The American Mathematical Monthly

ISSN: 0002-9890 (Print) 1930-0972 (Online) Journal homepage: https://www.tandfonline.com/loi/uamm20

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To cite this article: Guram Bezhanishvili & David Pengelley (2015) What Does “Less Than or Equal“ Really Mean?, The American Mathematical Monthly, 122:10, 983-989

To link to this article: https://doi.org/10.4169/amer.math.monthly.122.10.983

Published online: 13 Dec 2017.

What Does "Less Than or Equal" Really Mean?

Guram Bezhanishvili and David Pengelley

Abstract. The Cantor–Bernstein theorem is often stated as '$\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{b} \leq \mathfrak{a}$ imply $\mathfrak{a} = \mathfrak{b}$' for cardinalities. This suggestive form of the theorem may lead to a trap, into which many early 20th century mathematicians fell, unless we are very careful in interpreting $\leq$. The key is the subtle interplay between $<$ and $\leq$. Originally, following Cantor, $<$ was considered the primary relation, and $\leq$ was defined as the disjunction of $<$ and $=$. However, the above suggestive form of the Cantor–Bernstein theorem requires the modern definition of $\leq$. The uncertainty, sometimes confusion, and evolution due to these subtleties can fascinate and motivate both us and our students today.

Today, we compare cardinalities by simple containment. For sets A and B, we define $\mathfrak{a} \leq \mathfrak{b}$ to hold between their respective cardinalities if and only if A is equivalent[1] to a subset of B. We imagine that this natural notion was always just so, but it was not, and therein hangs a tale of uncertainty, sometimes confusion, and evolution over several decades of foundational subtleties in mathematics.

Consider what Felix Hausdorff wrote in 1927:

> The Equivalence Theorem can then be written in the suggestive form: If $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{a} \geq \mathfrak{b}$, then $\mathfrak{a} = \mathfrak{b}$. [**18**, p. 29] [**19**, p. 32]

This sentence seems at first sight eminently reasonable and clear, given Hausdorff's statement of the equivalence theorem (today called the Cantor–Bernstein theorem or Schröder-Bernstein theorem[2]):

> If each of two given sets is equivalent to a subset of the other, then the two given sets are themselves equivalent. [**18**, p. 27] [**19**, p. 30]

Surprisingly, though, this simple interpretation is nixed by the discovery that what Hausdorff meant by $\leq$ is not what we mean today! Rather, he had defined it in part of an earlier sentence:

> If A is equivalent to a subset of B (case (1) or (2)), then $\mathfrak{a} = \mathfrak{b}$ or $\mathfrak{a} < \mathfrak{b}$, which we write as the single expression $\mathfrak{a} \leq \mathfrak{b}$. [**18**, p. 29] [**19**, p. 32]

Thus, for Hausdorff $\mathfrak{a} < \mathfrak{b}$ was more fundamental than $\mathfrak{a} \leq \mathfrak{b}$, used in a disjunction to define $\leq$, quite unlike our approach today. So to understand what $\leq$ meant to him, we must first learn his meaning for $<$, contained in the following excerpt (where $\sim$ stands for equivalence).

http://dx.doi.org/10.4169/amer.math.monthly.122.10.983
MSC: Primary 03E10, Secondary 97E60

[1] Recall that two sets X and Y are equivalent provided they can be put in one-to-one correspondence.

[2] While Georg Cantor proved the theorem in special situations, Richard Dedekind actually had the first general proof in 1887, which he described more than a decade later in their 1899 correspondence [**15**, p. 449], even though he never published it. This nontrivial theorem received many different proofs. For a detailed history, see [**20, 25, 26**].

If A and B denote sets, and A_1 and B_1 any subsets of A and B respectively, then there are four [mutually exclusive] possibilities:

(1) There is an $A_1 \sim B$, and a $B_1 \sim A$.
(2) There is no $A_1 \sim B$, but there is a $B_1 \sim A$.
(3) There is an $A_1 \sim B$, but no $B_1 \sim A$.
(4) There is no $A_1 \sim B$, and no $B_1 \sim A$.

In case (1) we have, by the Equivalence Theorem, $\mathfrak{a} = \mathfrak{b}$; in case (2) it is natural to define $\mathfrak{a} < \mathfrak{b}$, and in case (3) to define $\mathfrak{a} > \mathfrak{b}$. In case (4), ... [we ...] have a fourth relation, which we write as $\mathfrak{a} \parallel \mathfrak{b}$, and call incomparability between $\mathfrak{a}$ and $\mathfrak{b}$, while in the first three cases $\mathfrak{a}$ and $\mathfrak{b}$ will be called comparable. [**18**, p. 28] [**19**, p. 31f]

This settles Hausdorff's definition of $<$. Now, finally, we can interpret what he says about $\leq$. He defines $\mathfrak{a} \leq \mathfrak{b}$ to mean "$\mathfrak{a} = \mathfrak{b}$ or $\mathfrak{a} < \mathfrak{b}$". But then from $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{a} \geq \mathfrak{b}$ it trivially follows that $\mathfrak{a} = \mathfrak{b}$, from the mutual exclusivity of $\mathfrak{a} < \mathfrak{b}$ and $\mathfrak{a} > \mathfrak{b}$. This seems absurd, Hausdorff claiming the equivalence theorem is a trivial consequence of a natural definition. Something is wrong here.

If we want to write the equivalence theorem in Hausdorff's suggestive form, then we have to define $\mathfrak{a} \leq \mathfrak{b}$ differently, as "*A* is equivalent to a subset of *B*", as is done today. The snag here is that the logical equivalence of the two statements "*A* is equivalent to a subset of *B*" and "$\mathfrak{a} = \mathfrak{b}$ or $\mathfrak{a} < \mathfrak{b}$" relies on the equivalence theorem itself, as Hausdorff even said above when addressing case (1). So to use this logical equivalence to create his suggestive form for the equivalence theorem is deceptive at best, circular at worst.

It turns out Hausdorff was far from the first to fall into this trap. Philip Jourdain in 1907 [**22**, pp. 355–356] had already explained what is really going on here. He named who had already stumbled, and he suggested a way of avoiding confusion about the meaning of $\leq$ and the role of the equivalence theorem.

It is important to be quite clear as to which is proved in this theorem. Bernstein, Whitehead, König, and Peano, owing to what is, I think, a misleading use of the symbol ($\leq$), have given rise to much confusion. These writers, namely, state the theorem in question as follows:

If $\mathfrak{m} \leq \mathfrak{n}$ and $\mathfrak{m} \geq \mathfrak{n}$, then $\mathfrak{m} = \mathfrak{n}$.

If $\mathfrak{m} \geq \mathfrak{n}$ means, as it is natural to suppose it does, ($\mathfrak{m} > \mathfrak{n}$ *or* $\mathfrak{m} = \mathfrak{n}$), this statement is a simple logical conclusion from the fact that each of the relations $>$, $<$, $=$ excludes the other two, and hardly needs a special 'theorem' in mathematics. But Bernstein, for example, seems to mean by $\mathfrak{m} \leq \mathfrak{n}$ the statement 'there is a part of N which is similar to M,' and thus either case (1) or (3) occurs.[3] By Cantor's definitions[4] of $<$ and $=$, we can state from this that '$\mathfrak{m} < \mathfrak{n}$ or $\mathfrak{m} = \mathfrak{n}$' when, and when only, we have proved the Schröder–Bernstein theorem.

It seems less misleading to give ($\leq$) the natural signification and to state the Schröder–Bernstein theorem (in the form it takes when M and N have cardinal numbers) as follows:

'If a part of N is similar to M, then

$$\mathfrak{m} \leq \mathfrak{n}.\text{'}$$

Even earlier, in 1898, when Émile Borel published Felix Bernstein's beautiful proof of the equivalence theorem in his book [**9**, pp. 106–107], he warned about this trap and how it is a dangerous artifact of our use of language:

[3] Jourdain's case (3) is case (2) in Hausdorff's listing, with $M = A$, $N = B$, $\mathfrak{m} = \mathfrak{a}$, and $\mathfrak{n} = \mathfrak{b}$.

[4] Hausdorff's definition of $<$ is adopted from Cantor.

> We will especially use this [theorem] in the following form: When an aliquot part of B has the same power as A, we say that *the power of* A *is less or equal to that of* B *and that the power of* B *is greater or equal to that of* A.[5] From then on, when we have shown that the power of a set A is both *greater or equal* and *less or equal* to that of a set B, we can affirm that *it is equal to it*. In this form, the theorem appears obvious; but it is important to notice that if it had not been demonstrated beforehand it would not be legitimate to introduce this language.

To summarize, there are two possible definitions of $\leq$. One is based on Hausdorff's definition of $<$ and is given by

$$\mathfrak{a} \leq \mathfrak{b} \text{ if and only if } \mathfrak{a} < \mathfrak{b} \text{ or } \mathfrak{a} = \mathfrak{b}.$$

The other definition is the modern one and is given by

$$\mathfrak{a} \leq \mathfrak{b} \text{ if and only if there is a subset of } B \text{ that is equivalent to } A.$$

Clearly, the disjunctive definition of $\leq$ implies the modern one. The converse requires the equivalence theorem. Notice, in fact, that Jourdain ends by correctly phrasing the equivalence theorem as saying that the modern meaning of $\leq$ implies the disjunctive meaning.[6] But Jourdain does not recommend adopting our modern definition of $\leq$. This leaves us wondering who first committed to the modern approach of introducing $\leq$ as the primary relation, without first introducing $<$, and why. The earliest reference we have found that gives the modern definition of $\leq$ is Giuseppe Peano's 1906 reformulation of the equivalence theorem [**27**]. The earliest textbooks we have found taking the modern approach are the 1939 summary *Théorie des ensembles (Fascicule de résultats)* by Bourbaki [**10**, p. 38], precursor to its appearance in the 1956 Bourbaki textbook *Théorie des ensembles* [**11**, p. 56] [**12**, p. 158f], and the 1941 *A survey of modern algebra* by Birkhoff and MacLane [**8**, p. 339]. Since then, it has become standard to define $\leq$ first.

But if we today define $\leq$ first, this leaves open the question of what $<$ should mean. Let us look at how some modern textbooks deal with this issue, having defined $\leq$ as the primary relation. There are two alternatives here, too.

In Jech [**21**, p. 23] and Kuratowski–Mostowski [**24**, p. 185], we see $<$ defined, perfectly naturally, as

$$\mathfrak{a} < \mathfrak{b} \text{ if and only if } \mathfrak{a} \leq \mathfrak{b} \text{ and } \mathfrak{a} \neq \mathfrak{b}.$$

However, notice that this natural definition is *not* Hausdorff's definition (his case (2)), which is unsettling.

On the other hand, in Kunen [**23**, p. 27], we find $<$ defined as

$$\mathfrak{a} < \mathfrak{b} \text{ if and only if } \mathfrak{a} \leq \mathfrak{b} \text{ and } \mathfrak{b} \not\leq \mathfrak{a},$$

which *is* Hausdorff's definition.

[5] There appears to have been a typographical error here: *A* and *B* were switched in the first clause, which we have reversed above.

[6] Waclaw Sierpinski discusses this extensively in his book [**30**, p. 80f] on cardinal and ordinal numbers. In fact, in the many editions of his books on set theory and cardinals, beginning with [**29**, p. 72f, 81f] in 1912, Sierpinski defines $<$ first then correctly deals with the tricky issue in defining and using $\leq$ (his 1910 handwritten set theory notes [**28**] do not yet discuss $\leq$).

In fact, Hausdorff's definition of $<$ was taken from Cantor's famous 1895 paper [**13**] [**14**, p. 89f]. In contrast, though, the natural definition above of $<$ was actually Cantor's first attempt in 1878 [**15**, p. 119]:

> If the two sets M and N are not of the same power, then either M has equal power with a component part of N, or N has equal power with a component part of M;[7] in the first case we call the power of M *smaller*, in the second case we call it *larger*, than the power of N.[8]

If we define $<$ using this natural definition, we find that its antisymmetry (that $\mathfrak{a} < \mathfrak{b}$ and $\mathfrak{b} < \mathfrak{a}$ cannot both occur) and transitivity ($\mathfrak{a} < \mathfrak{b}$ and $\mathfrak{b} < \mathfrak{c}$ imply $\mathfrak{a} < \mathfrak{c}$) are not obvious without the use of the equivalence theorem. On the other hand, if $<$ is defined as Cantor did in 1895, then antisymmetry and transitivity of $<$ are straightforward.

In 1887, Cantor wrote [**15**, p. 413]:

> If by some means it is determined, that two given sets M and N are not equivalent, then one of the following two cases occurs: either a component part N′ can be separated from N, so that $M \sim N'$, or a component part M′ can be separated from M, so that $M' \sim N$. In the first case $\overline{\overline{M}}$ [9] is called smaller than $\overline{\overline{N}}$, in the second we call $\overline{\overline{M}}$ larger than $\overline{\overline{N}}$.
>
> Here it cannot be sufficiently emphasized, that the exclusive behavior of the two cases, which underlies the definition of greater and lesser for cardinal numbers, depends vitally on the assumption made, that M and N do not have the same power. If the two sets were equivalent, it could certainly happen, that component parts M′ and N′ exist, for which both $\overline{\overline{M}} = \overline{\overline{N'}}$ and $\overline{\overline{M'}} = \overline{\overline{N}}$. One has the theorem: if M and N are two such sets, from which component parts M′ and N′ can be separated, for which one can show that $\overline{\overline{M}} = \overline{\overline{N'}}$ and $\overline{\overline{M'}} = \overline{\overline{N}}$, then M and N are equivalent sets.

This represents transitional thinking between what Cantor wrote in 1878 and 1895 since he recognizes here that antisymmetry of the natural definition of $<$ depends on the equivalence theorem. It is interesting to note how, in Cantor's writings, the equivalence theorem took on increasing significance with time [**15**, pp. 201, 257].

In 1895, Cantor changed his definition of $<$ [**14**, p. 89f].

> If for two sets[10] M and N with the cardinal numbers $\mathfrak{a} = \overline{\overline{M}}$ and $\mathfrak{b} = \overline{\overline{N}}$, both the conditions:
>
> (a) There is no part of M which is equivalent to N,
> (b) There is a part N_1 of N, such that $N_1 \sim M$,
>
> are fulfilled, it is obvious that these conditions still hold if in them M and N are replaced by two equivalent sets M′ and N′. Thus they express a definite relation of the cardinal numbers $\mathfrak{a}$ and $\mathfrak{b}$ to one another.
>
> ...

[7] Surprisingly, by "part" Cantor means proper subset, something we did not initially realize, which led to much confusion on our part. On the other hand, Cantor's contemporary Dedekind made a clear distinction between subset and proper subset by calling them "part" and "proper part." In their correspondence, they needed to acknowledge their different meanings for "part" since in some situations the distinction is highly relevant. For example, in a 1899 letter to Dedekind, Cantor writes "When I speak of 'part', I always mean 'proper part'." [**15**, p. 450] For this particular definition, however, the distinction does not influence the meaning.

[8] Notice here Cantor's implicit belief that the powers (cardinal numbers) of any two sets are inherently comparable. This is not necessary for the definition being made.

[9] The notation $\overline{\overline{M}}$ is Cantor's terminology for the cardinal number of the set M.

[10] This translation is by Jourdain. He calls sets "aggregates," but we choose to use "set" instead since it is by now an established word.

We express the relation of $\mathfrak{a}$ to $\mathfrak{b}$ characterized by (a) and (b) by saying: $\mathfrak{a}$ is "less" than $\mathfrak{b}$ or $\mathfrak{b}$ is "greater" than $\mathfrak{a}$; in signs

$$\mathfrak{a} < \mathfrak{b} \text{ or } \mathfrak{b} > \mathfrak{a}.$$

According to Hinkis [**20**, §5.1], Cantor's change in his definition of $<$ was precisely because he had difficulties giving an elementary proof that his 1878 definition is transitive. Another reason could be that the 1895 definition fits perfectly as one of the four mutually exclusive cases for comparing sets (Hausdorff's case (2)). These four cases were discussed by Cantor in a 1899 letter to Dedekind [**15**, p. 450] (see also [**9**, p. 102ff]).

It is clear that Cantor's 1895 definition implies the 1878 definition, while the converse needs the equivalence theorem. Indeed, in 1942 Paul Bernays [**6**] uses Cantor's more natural 1878 definition of $<$, rather than the 1895 definition but only after establishing the equivalence theorem first.

Confusion, resolution, and pedagogy. The seemingly simple question of how to define and work with inequality ($<$) and less than or equal ($\leq$) for cardinalities led to over half a century of uncertainty. Moreover, it sometimes led to confusion and evolution in research and textbooks by some of the best mathematicians of the era due to several wonderful foundational issues that are eminently accessible to undergraduate students and that can fascinate us today. The definition of $<$ evolved even in Cantor's own writings over a 17-year period as he grappled with subtleties of the situation.

Originally, $<$ was considered as the primary relation, and $\leq$ was defined as the disjunction of $<$ and $=$. Then the statement

$$\mathfrak{a} \leq \mathfrak{b} \text{ and } \mathfrak{a} \geq \mathfrak{b} \text{ imply } \mathfrak{a} = \mathfrak{b}$$

is trivially true. On the other hand, under the modern definition, the roles of the two relations are reversed. The relation $\leq$ is primary, and $<$ is defined as the conjunction of $\leq$ and either $\neq$ or $\ngeq$ (which are equivalent definitions by the equivalence theorem). The implication above then becomes a restatement of the equivalence theorem. The original and modern definitions of $\leq$ are equivalent by the equivalence theorem, but the modern approach has the advantage that many developments and proofs are streamlined.

Today's textbooks sweep these subtleties under the rug via the modern definition of $\leq$ and present a development that hides the foundational issues from both us and our students. But these very issues provide the richness and inherent interest that are at the core of what mathematics is, that we should want for our students, and that will help them think deeply about mathematics [**3, 4**].

In our own teaching, we have implemented a pedagogical philosophy of having students acquire fundamental mathematical ideas, knowledge, and practice by studying excerpts from primary historical sources [**1, 2, 5, 7**]. In teaching discrete mathematics, our intention was to teach infinity, cardinals, and $\leq$ from primary sources. To our amazement, we discovered that the earlier sources did not even define $\leq$ but, rather, focused on $<$. Thus, began our own journey that stimulated us and our students to think more deeply about matters that today's mathematicians imagine are straightforward and to find hidden gems of new interest and insight. In fact, primary sources often provoke many natural and important questions about definitions. Through guided reading and study of carefully selected primary source excerpts, our students (and we) can benefit from confronting and resolving conceptual issues, precisely because the sources have not been retrospectively sanitized to remove the issues and questions that got “cleared up” later and then often lost.

ACKNOWLEDGMENT. We thank Patricia Baggett, Nick Bezhanishvili, Andrzej Ehrenfeucht, Tiziana Giorgi, Tadeusz Litak, and Wacek Zawadowski for suggestions and assistance with translation and obtaining sources. We also thank interlibrary loan at New Mexico State University.

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A Simple Proof of the Uniform Continuity of Real-Valued Continuous Functions on Compact Intervals

Instead of using the Bolzano–Weierstrass theorem or the compactness property of the interval (as usually found in its standard proofs), here we use the notion of maximal intervals to prove that a real-valued continuous function on a compact interval is uniformly continuous. We shall take advantage of the following obvious fact: *If a function $f : [a, b] \to \mathbb{R}$ is uniformly continuous on $[a, b)$ and continuous at b, then it is uniformly continuous on $[a, b]$.*

Theorem 1. *A continuous function $f : [a, b] \to \mathbb{R}$ is uniformly continuous.*

Proof. Let $\varepsilon > 0$ be given, and define $[a, a) := \emptyset$, for convenience. Consider the collection

$$\mathcal{C}_\varepsilon := \{[a, t) \subseteq [a, b] : (\exists \delta_t > 0)(x_1, x_2 \in [a, t), |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon)\}.$$

Since $\emptyset \in \mathcal{C}_\varepsilon$, $\mathcal{C}_\varepsilon \neq \emptyset$. Let $[a, t_0) \in \mathcal{C}_\varepsilon$ be a maximum interval. Then $t_0 = b$ for, if $t_0 < b$, since f is continuous at t_0, we can choose $\delta_1 > 0$, $\delta_1 \leq \min\{\delta_{t_0}, b - t_0\}$, such that $|f(x) - f(t_0)| < \varepsilon/2$, for all $x \in I_{\delta_1} := [t_0 - \delta_1, t_0 + \delta_1] \cap [a, b]$. This implies that, if $x_1, x_2 \in I_{\delta_1}$, then $|f(x_1) - f(x_2)| \leq |f(x_1) - f(t_0)| + |f(x_2) - f(t_0)| < \varepsilon$. Now if $x_1, x_2 \in [a, t_0 + \delta_1)$ and $|x_1 - x_2| < \delta_1$, then either $x_1, x_2 \in [a, t_0)$ or $x_1, x_2 \in I_{\delta_1}$, and so $|f(x_1) - f(x_2)| < \varepsilon$, since $[a, t_0) \in \mathcal{C}_\varepsilon$. Thus $[a, t_0 + \delta_1) \in \mathcal{C}_\varepsilon$, contradicting the maximality of $[a, t_0)$. This completes the proof, since for any $\varepsilon > 0$, $[a, b) \in \mathcal{C}_\varepsilon$. ■

Remark.

1. As a consequence of the definition $[a, a) := \emptyset$, we carefully allow the possibility of the maximal interval $[a, t_0)$ to be $\emptyset$. In such a case, where $t_0 = a$ and $I_{\delta_1} = [a, a + \delta_1]$, we have a contradiction as shown above.
2. Those who would rather avoid the definition $[a, a) := \emptyset$ can still reason on $\mathcal{C}_\varepsilon \neq \emptyset$ by using the fact that f is continuous at a.

—Submitted by Haryono Tandra

http://dx.doi.org/10.4169/amer.math.monthly.122.10.989
MSC: Primary 26A03