theorem or a local Poincaré-type inequality holds in a domain, then it satisfies the measure density condition. For the definition of functions Ψ_n and ω_n, see Theorem 2.1.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ a domain and let Ψ be a Young function. Suppose that one of the following conditions holds.*

(a) *There exists $1 \leq p < n$ such that $\Psi(t)/t^p$ is decreasing and $W^{1,\Psi}(\Omega) \subset L^{\Psi_n}(\Omega)$.*

(b) *Ψ is doubling, there exists $p > n$ such that $\Psi(t)/t^p$ is increasing, and each $u \in W^{1,\Psi}(\Omega)$ satisfies*

$$|u(x) - u(y)| \leq C\omega_n(C^{-1}|x-y|^{-n})\|u\|_{W^{1,\Psi}(\Omega)}$$

for a.e. $x, y \in \Omega$ such that $|x - y| < 1$.

(c) *There is a function Φ satisfying $\Phi^{-1}(t) \leq C\Psi^{-1}(t)t^{-\varepsilon}$ for some $\varepsilon > 0$ and all $t \geq 1$, and a constant $C > 0$ such that*

$$\inf_{c\in\mathbb{R}} \|u - c\|_{L^{\Phi}_w(B\cap\Omega;\,|B|^{-1})} \leq Cr\|u\|_{W^{1,\Psi}(\Omega;\,|B|^{-1})}$$

for all balls $B = B(x, r)$ with $x \in \overline{\Omega}$ and $0 < r \leq 1$.

Then Ω satisfies measure density condition (1.1)*.*

Remark 1.3. Case (c) of Theorem 1.2 is new already when Ψ is a power function. It says that an inequality of type

$$\inf_{c\in\mathbb{R}} \|u - c\|_{L^{p+\varepsilon}(B\cap\Omega;\,|B|^{-1})} \leq Cr\|u\|_{W^{1,p}(\Omega;\,|B|^{-1})},$$

where $\varepsilon > 0$, implies the measure density condition. In particular, this improves the case $p = n$ of [15, Theorem 1], which says that the measure density condition follows from an inequality of type

$$\inf_{c\in\mathbb{R}} \|u - c\|_{L^{\exp(\cdot)^s}(B\cap\Omega;\,|B|^{-1})} \leq Cr\|u\|_{W^{1,n}(\Omega;\,|B|^{-1})},$$

where $s > 0$.

Example 1.4. Let $q = 1$ and $\alpha \geq 0$ or $q > 1$ and $\alpha \in \mathbb{R}$. Then there exist constants t_0, c_1 and c_2 such that

$$\Psi(t) = \begin{cases} c_1 t^q, & \text{if } t \leq t_0 \\ t^q \log^\alpha t + c_2, & \text{if } t \geq t_0 \end{cases}$$

is a Young function and satisfies the growth condition of (a), if $q < n$, and that of (b), if $q > n$. If $q = n$, then only condition (c) may hold.

Weakly differentiable functions with gradient in the Orlicz space L^Ψ, where Ψ increases slightly more slowly than the function $t \mapsto t^n$, are important in the theory of mappings of finite distortion, see for example [22], [24] and the references therein.

Notice that, if $q > 1$, the constants above can be chosen so that both Ψ and its conjugate are doubling N-functions.

The above theorem together with Sobolev-type embeddings for $W^{1,\Psi}(\mathbb{R}^n)$ gives Theorem 1.5 below. Note that if Ω is a $W^{1,\Psi}$-extension domain, then the trace operator

$$\mathcal{T} : W^{1,\Psi}(\mathbb{R}^n) \to W^{1,\Psi}(\Omega), \qquad \mathcal{T}u = u|_\Omega, \tag{1.2}$$

is surjective. Hence the following result shows that extension domains satisfy measure density condition provided Ψ is nice enough.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let Ψ be a doubling Young function. Suppose that one of the following conditions holds.*

(a) *There exists an extension operator $\mathcal{E} : W^{1,\Psi}(\Omega) \to W^{1,\Psi}(\mathbb{R}^n)$ such that*

$$\|\mathcal{E}u\|_{W^{1,\Psi}(\mathbb{R}^n;\lambda)} \leq C\|u\|_{W^{1,\Psi}(\Omega;\lambda)} \tag{1.3}$$

for all $\lambda > 0$.

(b) *The function* $\Psi(t)/t^p$ *is either decreasing for some* $p < n$ *or increasing for some* $p > n$*, and trace operator* (1.2) *is surjective.*

Then Ω *satisfies measure density condition* (1.1)*.*

Remark 1.6. In the proof of the above theorem we show that the extension satisfies one of the conditions of Theorem 1.2. In order to show that it satisfies condition (c), we have to assume that (1.3) holds for all $\lambda > 0$ instead of just for $\lambda = 1$. If $\Psi(t) = t^p$, then (1.3) with $\lambda = 1$ trivially implies (1.3) for all $\lambda > 0$. We do not know if this is true for general Ψ.

In the next two theorems, we give several characterizations for extension domains.

Theorem 1.7. *Let* $\Omega \subset \mathbb{R}^n$ *be a domain. Let* Ψ *and its conjugate be doubling* N*-functions. Then the following conditions are equivalent.*

(a) *There exists an extension operator* $\mathcal{E} : W^{1,\Psi}(\Omega) \to W^{1,\Psi}(\mathbb{R}^n)$ *such that* (1.3) *holds.*

(b) *There exists a linear extension operator* $\mathcal{E} : W^{1,\Psi}(\Omega) \to W^{1,\Psi}(\mathbb{R}^n)$ *such that* (1.3) *holds.*

(c) *The domain* Ω *satisfies measure density condition* (1.1)*,* $M^{1,\Psi}(\Omega) = W^{1,\Psi}(\Omega)$ *as sets and*

$$\|u\|_{M^{1,\Psi}(\Omega;\lambda)} \leq C\|u\|_{W^{1,\Psi}(\Omega;\lambda)}$$

for all $\lambda > 0$*.*

Theorem 1.8. *Let* $\Omega \subset \mathbb{R}^n$ *be a domain. Let* Ψ *and its conjugate be doubling* N*-functions. Suppose that the function* $\Psi(t)/t^p$ *is either decreasing for some* $p < n$ *or increasing for some* $p > n$*. Then the following conditions are equivalent.*

(a) *Trace operator* (1.2) *is surjective.*

(b) *There exists a bounded extension operator.*

(c) *There exists a bounded linear extension operator.*

(d) *The domain* Ω *satisfies measure density condition* (1.1) *and* $M^{1,\Psi}(\Omega) = W^{1,\Psi}(\Omega)$ *as sets and the norms are equivalent.*

In a general metric space we cannot speak about weak derivatives; hence there has been a need for characterizations of the classical Sobolev and Orlicz-Sobolev spaces that do not involve derivatives. We use spaces $N^{1,\Psi}(X)$ consisting of $L^{\Psi}(X)$-functions with upper gradients in $L^{\Psi}(X)$ and spaces $M^{1,\Psi}(X)$ defined using pointwise inequalities. The basic properties of the spaces $N^{1,\Psi}(X)$ are studied in [35], and of the spaces $M^{1,\Psi}(X)$ in [2], see also Section 2. We prove the extension results for these spaces and obtain our main theorems from these general results.

The paper is organized as follows. In Section 2, we introduce the notation and the standard assumptions and recall the definitions of Orlicz and Orlicz-Sobolev spaces. In Section 3, we show that measure density condition implies extension property for the space $M^{1,\Psi}$. The reverse result that the existence of an extension or a Poincaré type inequality implies measure density condition is proved in Section 4. In the last section, we provide characterizations for extension domains of Orlicz-Sobolev spaces in the metric setting.

Theorems in Section 1 follow from the more general theorems using the facts that the Euclidean space $\mathbb{R}^n$ with the Lebesgue measure is n-regular (and hence doubling and reverse doubling) and supports a Poincaré inequality and that for a doubling Young function $W^{1,\Psi}(\Omega) = N^{1,\Psi}(\Omega)$ for each domain Ω. Theorems 1.1, 1.2, 1.5, 1.7 and 1.8 follow from theorems 5.1, 4.1, 4.2, 5.2 and 5.3, respectively.

2. Notation and preliminaries

We assume that $X = (X, \mathrm{d}, \mu)$ is a metric measure space equipped with a metric d and a Borel regular outer measure μ such that $0 < \mu(B) < \infty$ for all balls $B = B(x, r) = \{y \in X : \mathrm{d}(y, x) < r\}$. For $0 < t < \infty$, we write $tB = B(x, tr)$. We assume that μ is *doubling*, which means that there is a constant $c_D > 0$, called *the doubling constant of* μ, such that

$$\mu(2B) \le c_D \mu(B) \tag{2.1}$$

for all balls B. The doubling condition implies that there exists a constant $c_0 > 0$ such that, for $s = \log_2 c_D$,

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \ge c_0 \Big(\frac{r}{R}\Big)^s, \tag{2.2}$$

whenever $0 < r \le R < \operatorname{diam} X$ and $y \in B(x, R)$, see [14, Lemma 14.6]. The smallest exponent for which (2.2) holds is denoted by s. In some results, we assume that X is complete. The doubling condition together with completeness implies that the space is proper, that is, closed balls are compact.

We say that the measure μ satisfies *a reverse doubling condition* if there is constant $C > 1$ such that

$$\mu(2B) \ge C\mu(2B) \tag{2.3}$$

for all balls B. This condition holds, for example, if μ is doubling and X is connected or if all annuli are nonempty.

The measure μ is Q-regular if there is a constant $c_R \ge 1$ such that

$$c_R r^Q \le \mu(B(x, r)) \le c_R r^Q$$

for all balls $B(x, r)$.

The Hardy-Littlewood maximal function of a locally integrable function u is

$$\mathcal{M}\, u(x) = \sup_{r>0} \fint_{B(x,r)} |u| \, d\mu,$$

where $u_B = \fint_B u \, d\mu = \mu(B)^{-1} \int_B u \, d\mu$ is the integral average of u over B.

By χ_E, we denote the characteristic function of a set E. In general, C will denote a positive constant whose value is not necessarily the same at each occurrence. By writing $C = C(\tau, \lambda)$, we indicate that the constant depends only on τ and λ. We say that two functions u and v are comparable if there is a constant $C \ge 1$ such that $C^{-1} v(t) \le u(t) \le C v(t)$ for all t (and similarly for measures and norms).

2.1. **Orlicz spaces.**

We will give a brief review of Orlicz spaces. For more details and proofs, see for example [27], [29]. A function $\Psi : [0, \infty) \to [0, \infty]$ is *a Young function* if

$$\Psi(s) = \int_0^s \psi(t) \, dt,$$

where $\psi \colon [0, \infty) \to [0, \infty]$ is an increasing, left continuous function which is neither identically zero nor identically infinite on $(0, \infty)$, and satisfies $\psi(0) = 0$. A Young function Ψ is convex, increasing, left-continuous, $\Psi(0) = 0$, and $\Psi(t) \to \infty$ as $t \to \infty$. A continuous Young function with properties $\Psi(t) = 0$ only if $t = 0$, $\Psi(t)/t \to \infty$ as $t \to \infty$, and $\Psi(t)/t \to 0$ as $t \to 0$ is *an N-function*. Below, Ψ is always a Young function. For a given Ψ, the function $\hat\Psi : [0, \infty) \to [0, \infty]$, $\hat\Psi(s) = \sup\{st - \Psi(t) : t \ge 0\}$, is *the conjugate function* of Ψ.

Convexity and the property $\Psi(0)=0$ imply that

$$\begin{aligned}&\Psi(\alpha t)\le \alpha\Psi(t),\ \text{ if } 0\le\alpha\le 1,\ \text{ and}\\&\Psi(\beta t)\ge \beta\Psi(t),\ \text{ if } \beta\ge 1.\end{aligned}\tag{2.4}$$

A function Ψ is *doubling* (satisfies the Δ_2-condition) if there is a constant $c_\Psi>0$ such that

$$\Psi(2t)\le c_\Psi\Psi(t)$$

for each $t\ge 0$. The smallest such constant is at least 2 by (2.4), and is called *the doubling constant of* Ψ. The doubling condition implies that Ψ is strictly increasing and continuous.

Given Ψ and a measurable set $\Omega\subset X$, *the Orlicz space* $L^\Psi(\Omega)$ consists of measurable functions $u\colon\Omega\to[-\infty,\infty]$ for which

$$\int_\Omega \Psi(\alpha|u|)\,d\mu<\infty$$

for some $\alpha>0$. If Ψ is doubling, then $L^\Psi(\Omega)$ coincides with the set of functions u for which $\int_\Omega\Psi(|u|)\,d\mu$ is finite. The space $L^\Psi(\Omega)$ is a Banach space with *the Luxemburg norm*,

$$\|u\|_{L^\Psi(\Omega)}=\inf\Big\{k>0:\int_\Omega\Psi\big(k^{-1}|u|\big)\,d\mu\le 1\Big\}.$$

The *weak Luxemburg norm* is the number

$$\|u\|_{L^\Psi_w(\Omega)}=\inf\Big\{k>0:\sup_{t>0}\Psi(t)\mu\big(\{x\in\Omega:k^{-1}|u(x)|>t\}\big)\le 1\Big\}.$$

By $\|u\|_{L^\Psi(\Omega;\lambda)}$, we denote the norm with respect to the weighted measure $\lambda\,d\mu$, where $\lambda>0$. Similar notation is used also for the other norms used in the paper. For a measurable set Ω with $0<\mu(\Omega)<\infty$,

$$\|\chi_\Omega\|_{L^\Psi(X)}=\|\chi_\Omega\|_{L^\Psi_w(X)}=\Big(\Psi^{-1}\Big(\tfrac{1}{\mu(\Omega)}\Big)\Big)^{-1}.\tag{2.5}$$

If Ψ is real-valued, then each function of $L^\Psi(X)$ is locally integrable. Using (2.4), it is easy to see that if $\|u\|_{L^\Psi(\Omega)}\le 1$, then

$$\int_\Omega\Psi(|u|)\,d\mu\le\|u\|_{L^\Psi(\Omega)}.\tag{2.6}$$

A function Ψ satisfies ∇_2-condition, if $2C\Psi(t)\le\Psi(Ct)$ for each $t\ge 0$ with a fixed constant $C>1$. For an N-function, the inequality above is equivalent to the doubling condition of the conjugate function. If Ψ is doubling and satisfies the ∇_2-condition, then the maximal function satisfies

$$\int_X\Psi(\mathcal{M}\,u)\,d\mu\le C\int_X\Psi(Cu)\,d\mu,$$

or equivalently,

$$\|\mathcal{M}\,u\|_{L^\Psi(X;\lambda)}\le C\|u\|_{L^\Psi(X;\lambda)},$$

for each $\lambda>0$, see [25, Theorem 1.2.2], [11], [6].

2.2. **Orlicz-Sobolev spaces.** We recall the definitions and some properties of Orlicz-Sobolev spaces in metric measure spaces. The first definition is given using pairs of integrable functions and upper gradients. For proofs, see [35], and for discussion on upper gradients, also for example [12], [20] and [31].

A Borel measurable function $g \geq 0$ is *an upper gradient* of u in an open set Ω if for all rectifiable curves γ joining points x and y in Ω, we have

$$|u(x) - u(y)| \leq \int_\gamma g\, ds. \tag{2.7}$$

whenever both $u(x)$ and $u(y)$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise.

The Sobolev space $N^{1,\Psi}(\Omega)$ consists of functions $u \in L^\Psi(\Omega)$ which have a Ψ-weak upper gradient $g \in L^\Psi(\Omega)$ in Ω. Being a Ψ-weak upper gradient means that inequality (2.7) holds for u and a measurable function $g \geq 0$ except for a family of compact, rectifiable curves in Ω with zero Ψ-modulus. If there is no risk of confusion, Ψ-weak upper gradients are called weak upper gradients. For definition and properties of Ψ-modulus, see [35].

The space $N^{1,\Psi}(\Omega)$ is a Banach space with the norm

$$\|u\|_{N^{1,\Psi}(\Omega)} = \|u\|_{L^\Psi(\Omega)} + \inf \|g\|_{L^\Psi(\Omega)},$$

where the infimum is taken over weak upper gradients of u. Note that if $\Psi(t) = t^p$, $p \geq 1$, we obtain the Sobolev space $N^{1,p}(\Omega)$, defined by Shanmugalingam [31].

If $\Omega \subset \mathbb{R}^n$ is a domain and Ψ is a doubling Young function, then $N^{1,\Psi}(\Omega) = W^{1,\Psi}(\Omega)$ as Banach spaces and the norms are equivalent, see ([35, Theorem 6.19]).

The second definition is given using pointwise inequalities. A measurable function $g \geq 0$ is *a generalized gradient* of a measurable function u, $g \in \mathrm{D}(u)$, if there is a set $E \subset X$ with $\mu(E) = 0$ such that

$$|u(x) - u(y)| \leq \mathrm{d}(x, y)\big(g(x) + g(y)\big) \tag{2.8}$$

for all $x, y \in X \setminus E$. The Orlicz-Sobolev space $M^{1,\Psi}(X)$ consists of functions $u \in L^\Psi(X)$ for which there exists a function $g \in L^\Psi(X) \cap \mathrm{D}(u)$. The space $M^{1,\Psi}(X)$, equipped with the norm

$$\|u\|_{M^{1,\Psi}(X)} = \|u\|_{L^\Psi(X)} + \inf \|g\|_{L^\Psi(X)},$$

where the infimum is taken over all functions $g \in L^\Psi(X) \cap \mathrm{D}(u)$, is a Banach space, see [2, Theorem 3.6]. For a closed set $F \subset X$, $M^{1,\Psi}(F)$ is the Orlicz-Sobolev space on the metric space F where the metric is inherited from X and the measure is the restriction of μ to F. Extension for spaces $N^{1,\Psi}$ and $M^{1,\Psi}$ and for closed sets are defined similarly as for $W^{1,\Psi}$.

2.3. **Poincaré inequalities.** A pair (u, g), where u is locally integrable and $g \geq 0$, satisfies *a Poincaré inequality*, if there exist constants $c_P > 0$ and $\tau \geq 1$ such that

$$\fint_B |u - u_B|\, d\mu \leq c_P r \fint_{\tau B} g\, d\mu \tag{2.9}$$

for each ball $B = B(x, r)$. More generally, if Ψ is a Young function, a pair as above satisfies *a Ψ-Poincaré inequality*, if

$$\fint_B |u - u_B|\, d\mu \leq c_P r \Psi^{-1}\Big(\fint_{\tau B} \Psi(g)\, d\mu\Big) \tag{2.10}$$

for each ball B. The space X *supports a Poincaré inequality* (respectively Ψ*-Poincaré inequality)* if (2.9) ((2.10)) holds for each locally integrable function u and every upper gradient g of u in each ball with fixed constants.

Notice that (2.9) implies (2.10) by the Jensen inequality. If Ψ is doubling, it implies a $(1,p)$-Poincaré inequality for all $p \geq \log_2 c_\Psi$, see [35, Theorem 5.7].

If X supports a Ψ-Poincaré inequality, it is necessarily connected and hence μ satisfies reverse doubling condition (2.3). A complete (and hence proper) space supporting a Ψ-Poincaré inequality is *quasiconvex*, that is, there exists a constant $C \geq 1$ such that every pair x, y of points can be connected by a curve whose length does not exceed $C\,\mathrm{d}(x,y)$. It follows that the length metric

$$d_l(x,y) = \inf\{l(\gamma) : \gamma \text{ connects } x \text{ to } y\}$$

is bi-Lipschitz equivalent to the original metric and the resulting space (X, d_l) is *geodesic*, that is, every pair of points can be connected by a curve whose length equals their distance. Note that doubling condition (2.1), reverse doubling condition (2.3), measure density condition (3.1) and Poincaré inequality (2.9) are invariant under a bi-Lipschitz change of a metric, see [19, Chapter 9], [12, Theorem 3.9]. The same is true for a Ψ-Poincaré inequality provided Ψ is doubling. Also, clearly, $N^{1,\Psi}(X; d_l) = N^{1,\Psi}(X; \mathrm{d})$ and $M^{1,\Psi}(X; d_l) = M^{1,\Psi}(X; \mathrm{d})$ with equivalent norms.

If Ψ and the space X are nice enough, then the definitions of $N^{1,\Psi}(X)$ and $M^{1,\Psi}(X)$ give the same space. If Ψ is a doubling N-function, the maximal operator is bounded in $L^\Psi(X)$ and X supports a Poincaré inequality, then $N^{1,\Psi}(X) = M^{1,\Psi}(X)$ and the norms are comparable, see [36, Theorem 4].

2.4. **Sobolev-type embeddings.** When showing that extension property implies measure density, we need Sobolev-type inequalities for Orlicz-Sobolev spaces. Embedding theorems for $W^{1,\Psi}(\Omega)$ were first proved by Donaldson and Trudinger in [9] and by Adams in [1], and improved by Cianchi in [4, 5]. Corresponding inequalities in the setting of metric measure spaces were recently shown to follow from a Ψ-Poincaré inequality, see [17, 18].

Theorem 2.1 ([18])**.** *Assume that Ψ is a Young function, $B = B(x_0, r)$ is a ball, $\delta > 0$, $\tau \geq 1$, and that a pair $(u/\|g\|_{L^\Psi(\hat{B})}, g/\|g\|_{L^\Psi(\hat{B})})$ satisfies a Ψ-Poincaré inequality in $\hat{B} = (1+\delta)\tau B$.*

1) *If*

$$\int_0^1 \frac{\Psi^{-1}(t)}{t^{1+1/s}}\,dt < \infty \quad \text{and} \quad \int_0^\infty \frac{\Psi^{-1}(t)}{t^{1+1/s}}\,dt = \infty, \tag{2.11}$$

then

$$\|u - u_B\|_{L_w^{\Psi_s}(B)} \leq Cr\mu(B)^{-1/s}\|g\|_{L^\Psi(\hat{B})}, \tag{2.12}$$

where

$$\Psi_s^{-1}(z) = \int_0^z \frac{\Psi^{-1}(t)}{t^{1+1/s}}\,dt. \tag{2.13}$$

2) *If*

$$\int^\infty \frac{\Psi^{-1}(t)}{t^{1+1/s}}\,dt < \infty, \tag{2.14}$$

then, for Lebesgue points $x, y \in B$ of u,

$$|u(x) - u(y)| \leq Cr\mu(B)^{-1/s}\|g\|_{L^\Psi(\hat{B})}\omega_s(\mu(B)^{-1}r^s d(x,y)^{-s}), \tag{2.15}$$

where

$$\omega_s(z) = \int_z^\infty \frac{\Psi^{-1}(t)}{t^{1+1/s}}\,dt. \tag{2.16}$$

Here, $C = C(c_0, s, c_P, \tau, \delta)$.

3. Extension from measure density

In this section, we show that the measure density condition implies the extendability of functions in the class $M^{1,\Psi}$ if μ satisfies a reverse doubling condition or if the maximal operator is bounded.

We say that a measurable set $E \subset X$ satisfies a measure density condition if there is a constant $c_E > 0$ such that

$$\mu(E \cap B(x,r)) \geq c_E \mu(B(x,r)) \tag{3.1}$$

for all $x \in \overline{E}$, and for all $0 < r \leq 1$. Note that by the doubling condition of μ, the measure density condition holds for balls of uniformly bounded radius. The measure density condition says that the set cannot be too thin near the boundary. It also implies that $\mu(\partial E) = 0$, see [15, Lemma 9]. Hence when studying extension for space $M^{1,\Psi}(E)$, we may assume that the set E is closed.

Whitney type covering is usually an essential tool when showing that a set has an extension property. We recall a covering result for doubling metric spaces from [7, Theorem III.1.3], [28, Lemma 2.9].

Lemma 3.1. *Let* $U \subset X$, $U \neq X$ *be an open set. There are balls* $B_i = B(x_i, r_i)$, $i \in \mathbb{N}$, *where* $r_i = \operatorname{dist}(x_i, X \setminus U)/10$, *such that*

1. *the balls* $\frac{1}{5}B_i$ *are pairwise disjoint,*
2. $U = \cup_i B_i$,
3. $5B_i \subset U$,
4. *if* $x \in 5B_i$, *then* $5r_i \leq \operatorname{dist}(x, X \setminus U) \leq 15r_i$,
5. *there is* $x_i^* \in X \setminus U$ *such that* $\mathrm{d}(x_i, x_i^*) < 15r_i$, *and*
6. $\sum_{i=1}^\infty \chi_{5B_i}(x) \leq M$ *for all* $x \in U$.

For a Whitney covering, we can associate a Lipschitz partition of unity, see [28, Lemma 2.1]. There is a sequence $(\varphi_i)_{i\in\mathbb{N}}$ of non-negative functions such that

1. $\operatorname{supp} \varphi_i \subset 2B_i$ for all i,
2. $\varphi_i(x) \geq M^{-1}$ for all $x \in B_i$ and for all i,
3. there is a constant K such that each φ_i is Kr_i^{-1} Lipschitz,
4. $\sum_{i=1}^\infty \varphi_i(x) = \chi_U(x)$ for all x.

Below, we want to extend functions defined in a closed set F to functions defined in X. We may assume that $F \neq X$ since otherwise there is nothing to do. For that, let $u \in M^{1,\Psi}(F)$ and let $g \in D(u) \cap L^\Psi(F)$ such that

$$\|g\|_{L^\Psi(F)} \leq 2\|u\|_{M^{1,\Psi}(F)}.$$

Let $\{B_i\}_{i\in\mathbb{N}}$ be a Whitney covering of $X \setminus F$ and let $(\varphi_i)_{i\in\mathbb{N}}$ be the corresponding partition of unity. For each i, let x_i^* be as in Lemma 3.1 and define

$$B_i^* = B(x_i^*, r_i).$$

If the maximal operator is bounded in $L^\Psi(X)$, then the extension property follows quite easily.

Theorem 3.2. *Let $F \subset X$ be a closed set that satisfies measure density condition* (3.1) *and let Ψ be a doubling Young function for which the maximal operator is bounded in $L^\Psi(X)$. Then there is a bounded linear extension operator of $M^{1,\Psi}(F)$ into $M^{1,\Psi}(X)$.*

Idea of the proof. We follow the proof of the first part of [15, Theorem 6] which shows that there is a bounded linear extension operator from $M^{1,p}(F)$ to $M^{1,p}(X)$ for $1 < p < \infty$ and use the boundedness of the maximal operator in L^Ψ instead of L^p. The function is first extended to a neighbourhood V of F using discrete convolution and small Whitney balls of radius at most 1, by setting

$$\tilde{E}u(x) = \begin{cases} u(x), & \text{if } x \in F, \\ \sum_{i \in J} \varphi_i(x) u_{B_i^* \cap F}, & \text{if } x \in X \setminus F, \end{cases}$$

where $J = \{i : r_i \leq 1\}$. Then the extension to X is done using a cut-off function and the Leibniz rule [13, Lemma 5.20]. □

The proof of case $p = 1$ for $M^{1,p}$-spaces in [15, Theorem 6] is much more difficult because the boundedness of the maximal operator cannot be used.

To obtain the extension property for more general Ψ, we assume that μ satisfies the reverse doubling condition. We will define the extension as in Shvartsman [33]. The sets used in the discrete convolution are obtained using the reverse doubling condition. For each Whitney ball B_i, there is Borel set $H_i \subset F$ whose measure is comparable with the measure of B_i. The sets H_i are of bounded overlap; this is important in showing that the extension and its gradient are integrable. Compared to sets $B_i^* \cap F$ used in the proof of Theorem 3.2, the sets H_i are defined by throwing away small balls that intersect the set $\varepsilon B_i^* \cap F$ for ε small enough.

Theorem 3.3. *[*[33]*, Theorem 2.6] Assume that μ satisfies reverse doubling condition* (2.3)*. Let $F \subset X$ be a closed set and let $\{B_i\}_{i \in \mathbb{N}}$ be a Whitney covering of $X \setminus F$. Then there are Borel sets $\{H_i\}_{i \in \mathbb{N}}$, and constants $c_1, c_2, c_3 > 0$ such that*

(1) *$H_i \subset (c_1 B_i \cap F)$,*
(2) *$\mu(H_i) \geq c_2 \mu(B_i)$ if $r_i \leq 1$,*
(3) *$\sum_{i=1}^\infty \chi_{H_i} \leq c_3$.*

Idea of the proof. For $0 < \varepsilon \leq 1$, define

$$H_i = (\varepsilon B_i^* \cap F) \setminus \bigcup_{j \in I_i} \varepsilon B_j^*,$$

where $I_i = \{j : \varepsilon B_j^* \cap \varepsilon B_i^* \neq \emptyset,\ r_j \leq \varepsilon r_i\}$ if $r_i \leq 1$. If $r_i > 1$, then $H_i = \emptyset$. The condition (1) follows from the properties of the Whitney covering because it is easy to see that $H_i \subset \varepsilon B_i^* \subset 16 B_i$. For (2), show first that $B_j \subset 18 \varepsilon B_i^*$ for each $j \in I_i$. The claim follows using the bounded overlap of balls B_i, reverse doubling and the measure density condition and choosing ε small enough. For (3), note that if H_i and H_j intersect, then $i \notin I_j$ and $j \notin I_i$ and the measures of B_i and B_j are comparable. The claim follows by using the bounded overlap of balls B_i. □

Theorem 3.4. *Assume that μ satisfies reverse doubling condition* (2.3) *and that Ψ is doubling. If a closed set $F \subset X$ satisfies measure density condition* (3.1)*, then there is a linear extension operator $\mathcal{E} : M^{1,\Psi}(F) \to M^{1,\Psi}(X)$ such that*

$$\|\mathcal{E}u\|_{M^{1,\Psi}(X;\lambda)} \leq C \|u\|_{M^{1,\Psi}(F;\lambda)} \tag{3.2}$$

for all $u \in M^{1,\Psi}(F)$ and for all $\lambda > 0$.

The first part of proof follows the proof of [33, Theorem 1.3]. The extension is defined similarly as well as the generalized gradient of the extension. We prove only the norm estimates for the extension and for the gradient.

Proof. Define the extension of u by setting

$$\mathcal{E}u(x) = \begin{cases} u(x), & \text{if } x \in F, \\ \sum_{i=1}^{\infty} \varphi_i(x) u_{H_i}, & \text{if } x \in X \setminus F. \end{cases}$$

Using standard techniques and the properties of Whitney covering, one can show that the function

$$\rho(x) = \begin{cases} g(x), & \text{if } x \in F, \\ \sum_{i=1}^{\infty} (g_{H_i} + |u_{H_i}|)\chi_{2B_i}(x), & \text{if } x \in X \setminus F \end{cases}$$

belongs to $D(\mathcal{E}u)$. Here the last two properties of Theorem 3.3 are not needed.

We prove only the unweighted case $\lambda = 1$. The other cases are similar. Next we show that $\|\mathcal{E}u\|_{L^\Psi(X)} \le C\|u\|_{L^\Psi(F)}$. It suffices to show this when $\|u\|_{L^\Psi(F)} = 1$. The general case then follows by using function $v = u/\|u\|_{L^\Psi(F)}$. Since $\sum_{i=1}^{\infty} \varphi_i(x) = 1$ and there are only bounded number of non-zero terms in the sum $\sum_{i=1}^{\infty} \varphi_i(x) u_{H_i}$ for each $x \in X \setminus F$, the Jensen inequality implies that for all $x \in X \setminus F$,

$$\Psi(|\mathcal{E}u(x)|) \le \sum_{i=1}^{\infty} \varphi_i(x) \fint_{H_i} \Psi(|u|)\, d\mu = \mathcal{E}(\Psi(|u|))(x).$$

Using the fact that $\varphi_i = 0$ outside $2B_i$, the doubling condition, comparability of the measures of H_i and B_i and the bounded overlap of the sets H_i, we obtain

$$\begin{aligned} \int_{X \setminus F} \Psi(|\mathcal{E}u|)\, d\mu &\le \sum_{i=1}^{\infty} \mu(2B_i) \fint_{H_i} \Psi(|u|)\, d\mu \le C \sum_{i=1}^{\infty} \int_{H_i} \Psi(|u|)\, d\mu \\ &= C \int_F \Psi(|u|) \Big(\sum_{i=1}^{\infty} \chi_{H_i} \Big)\, d\mu \le k \int_F \Psi(|u|)\, d\mu, \end{aligned}$$

where $k \ge 1$. Hence, using the assumption $\|u\|_{L^\Psi(F)} = 1$ and (2.6), we have that

$$\int_{X \setminus F} \Psi(|\mathcal{E}u|)\, d\mu \le k.$$

Since $\mathcal{E}u|_F = u$, (2.4) together with the definition of Luxemburg norm shows that

$$\|\mathcal{E}u\|_{L^\Psi(X)} \le 2k = 2k\|u\|_{L^\Psi(F)}.$$

To show that $\|\rho\|_{L^\Psi(X)} \le C\|g\|_{L^\Psi(F)}$, it suffices to show that for each non-negative function $v \in L^\Psi(F)$ with $\|v\|_{L^\Psi(F)} = 1$,

$$\|V\|_{L^\Psi(X)} \le C\|v\|_{L^\Psi(F)},$$

where $V(x) = \sum_{i=1}^{\infty} v_{H_i} \chi_{2B_i}(x)$ for $x \in X \setminus F$ and $V|_F = v$. For that, let $x \in X \setminus F$ and $a = \sum_{i=1}^{\infty} \chi_{2B_i}(x)$. The doubling condition of Ψ, the bounded overlap of the balls $2B_i$ and the Jensen inequality imply that

$$\Psi(V(x)) = \Psi\Big(\frac{a \sum_{i=1}^{\infty} v_{H_i} \chi_{2B_i}(x)}{a} \Big) \le C a^{-1} \sum_{i=1}^{\infty} \chi_{2B_i}(x) \fint_{H_i} \Psi(v)\, d\mu.$$

Hence, using the doubling condition of μ, the comparability of the measures of H_i and B_i and the bounded overlap of the sets H_i, we obtain

$$\int_{X\setminus F} \Psi(V)\,d\mu \le C\sum_{i=1}^{\infty}\int_{H_i}\Psi(v)\,d\mu \le C\int_F \Psi(v)\,d\mu.$$

The norm estimate follows similarly as for $\mathcal{E}u$ above. Using the definition of ρ, we have that

$$\|\rho\|_{L^\Psi(X)} \le C\|g\|_{L^\Psi(F)}.$$

Hence there is a bounded linear extension of $M^{1,\Psi}(F)$ to $M^{1,\Psi}(X)$. □

4. Measure density from extension

In this section, we show that the existence of an extension or a Poincaré type inequality implies measure density condition. The proofs are modifications of the proof of [15, Proposition 13].

Recall that a space X is a geodesic space if every two points $x, y \in X$ can be joined by a curve whose length equals $d(x,y)$. As discussed above, geodesity follows from completeness and Poincaré inequality by changing the metric. Geodesity is needed to show that balls satisfy measure density condition, see [15, Proof of Proposition 13]. Thus $\mu(\partial B) = 0$ for each ball B.

Below, functions Ψ_Q and ω_Q are as in Theorem 2.1.

Theorem 4.1. *Let X be Q-regular, geodesic metric measure space. Suppose that one of the following conditions holds for a domain $\Omega \subset X$.*

(a) *There exists $p < Q$ such that $\Psi(t)/t^p$ is decreasing and $N^{1,\Psi}(\Omega) \subset L^{\Psi_Q}(\Omega)$.*

(b) *Ψ is doubling, there exists $p > Q$ such that $\Psi(t)/t^p$ is increasing and every $u \in N^{1,\Psi}(\Omega)$ satisfies*

$$|u(x) - u(y)| \le C\omega_Q(C^{-1}d(x,y)^{-Q})\|u\|_{N^{1,\Psi}(\Omega)} \tag{4.1}$$

for a.e. $x, y \in \Omega$ such that $\mathrm{d}(x,y) < 1$.

(c) *There is a function Φ satisfying*

$$\Phi^{-1}(t) \le C\Psi^{-1}(t)t^{-\varepsilon} \tag{4.2}$$

for some $\varepsilon > 0$ and all $t \ge 1$, and a constant $C > 0$ such that

$$\inf_{c\in\mathbb{R}} \|u - c\|_{L^\Phi_w(B\cap\Omega;\,|B|^{-1})} \le Cr\|u\|_{W^{1,\Psi}(\Omega;\,|B|^{-1})} \tag{4.3}$$

for all balls $B = B(x,r)$ with $x \in \overline{\Omega}$ and $0 < r \le 1$.

Then Ω satisfies measure density condition (3.1).

Proof. Let $B = B(x,r)$ with $x \in \overline{\Omega}$ and $0 < r \le 1$.
(a): As $\Psi(t)/t^p$ is decreasing for some $p < Q$, it is easy to see that (2.11) holds. Since boundaries of balls have zero measure, we can choose radii $0 < \tilde{\tilde{r}} < \tilde{r} < r$ such that

$$\mu(B(x,\tilde{\tilde{r}})\cap\Omega) = \tfrac{1}{2}\mu(B(x,\tilde{r})\cap\Omega) = \tfrac{1}{4}\mu(B(x,r)\cap\Omega)$$

and define

$$A(\tilde{r},\tilde{\tilde{r}}) = B(x,\tilde{r}) \setminus B(x,\tilde{\tilde{r}}), \qquad A(r,\tilde{r}) = B(x,r)\setminus B(x,\tilde{r}).$$

As in [15, Proposition 13], it suffices to show that

$$\tilde{r} - \tilde{\tilde{r}} \le C\mu(B(x,r)\cap\Omega)^{1/Q} \tag{4.4}$$

because then, using estimate (4.4) for radii $r_0 = r$, $r_{j+1} = \tilde{r}_j$, we obtain

$$\tilde{r}^Q \le C\mu(B(x,r)\cap\Omega).$$

Since by [15, Lemma 14] we may assume that $\tilde{r} \ge 1/10r$, the measure density condition follows from the Q-regularity.

To show that (4.4) holds, we define $u\colon \Omega \to \mathbb{R}$ by setting

$$u(y) = \begin{cases} 1, & \text{if } y \in B(x,\tilde{\tilde{r}})\cap\Omega, \\ \frac{\tilde{r}-\mathrm{d}(x,y)}{\tilde{r}-\tilde{\tilde{r}}}, & \text{if } y \in A(\tilde{r},\tilde{\tilde{r}})\cap\Omega, \\ 0, & \text{if } y \in \Omega\setminus B(x,\tilde{r}). \end{cases}$$

The function u is $1/(\tilde{r}-\tilde{\tilde{r}})$-Lipschitz and hence

$$g = \frac{1}{\tilde{r}-\tilde{\tilde{r}}}\,\chi_{\overline{A(\tilde{r},\tilde{\tilde{r}})\cap\Omega}}$$

is an upper gradient of u. Using (2.5) and the fact that $0 < \tilde{r}-\tilde{\tilde{r}} < 1$, we obtain

$$\begin{aligned} \|u\|_{N^{1,\Psi}(\Omega)} &\le \|u\|_{L^\Psi(\Omega)} + \|g\|_{L^\Psi(\Omega)} \\ &\le \Psi^{-1}(\mu(B\cap\Omega)^{-1})^{-1} + (\tilde{r}-\tilde{\tilde{r}})^{-1}\Psi^{-1}(\mu(B\cap\Omega)^{-1})^{-1} \\ &\le C(\tilde{r}-\tilde{\tilde{r}})^{-1}\Psi^{-1}(\mu(B\cap\Omega)^{-1})^{-1}. \end{aligned} \tag{4.5}$$

Since $N^{1,\Psi}(\Omega) \subset L^{\Psi_Q}(\Omega)$, it follows from the closed graph theorem that there is constant $C > 0$ such that $\|v\|_{L^{\Psi_Q}(\Omega)} \le C\|v\|_{N^{1,\Psi}(\Omega)}$ for all $v \in N^{1,\Psi}(\Omega)$. Thus, using (2.5), we have

$$C\|u\|_{N^{1,\Psi}(\Omega)} \ge \|u\|_{L^{\Psi_Q}(\Omega)} \ge \|\chi_{B(x,\tilde{\tilde{r}})\cap\Omega}\|_{L^{\Psi_Q}(\Omega)} \ge \Psi_Q^{-1}(4\mu(B\cap\Omega)^{-1})^{-1}. \tag{4.6}$$

By [18, Theorem 1.5], $\Psi_Q^{-1}(t)$ is comparable to $\Psi^{-1}(t)t^{-1/Q}$ for $t \ge 1$. Hence (4.4) follows using estimates (4.5) and (4.6).

(b): As there is $p > Q$ such that $\Psi(t)/t^p$ is increasing, it is easy to see that (2.14) holds. Moreover, $\omega_Q(t)$ is comparable to $\Psi^{-1}(t)t^{-1/Q}$ by [18, Theorem 1.5].

We may assume that $\Omega\setminus B \ne \emptyset$. We define $u\colon \Omega \to \mathbb{R}$ by setting

$$u(y) = \max\Big\{1 - \frac{\mathrm{d}(x,y)}{r}, 0\Big\}.$$

The function u is $1/r$-Lipschitz and

$$g(y) = \frac{1}{r}\,\chi_{\overline{B(x,r)\cap\Omega}}$$

is an upper gradient of u. Since $u(x) = 1$ and $u(y) = 0$ for some $y \in (\Omega\setminus B)\cap 2B$, using (4.1), the fact that ω_Q is decreasing and (2.5), we obtain

$$\begin{aligned} 1 \le |u(x) - u(y)| &\le C\|u\|_{N^{1,\Psi}(\Omega)}\omega_Q(C^{-1}\,\mathrm{d}(x,y)^{-Q}) \\ &\le C\|u\|_{N^{1,\Psi}(\Omega)}\omega_Q(Cr^{-Q}) \\ &\le Cr^{-1}\Psi^{-1}(\mu(B\cap\Omega)^{-1})^{-1}\omega_Q(Cr^{-Q}) \\ &\le C\Psi^{-1}(\mu(B\cap\Omega)^{-1})^{-1}\Psi^{-1}(Cr^{-Q}). \end{aligned}$$

The doubling property of Ψ then implies that

$$\mu(B\cap\Omega)^{-1} \le \Psi(C\Psi^{-1}(C^{-1}r^{-Q})) \le Cr^{-Q},$$

from which (3.1) follows easily by the Q-regularity.

(c): Let $0 < \tilde{\tilde{r}} < \tilde{r} < r \le 1$, u and g be as in the case (a). As in (4.5), we obtain

$$\begin{aligned}\|u\|_{N^{1,\Psi}(\Omega;\,\mu(B)^{-1})} &\le \|u\|_{L^\Psi(\Omega;\,\mu(B)^{-1})} + \|g\|_{L^\Psi(\Omega;\,\mu(B)^{-1})}\\ &\le C(\tilde{r}-\tilde{\tilde{r}})^{-1}\Psi^{-1}\Big(\frac{\mu(B)}{\mu(B\cap\Omega)}\Big)^{-1}.\end{aligned}$$

Let $c \in \mathbb{R}$. Since $u = 1$ in $B(x,\tilde{\tilde{r}}) \cap \Omega$ and $u = 0$ in $A(r,\tilde{r}) \cap \Omega$, we have that $|u - c| \ge 1/2$ in $B(x,\tilde{\tilde{r}}) \cap \Omega$ or $|u - c| \ge 1/2$ in $A(r,\tilde{r}) \cap \Omega$. Thus, using (2.5), we obtain

$$\begin{aligned}&\|u-c\|_{L^\Phi_w(B\cap\Omega;\,\mu(B)^{-1})}\\ &\ge \tfrac12 \min\Big\{\|\chi_{B(x,\tilde{\tilde{r}})\cap\Omega}\|_{L^\Phi_w(B\cap\Omega;\,\mu(B)^{-1})}, \|\chi_{A(r,\tilde{r})\cap\Omega}\|_{L^\Phi_w(B\cap\Omega;\,\mu(B)^{-1})}\Big\}\\ &\ge \tfrac12 \Phi^{-1}\Big(4\frac{\mu(B)}{\mu(B\cap\Omega)}\Big)^{-1}.\end{aligned}$$

These estimates, together with (4.3), (4.2), and the Q-regularity yield

$$\begin{aligned}\tilde{r} - \tilde{\tilde{r}} &\le Cr\Phi^{-1}\Big(4\frac{\mu(B)}{\mu(B\cap\Omega)}\Big)\Psi^{-1}\Big(\frac{\mu(B)}{\mu(B\cap\Omega)}\Big)^{-1}\\ &\le Cr\Big(\frac{\mu(B)}{\mu(B\cap\Omega)}\Big)^{-\varepsilon} \le Cr^{1-\varepsilon Q}\mu(B\cap\Omega)^\varepsilon.\end{aligned} \tag{4.7}$$

Let $r_0 = r$ and $r_i = \tilde{r}_{i-1}$ for $i > 0$. We may assume that $\varepsilon \le 1/Q$ (because if (4.2) holds for $\varepsilon_0 > 0$, then it holds for all exponents $0 < \varepsilon \le \varepsilon_0$). Now (4.7) applied for $(r_{i+1} - r_{i+2})$ gives

$$\begin{aligned}\tilde{r} = r_1 &= \sum_{i=0}^\infty (r_{i+1} - r_{i+2}) \le C\sum_{i=0}^\infty r_i^{1-\varepsilon Q}\mu(B(x,r_i)\cap\Omega)^\varepsilon\\ &\le Cr^{1-\varepsilon Q}\mu(B\cap\Omega)^\varepsilon \sum_{i=0}^\infty (2^{-i})^\varepsilon \le Cr^{1-\varepsilon Q}\mu(B\cap\Omega)^\varepsilon.\end{aligned} \tag{4.8}$$

Now we use the assumption $\tilde{r} \ge 1/10r$, multiply the inequality (4.8) by $r^{\varepsilon Q-1}$ and raise it to the power $1/(\varepsilon Q)$ and we obtain

$$r \le C\mu(B\cap\Omega)^{1/Q}.$$

The measure density condition follows from the Q-regularity. □

Theorem 4.2. *Let X be a Q-regular complete metric measure space that supports a Ψ-Poincaré inequality, let $\Omega \subset X$ be a domain and let Ψ be doubling. Suppose that one the following conditions holds.*

(a) *There exists an extension operator $\mathcal{E} : N^{1,\Psi}(\Omega) \to N^{1,\Psi}(X)$ such that*

$$\|\mathcal{E}u\|_{N^{1,\Psi}(X;\lambda)} \le C\|u\|_{N^{1,\Psi}(\Omega;\lambda)} \tag{4.9}$$

for all $\lambda > 0$.

(b) *The function $\Psi(t)/t^p$ is either decreasing for some $p < Q$ or increasing for some $p > Q$, and the trace operator*

$$\mathcal{T} : N^{1,\Psi}(X) \to N^{1,\Psi}(\Omega), \qquad \mathcal{T}u = u|_\Omega, \tag{4.10}$$

is surjective.

Then Ω satisfies measure density condition (3.1).

Note again that if Ω is an $N^{1,\Psi}$-extension domain, then the trace operator is surjective and hence extension domains satisfy the measure density condition.

Proof. Since X is complete and supports a Ψ-Poincaré inequality, we may assume that X is geodesic.

(a) Case 1: Suppose that

$$\int^{\infty} \frac{\Psi^{-1}(t)}{t^{1+1/Q}}\, dt = \infty. \tag{4.11}$$

Then we can modify Ψ near zero so that the modified function $\tilde{\Psi}$ satisfies (2.11) and

$$\|v\|_{N^{1,\tilde{\Psi}}(\hat{B};\, \mu(B)^{-1})} \le C\|v\|_{N^{1,\Psi}(\hat{B};\, \mu(B)^{-1})}$$

for all balls B. Here $\hat{B}$ is as in Theorem 2.1. Note that such a modification can be done because the measure of B is 1 and the measure of $\hat{B}$ is comparable to a constant by the doubling condition of μ.

Let $u \in N^{1,\Psi}(\Omega)$ and let $B = B(x,r)$ with $x \in \overline{\Omega}$ and $0 < r \le 1$. By assumption, there exists an extension v of u such that

$$\|v\|_{N^{1,\Psi}(X;\, \mu(B)^{-1})} \le C\|u\|_{N^{1,\Psi}(\Omega;\, \mu(B)^{-1})}.$$

The weighted measure $\mu(B)^{-1}\mu$ is doubling with the same doubling constant as μ. Using (2.12) for the measure $\mu(B)^{-1}\mu$, we obtain

$$\begin{aligned}\|v - v_B\|_{L_w^{\tilde{\Psi}_Q}(B;\, \mu(B)^{-1})} &\le Cr\|v\|_{N^{1,\tilde{\Psi}}(\hat{B};\, \mu(B)^{-1})} \le Cr\|v\|_{N^{1,\Psi}(\hat{B};\, \mu(B)^{-1})} \\ &\le Cr\|v\|_{N^{1,\Psi}(X;\, \mu(B)^{-1})} \le Cr\|u\|_{N^{1,\Psi}(\Omega;\, \mu(B)^{-1})},\end{aligned}$$

which implies (4.3) with $\Phi = \tilde{\Psi}_Q$. By Theorem 4.1, it suffices to prove the following lemma.

Lemma 4.3. *The function $\Phi = \tilde{\Psi}_Q$ satisfies* (4.2)*.*

Proof. It suffices to show that (4.2) holds for large t. As

$$\Psi_Q^{-1}(t) = \int_0^t \frac{\Psi^{-1}(s)}{s^{1+1/Q}}\, ds,$$

(4.11) implies that $\tilde{\Psi}_Q^{-1}(t) \to \infty$ as $t \to \infty$. Thus

$$\tilde{\Psi}_Q^{-1}(t) \le 2\int_1^t \frac{\Psi^{-1}(s)}{s^{1+1/Q}}\, ds$$

for large t. By [35, Lemma 2.7], the doubling property of Ψ implies that if $p \ge \log_2 c_\Psi$ and $s \le t$, then

$$\frac{\Psi^{-1}(s)}{s^{1/p}} \le C\frac{\Psi^{-1}(t)}{t^{1/p}}.$$

Hence for $p > \max\{Q, \log_2 c_\Psi\}$, we have that

$$\begin{aligned}\int_1^t \frac{\Psi^{-1}(s)}{s^{1+1/Q}}\, ds &\le C\Psi^{-1}(t)t^{-1/p}\int_1^t s^{1/p-1/Q-1}\, ds \\ &\le C(1/Q - 1/p)\Psi^{-1}(t)t^{-1/p}.\end{aligned}$$

The claim follows by setting $\varepsilon = 1/p$. ∎

CASE 2: Suppose that (2.14) holds. Let $B = B(x,r)$ with $x \in \overline{\Omega}$ and $0 < r \leq 1$. We may assume that $\Omega \setminus B \neq \emptyset$. As in the proof of Theorem 4.1, we use the function $u\colon \Omega \to \mathbb{R}$,

$$u(y) = \max\Big\{1 - \frac{\mathrm{d}(x,y)}{r}, 0\Big\}.$$

The function u has an upper gradient

$$g(y) = \tfrac{1}{r}\chi_{\overline{B}(x,r)\cap\Omega}$$

and $v(x) = u(x) = 1$ and $v(y) = u(y) = 0$ for some $y \in (\Omega \setminus B) \cap 2B$, where $v \in N^{1,\Psi}(X)$ is the extension of u. We will apply (2.15) to v and the measure

$$\nu = \mu(B)^{-1}\mu.$$

Since $\nu(B) = 1$, ω_Q is decreasing and $r \leq 1$, using the Q-regularity and (2.5), we obtain

$$\begin{aligned}
1 \leq |v(x) - v(y)| &\leq Cr\,\nu(B)^{-1}\|v\|_{N^{1,\Psi}(X;\,\nu)}\,\omega_Q\big(\nu(B)^{-1}r^Q\,\mathrm{d}(x,y)^{-Q}\big)\\
&\leq Cr\|v\|_{N^{1,\Psi}(X;\,\nu)}\,\omega_Q\big(r^Q\,\mathrm{d}(x,y)^{-Q}\big)\\
&\leq Cr\|u\|_{N^{1,\Psi}(\Omega;\,\nu)}\,\omega_Q\big(2^{-Q}\big)\\
&\leq C\Psi^{-1}(\nu(B(x,r)\cap\Omega)^{-1})^{-1}\\
&\leq C\Psi^{-1}\Big(\frac{\mu(B(x,r))}{\mu(B(x,r)\cap\Omega)}\Big)^{-1},
\end{aligned}$$

which implies the claim.

(b) The surjectivity of the trace operator implies that the space $N^{1,\Psi}(\Omega)$ is isomorphic to $N^{1,\Psi}(X)/\ker\mathcal{T}$. Hence there is $C > 0$ such that for every $u \in N^{1,\Psi}(\Omega)$ there is $v \in N^{1,\Psi}(X)$ such that $v|_\Omega = u$ and

$$\|v\|_{N^{1,\Psi}(X)} \leq C\|u\|_{N^{1,\Psi}(\Omega)}.$$

This means that there is a bounded but not necessarily linear extension operator. Let $u \in N^{1,\Psi}(\Omega)$ and let v be such an extension.

CASE 1: Suppose that $\Psi(t)/t^p$ is decreasing for some $p < Q$. Then (2.11) holds. Let $x \in \Omega$. Choose the radii $0 < \tilde{\tilde{r}} < \tilde{r} < r \leq 1$ and define the function u as in the part (a) of the proof of Theorem 4.1. Inequality (2.12), the Q-regularity and estimate (4.5) imply that

$$\begin{aligned}
\|v - v_B\|_{L_w^{\Psi_Q}(B)} &\leq Cr\mu(B)^{-1/Q}\|v\|_{N^{1,\Psi}(X)}\\
&\leq C\|v\|_{N^{1,\Psi}(\Omega)} \leq C(\tilde{r} - \tilde{\tilde{r}})^{-1}\Psi^{-1}(\mu(B\cap\Omega)^{-1})^{-1}.
\end{aligned}$$

Since $v = u = 1$ in $B(x,\tilde{\tilde{r}}) \cap \Omega$ and $v = u = 0$ in $A(r,\tilde{r}) \cap \Omega$, we have that $|v - v_B| \geq 1/2$ in $B(x,\tilde{\tilde{r}}) \cap \Omega$ or $|v - v_B| \geq 1/2$ in $A(r,\tilde{r}) \cap \Omega$. Thus

$$\begin{aligned}
\|v - v_B\|_{L_w^{\Psi_Q}(B)} &\geq \tfrac{1}{2}\min\Big\{\|\chi_{B(x,\tilde{\tilde{r}})\cap\Omega}\|_{L_w^{\Psi_Q}(B)}, \|\chi_{A(r,\tilde{r})\cap\Omega}\|_{L_w^{\Psi_Q}(B)}\Big\}\\
&\geq \tfrac{1}{2}\Psi_Q^{-1}(4\mu(B\cap\Omega)^{-1})^{-1}.
\end{aligned}$$

The rest of the proof follows the proof of case (a) of Theorem 4.1.

CASE 2: Suppose that $\Psi(t)/t^p$ is increasing for some $p > Q$. Then the condition (2.14) holds. By Theorem 2.1, v satisfies (2.15), which, by the Q-regularity, implies (4.1) for u. Hence the claim follows from Theorem 4.1. □

Now we formulate the counterparts of above two theorems for the space $M^{1,\Psi}$. The test functions u used in the proof of Theorem 4.1 clearly belong to $M^{1,\Psi}(\Omega)$ and the functions g used as upper gradients of u belong to $D(u)$. Therefore, we have the following.

Theorem 4.4. *Theorem 4.1 remains valid, if we replace $N^{1,\Psi}$ by $M^{1,\Psi}$.*

Note that, by integrating (2.8) twice, we see that every pair $u \in M^{1,\Psi}(X)$ and $g \in D(u)$ satisfies the Poincaré inequality and hence the Ψ-Poincaré inequality for any Young function Ψ. Thus, the proof of Theorem 4.2 yields the following.

Theorem 4.5. *Let X a Q-regular geodesic metric measure space, let $\Omega \subset X$ be a domain and let Ψ be doubling. Suppose that one the following conditions holds.*

(a) *There exists an extension operator $\mathcal{E} : M^{1,\Psi}(\Omega) \to M^{1,\Psi}(X)$ such that*

$$\|\mathcal{E}u\|_{M^{1,\Psi}(X;\lambda)} \le C\|u\|_{M^{1,\Psi}(\Omega;\lambda)}$$

for all $\lambda > 0$.

(b) *The function $\Psi(t)/t^p$ is either decreasing for some $p < Q$ or increasing for some $p > Q$, and the trace operator*

$$\mathcal{T} : M^{1,\Psi}(X) \to M^{1,\Psi}(\Omega), \qquad \mathcal{T}u = u|_\Omega \tag{4.12}$$

is surjective.

Then Ω satisfies measure density condition (3.1).

5. Characterizations of extension domains

In this section, we give characterizations of extension domains for Orlicz-Sobolev spaces in the metric setting. Notice that our results hold in Heisenberg and Carnot groups since they are Ahlfors Q-regular spaces for a suitable Q and they support a Poincaré inequality.

The first theorem of this section is a general version of Theorem 1.1. It characterizes the trace space of $N^{1,\Psi}(X)$.

Theorem 5.1. *Assume that X supports a Poincaré inequality and that Ψ and its conjugate are doubling N-functions. If $E \subset X$ is a measurable set that satisfies measure density condition* (3.1)*, then*

$$N^{1,\Psi}(X)|_E = M^{1,\Psi}(E)$$

as sets and the norms are equivalent. Moreover, there is a bounded linear extension operator

$$\mathcal{E} \colon M^{1,\Psi}(E) \to M^{1,\Psi}(X) = N^{1,\Psi}(X).$$

Note that μ satisfies reverse doubling condition (2.3) because the validity of a Poincaré inequality implies that X is connected.

Proof. The assumptions on Ψ and X imply that if $u \in N^{1,\Psi}(X)$, then $u \in M^{1,\Psi}(X)$ and $g|_E$ is a generalized gradient of $u|_E$ whenever g is a generalized gradient of u, see [36, Theorem 4]. Hence $N^{1,\Psi}(X)|_E \subset M^{1,\Psi}(E)$ and

$$\|u|_E\|_{M^{1,\Psi}(E)} \le \|u\|_{M^{1,\Psi}(X)} \le C\|u\|_{N^{1,\Psi}(X)}.$$

Since E satisfies the measure density condition, Theorem 3.4 implies that there is a bounded linear extension operator from $M^{1,\Psi}(E) = M^{1,\Psi}(\overline{E})$ to $M^{1,\Psi}(X) =$

$N^{1,\Psi}(X)$. Hence each function of $M^{1,\Psi}(E)$ is a restriction of a function from $N^{1,\Psi}(X)$ with the norm inequality. The claim follows. □

Next two results are general versions of Theorems 1.7 and 1.8.

Theorem 5.2. *Let X be a Q-regular complete metric measure space that supports a Poincaré inequality and let $\Omega \subset X$ be a domain. Let Ψ and its conjugate be doubling N-functions. Then the following conditions are equivalent.*

(a) *There exists an extension operator $\mathcal{E} : N^{1,\Psi}(\Omega) \to N^{1,\Psi}(X)$ such that* (4.9) *holds.*
(b) *There exists a linear extension operator $\mathcal{E} : N^{1,\Psi}(\Omega) \to N^{1,\Psi}(X)$ such that* (4.9) *holds.*
(c) *The domain Ω satisfies measure density condition* (3.1)*, $M^{1,\Psi}(\Omega) = N^{1,\Psi}(\Omega)$ as sets and*
$$\|u\|_{M^{1,\Psi}(\Omega;\lambda)} \leq C\|u\|_{N^{1,\Psi}(\Omega;\lambda)} \tag{5.1}$$
for all $\lambda > 0$.

Proof. (a) $\Longrightarrow$ (c): By Theorem 4.2, Ω satisfies the measure density condition.

Since X supports the Poincaré inequality, the assumptions on Ψ imply that $N^{1,\Psi}(X) = M^{1,\Psi}(X)$ and that
$$\frac{1}{2}\|v\|_{N^{1,\Psi}(X;\lambda)} \leq \|v\|_{M^{1,\Psi}(X;\lambda)} \leq C\|v\|_{N^{1,\Psi}(X;\lambda)} \tag{5.2}$$
for all $\lambda > 0$ and for all functions v, see [36]. Hence $\mathcal{E}u$, the extension of $u \in N^{1,\Psi}(\Omega)$ belongs to $M^{1,\Psi}(X)$, and so $u = \mathcal{E}u|_\Omega \in M^{1,\Psi}(\Omega)$. Therefore $N^{1,\Psi}(\Omega) \subset M^{1,\Psi}(\Omega)$. Since $M^{1,\Psi}(\Omega) \subset N^{1,\Psi}(\Omega)$ by [35], we have that $M^{1,\Psi}(\Omega) = N^{1,\Psi}(\Omega)$. Moreover, by (5.2) and (4.9),
$$\|u\|_{M^{1,\Psi}(\Omega;\lambda)} \leq \|\mathcal{E}u\|_{M^{1,\Psi}(X;\lambda)} \leq C\|\mathcal{E}u\|_{N^{1,\Psi}(X;\lambda)} \leq C\|u\|_{N^{1,\Psi}(\Omega;\lambda)}$$
for all $\lambda > 0$.

(c) $\Longrightarrow$ (b): Since $\mu(\partial\Omega) = 0$, we have that $M^{1,\Psi}(\Omega) = M^{1,\Psi}(\overline{\Omega})$. Since μ is Q-regular, it satisfies the reverse doubling condition. By Theorem 3.4, there exists a linear extension $\mathcal{E} : N^{1,\Psi}(\Omega) = M^{1,\Psi}(\Omega) \to M^{1,\Psi}(X) = N^{1,\Psi}(X)$ that satisfies (3.2). From (5.2), (3.2) and (5.1), it follows that $\mathcal{E}$ satisfies (4.9).

The last implication (b) $\Longrightarrow$ (a) is trivial. □

Theorem 5.3. *Let X be a Q-regular complete metric measure space that supports a Poincaré inequality and let $\Omega \subset X$ be a domain. Assume that Ψ and its conjugate are doubling N-functions and that $\Psi(t)/t^p$ is either decreasing for some $p < Q$ or increasing for some $p > Q$. Then the following conditions are equivalent.*

(a) *The trace operator* (4.10) *is surjective.*
(b) *There exists a bounded extension operator $\mathcal{E} : N^{1,\Psi}(\Omega) \to N^{1,\Psi}(X)$.*
(c) *There exists a bounded linear extension operator $\mathcal{E} : N^{1,\Psi}(\Omega) \to N^{1,\Psi}(X)$.*
(d) *The domain Ω satisfies measure density condition* (3.1) *and $M^{1,\Psi}(\Omega) = N^{1,\Psi}(\Omega)$ as sets and the norms are equivalent.*

Proof. (a) $\Longrightarrow$ (b) as in the proof of Theorem 4.2. The proof of (b) $\Longrightarrow$ (d) $\Longrightarrow$ (c) is analogous to the proof of Theorem 5.2. Trivially, (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a). □

The following theorem shows that the measure density condition characterizes extension domains for the space $M^{1,\Psi}$. As discussed in the introduction, this cannot be the case for spaces $N^{1,\Psi}$ and $W^{1,\Psi}$ as the unit disc in plane with radius removed shows. The reason for that difference is that $M^{1,\Psi}$ spaces do not see sets of zero measure. It follows directly from the definition that $M^{1,\Psi}(F) = M^{1,\Psi}(F \setminus E)$ as sets whenever the measure of the set E is zero.

Theorem 5.4. *Let X a Q-regular geodesic metric measure space, let $\Omega \subset X$ be a domain, and let Ψ be doubling. Then the following conditions are equivalent.*

(a) *There exists an extension operator $\mathcal{E} : M^{1,\Psi}(\Omega) \to M^{1,\Psi}(X)$ such that* (3.2) *holds.*
(b) *There exists a linear extension operator $\mathcal{E} : M^{1,\Psi}(\Omega) \to M^{1,\Psi}(X)$ such that* (3.2) *holds.*
(c) *The domain Ω satisfies measure density condition* (3.1).

If, in addition, $\Psi(t)/t^p$ is either decreasing for some $p < Q$ or increasing for some $p > Q$, then also the following conditions are equivalent.

(d) *The trace operator* (4.12) *is surjective.*
(e) *There exists a bounded extension operator.*
(f) *There exists a bounded linear extension operator.*

Proof. Implications (b) $\Longrightarrow$ (a) and (b) $\Longrightarrow$ (f) $\Longrightarrow$ (e) $\Longrightarrow$ (d) are trivial. By Theorem 4.5, (a) implies (c). If (c) holds, then $M^{1,\Psi}(\Omega) = M^{1,\Psi}(\overline{\Omega})$, and so (b) follows from Theorem 3.4.

If $\Psi(t)/t^p$ is decreasing for some $p < Q$ or $\Psi(t)/t^p$ increasing for some $p > Q$, then (d) implies (c) by Theorem 4.5. $\square$

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