

SINGULAR SOLUTIONS TO p -LAPLACIAN TYPE EQUATIONS

TERO KILPELÄINEN

ABSTRACT. We construct singular solutions to equations

$$\operatorname{div} \mathcal{A}(x, \nabla u) = 0,$$

similar to the p -Laplacian, that tend to ∞ on a given closed set of p -capacity zero. Moreover, we show that every G_δ -set of vanishing p -capacity is the infinity set of some \mathcal{A} -superharmonic function.

1. Introduction

Suppose that u is a solution of the p -Laplacian equation

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$$

in an open subset Ω of \mathbf{R}^n . If $u(x)$ tends continuously to ∞ as x approaches the boundary $\partial\Omega$ of Ω , then it is easily seen that the complement of Ω is of p -capacity zero, i.e.

$$C_p(\mathbf{R}^n \setminus \Omega) = 0;$$

see e.g. [HK, 3.4], [HKM, 10.5 and 10.6], or [R, Thm 5.9]. The p -capacity of a set E is defined as

$$C_p(E) = \inf \int_{\mathbf{R}^n} (|\nabla \varphi|^p + |\varphi|^p) dx,$$

where φ runs through all $\varphi \in W_{\text{loc}}^{1,p}(\mathbf{R}^n)$ such that $\varphi \geq 1$ on an open neighborhood of E . In this paper we are interested in the converse problem: given a set E of p -capacity zero, can one construct a solution to the p -Laplacian whose singularity set E is?

In the precise treatment we use the language of nonlinear potential theory [HKM]. We consider more general equations

$$(1.1.) \quad \operatorname{div} \mathcal{A}(x, \nabla u) = 0$$

1991 *Mathematics Subject Classification.* 35J60, 31C45.

The research is financed by the Academy of Finland (Project #8597)

that are similar to the p -Laplacian; see Section 2.1. The continuous solutions to (1.1) are called \mathcal{A} -harmonic and \mathcal{A} -superharmonic functions are defined via a comparison with the \mathcal{A} -harmonic functions. The precise definitions and properties of \mathcal{A} -harmonic and \mathcal{A} -superharmonic functions are listed in Section 2.1 below. Roughly speaking, \mathcal{A} -superharmonic functions u are but the solutions of

$$(1.2) \quad -\operatorname{div} \mathcal{A}(x, \nabla u) = \mu$$

with nonnegative Radon measures μ .

It has been known for about a decade that sets of p -capacity zero can be characterized as \mathcal{A} -polar sets; a set E is called \mathcal{A} -polar if there is an \mathcal{A} -superharmonic function u on \mathbf{R}^n such that $u = \infty$ on E . This was first established by Lindqvist and Martio for the $p = n$ case in [LM] and later for all p 's in [HK] (see [HKM, Ch. 10]). Note that the definition of an \mathcal{A} -polar set does not require that it be exactly the infinity set for some \mathcal{A} -superharmonic function. Since each \mathcal{A} -superharmonic function u is lower semicontinuous, we have that its set of infinity is a G_δ -set, a countable intersection of open sets:

$$\{x: u(x) = \infty\} = \bigcap_j \{x: u(x) > j\}.$$

Therefore it is natural to ask whether, for a given G_δ set E of p -capacity zero, there exists an \mathcal{A} -superharmonic u that is ∞ *exactly* on E . The first result in this direction is Theorem 1.7 in [K] which states that an \mathcal{A} -superharmonic function can be chosen to be ∞ on E but finite at a given point outside E . In this paper we give a complete affirmative answer to the question and prove:

1.3. Theorem. *Suppose that E is a G_δ -set of p -capacity zero. Then there is an \mathcal{A} -superharmonic function u in \mathbf{R}^n such that*

$$E = \{x: u(x) = \infty\}.$$

Moreover, if $1 < p < n$, then u can be chosen to be positive.

We want to emphasize that according to Theorem 1.3 the “true \mathcal{A} -polarity” is independent of the actual operator: if \mathcal{A}_1 and \mathcal{A}_2 are two mappings that satisfy the assumptions listed in Section 2.1 and u_1 is \mathcal{A}_1 -superharmonic, then there is an \mathcal{A}_2 -superharmonic u_2 in \mathbf{R}^n such that

$$\{x: u_1(x) = \infty\} = \{x: u_2(x) = \infty\}.$$

Moreover, we show that the function u given by Theorem 1.3 can be chosen to be \mathcal{A} -harmonic outside E if E is closed:

1.4. Theorem. *Let E be a relatively closed subset of an open set Ω . If E is of p -capacity zero, then there is a continuous \mathcal{A} -superharmonic function u in Ω such that*

$$E = \{x \in \Omega: u(x) = \infty\}$$

and u is \mathcal{A} -harmonic in $\Omega \setminus E$.

Since \mathcal{A} -superharmonic functions solve equations like (1.2), we may interpret Theorem 1.4 as follows: there is a Radon measure μ supported on any given closed set E of p -capacity zero so that μ is concentrated at each point of E . The precise meaning of this statement will be made clear later.

In the classical linear case Theorem 1.4 was first proven by Evans [E] whence such a function is often called an Evans potential. Later Choquet [C] extended it for a general G_δ -set E of capacity zero. In the case where E is countable and compact Theorem 1.4 is established in Holopainen's thesis [H]. In Section 3 we prove a slight more general result than Theorem 1.4: any G_δ -set of n p -capacity zero that is also an F_σ -set (a countable union of compact sets) is the infinity set for some \mathcal{A} -superharmonic function.

Our method of proof is based on the potential estimate of the author and Malý [KM2] (see Theorem 2.12 below) that allows us to convert the construction of solutions of nonlinear equations into the construction of Radon measures with certain density properties. Indeed, there is a correspondence between Radon measures μ and \mathcal{A} -superharmonic functions u by

$$\mu = -\operatorname{div} \mathcal{A}(x, \nabla u).$$

Moreover, the local behavior of an \mathcal{A} -superharmonic function u whose ‘‘Riesz mass’’ is μ can be controlled in terms of a nonlinear potential, the Wolff potential of μ ,

$$\mathbf{W}_{1,p}^\mu(x, r) = \int_0^r \left(\frac{\mu(B(x, t))}{t^{n-p}} \right)^{1/(p-1)} \frac{dt}{t}.$$

In particular, it was proven in [KM2] that $u(x) = \infty$ exactly when $\mathbf{W}_{1,p}^\mu(x, r) = \infty$. So the proof of Theorem 1.3 boils down in constructing a measure μ such that E is the set of infinity of its Wolff potential and then pick an \mathcal{A} -superharmonic function whose Riesz mass μ is.

The main new trick in this paper is the ‘‘sweeping’’ of the nonlinear Riesz mass onto E so that the Wolff potential does not get essentially smaller; this is done in Section 2.14 below.

I would like to thank Seppo Rickman whose questions made me think these problems once more.

2. Measures, potentials, and \mathcal{A} -superharmonic functions

2.1. Preliminaries. Throughout the paper we let Ω denote an open set in \mathbf{R}^n and $1 < p \leq n$ is a fixed number; note that the case $p > n$ is trivial since then no nonempty set is of p -capacity zero. Moreover, we assume that $\mathcal{A} : \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a mapping which satisfies the following assumptions for some constants $0 < \alpha \leq \beta < \infty$:

$$(2.2) \quad \begin{aligned} & \text{the function } x \mapsto \mathcal{A}(x, \xi) \text{ is measurable for all } \xi \in \mathbf{R}^n, \text{ and} \\ & \text{the function } \xi \mapsto \mathcal{A}(x, \xi) \text{ is continuous for a.e. } x \in \mathbf{R}^n; \end{aligned}$$

for all $\xi \in \mathbf{R}^n$ and a.e. $x \in \mathbf{R}^n$

$$(2.3) \quad \mathcal{A}(x, \xi) \cdot \xi \geq \alpha |\xi|^p,$$

$$(2.4) \quad |\mathcal{A}(x, \xi)| \leq \beta |\xi|^{p-1},$$

$$(2.5) \quad (\mathcal{A}(x, \xi) - \mathcal{A}(x, \zeta)) \cdot (\xi - \zeta) > 0$$

whenever $\xi \neq \zeta$, and

$$(2.6) \quad \mathcal{A}(x, \lambda \xi) = \lambda |\lambda|^{p-2} \mathcal{A}(x, \xi)$$

for all $\lambda \in \mathbf{R}$, $\lambda \neq 0$.

We define the divergence of $\mathcal{A}(x, \nabla u)$ in the sense of distributions, i.e. if $\varphi \in C_0^\infty(\Omega)$, then

$$\operatorname{div} \mathcal{A}(x, \nabla u)(\varphi) = - \int_{\Omega} \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx,$$

where $u \in W_{\operatorname{loc}}^{1,p}(\Omega)$. A solution $u \in W_{\operatorname{loc}}^{1,p}(\Omega)$ to the equation

$$(2.7) \quad \operatorname{div} \mathcal{A}(x, \nabla u) = 0$$

always has a continuous representative; we call continuous solutions $u \in W_{\operatorname{loc}}^{1,p}(\Omega) \cap C(\Omega)$ of (2.7) \mathcal{A} -harmonic in Ω .

A lower semicontinuous function $u: \Omega \rightarrow (-\infty, \infty]$ is called \mathcal{A} -superharmonic if u is not identically infinite in each component of Ω , and if for all open $D \subset\subset \Omega$ and all $h \in C(\overline{D})$, \mathcal{A} -harmonic in D , $h \leq u$ on ∂D implies $h \leq u$ in D .

The following connection between \mathcal{A} -superharmonic functions and supersolutions of (2.7) is fundamental.

2.8. Proposition [HKM, 7.25]. (i) If $u \in W_{\operatorname{loc}}^{1,p}(\Omega)$ is such that

$$- \operatorname{div} \mathcal{A}(x, \nabla u) \geq 0,$$

then there is an \mathcal{A} -superharmonic function v such that $u = v$ a.e. Moreover,

$$(2.9) \quad v(x) = \operatorname{ess} \liminf_{y \rightarrow x} v(y) \quad \text{for all } x \in \Omega.$$

(ii) If v is \mathcal{A} -superharmonic, then (2.9) holds. Moreover, $- \operatorname{div} \mathcal{A}(x, \nabla v) \geq 0$ if $v \in W_{\operatorname{loc}}^{1,p}(\Omega)$.

(iii) If v is \mathcal{A} -superharmonic and locally bounded, then $v \in W_{\operatorname{loc}}^{1,p}(\Omega)$ and

$$- \operatorname{div} \mathcal{A}(x, \nabla v) \geq 0.$$

Because an \mathcal{A} -superharmonic function does not necessarily belong to $W_{\operatorname{loc}}^{1,p}(\Omega)$, we extend the definition for the divergence of $\mathcal{A}(x, \nabla u)$: If u is an \mathcal{A} -superharmonic function in Ω , then we define

$$- \operatorname{div} \mathcal{A}(x, \nabla u)(\varphi) = \int_{\Omega} \lim_{k \rightarrow \infty} \mathcal{A}(x, \nabla \min(u, k)) \cdot \nabla \varphi \, dx,$$

$\varphi \in C_0^\infty(\Omega)$. By [KM1, 1.15]

$$\lim_{k \rightarrow \infty} \mathcal{A}(x, \nabla \min(u, k))$$

is locally integrable and hence $\operatorname{div} \mathcal{A}(x, \nabla u)$ is its divergence. (Since the truncations $\min(u, k)$ are in $W_{\text{loc}}^{1,p}(\Omega)$ and

$$\nabla \min(u, k) = \nabla \min(u, j)$$

a.e. in $\{u < \min(k, j)\}$, the limit exists. It is equal to $\mathcal{A}(x, \nabla u)$ if $u \in W_{\text{loc}}^{1,1}(\Omega)$, which is always the case if $p > 2 - 1/n$.) Our definition treats the difficulty that arises from the fact that for $p \leq 2 - 1/n$ the distributional gradient ∇u need not be a function. Indeed, the above definition of $\operatorname{div} \mathcal{A}(x, \nabla u)$ is merely a technical tool to treat all p 's simultaneously.

Since $-\operatorname{div} \mathcal{A}(x, \nabla u)$ is a nonnegative distribution in Ω for an \mathcal{A} -superharmonic u it follows that there is a nonnegative Radon measure μ such that

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = \mu$$

in Ω ; this measure μ is sometimes referred to as the *Riesz mass of u* . Conversely, given a finite measure μ in bounded Ω , there is an \mathcal{A} -superharmonic function u such that $-\operatorname{div} \mathcal{A}(x, \nabla u) = \mu$ in Ω and $\min(u, k) \in W_0^{1,p}(\Omega)$ for all integers k . We refer to [KM1] and [HKM, Ch. 7] for details.

The existence of \mathcal{A} -superharmonic solutions to $-\operatorname{div} \mathcal{A}(x, \nabla u) = \mu$ in bounded Ω is not adequate for us. Hence we establish:

2.10. Theorem. *Let μ be a finite Radon measure in \mathbf{R}^n . Then there is an \mathcal{A} -superharmonic function u in \mathbf{R}^n such that*

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = \mu$$

in \mathbf{R}^n . If $1 < p < n$, then u can be chosen to be positive.

Proof. The case $1 < p < n$ is easy and it follows by employing an argument similar to that used in [KM1]; an existence result is also proven in [BBGGPV, Thm 8.1] except for the fact that u is \mathcal{A} -superharmonic. The details are left to the reader.

We outline how the argument of [DHM] should be modified to obtain our theorem in the case $p = n$. (I thank Stefan Müller for showing me an early draft of the paper [DHM]).

Choose a sequence $\mu_k \in C_0^\infty(B(0, k))$ of nonnegative functions (measures) such that $\mu_k \rightarrow \mu$ weakly in the sense of measures. Let v_k be the \mathcal{A} -superharmonic solution of the problem

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla v_k) = \mu_k & \text{on } B(0, k) \\ v_k = 0 & \text{on } \partial B(0, k). \end{cases}$$

Using a rescaling argument similarly as in [DHM] we infer that for $u_k = v_k - c_k$, where c_k is a constant, it holds that

$$\int_{B(0,1)} u_k \, dx = 0$$

and

$$[u_k]_{BMO} \leq c \|\mu_k\|^{1/(n-1)} \leq c(\|\mu\| + 1)^{1/(n-1)}.$$

Since u_k is bounded in BMO and since its mean value in the unit ball vanishes, it follows that u_k is bounded in $L^n(B)$, where B is any large ball. Now we obtain from the estimate [HKM, 3.36] that the sequence u_k is uniformly bounded from below in $\frac{1}{2}B$. Hence it follows from [KM1, 1.15] that a subsequence of u_k converges a.e. to an \mathcal{A} -superharmonic function u in \mathbf{R}^n . Moreover, $\nabla u_j \rightarrow \nabla u$ both a.e. pointwise and in $L^q_{loc}(\mathbf{R}^n)$ for $q < n$. In conclusion,

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = \mu$$

in \mathbf{R}^n , as desired. \square

2.11. Wolff potentials. The fact that an \mathcal{A} -superharmonic function can be locally estimated in terms of its Riesz mass is very useful. In our problem these estimates enable us to change the construction of \mathcal{A} -superharmonic functions (solutions to nonlinear equations) to a much easier task: to construct certain Radon measures.

To make this precise we recall that *the Wolff potential of the measure μ* is

$$\mathbf{W}_{1,p}^\mu(x_0, r) = \int_0^r \left(\frac{\mu(B(x_0, t))}{t^{n-p}} \right)^{1/(p-1)} \frac{dt}{t},$$

$r > 0$. We next record the fundamental potential estimate.

2.12. Theorem [KM2, 1.6]. *Suppose that u is a nonnegative \mathcal{A} -superharmonic function in $B(x_0, 3r)$. If $\mu = -\operatorname{div} \mathcal{A}(x, \nabla u)$, then*

$$c_1 \mathbf{W}_{1,p}^\mu(x_0; r) \leq u(x_0) \leq c_2 \inf_{B(x_0, r)} u + c_3 \mathbf{W}_{1,p}^\mu(x_0; 2r),$$

where c_1 , c_2 , and c_3 are positive constants, depending only on n , p , and the structural constants α and β .

In particular, $u(x_0) < \infty$ if and only if $\mathbf{W}_{1,p}^\mu(x_0; r) < \infty$ for some $r > 0$.

The following simple lemma will be used to estimate $\inf u$ that appears in the potential estimate above.

2.13. Lemma [KM2, 3.9]. *Suppose that $u \in W_0^{1,p}(\Omega)$ is \mathcal{A} -superharmonic in Ω and $\mu = -\operatorname{div} \mathcal{A}(x, \nabla u)$. Then for $\lambda > 0$ it holds that*

$$\lambda^{p-1} \operatorname{cap}_p(\{x \in \Omega : u(x) > \lambda\}, \Omega) \leq \frac{\mu(\Omega)}{\alpha}.$$

Recall that $\operatorname{cap}_p(E, \Omega)$ stands for the relative p -capacity of E in Ω which for $E \subset \Omega$ is defined as

$$\operatorname{cap}_p(E, \Omega) = \inf_{\substack{G \subset \Omega \text{ open} \\ E \subset G}} \sup_{\substack{K \subset G \\ K \text{ compact}}} * \operatorname{cap}_p(K, \Omega),$$

where

$$* \operatorname{cap}_p(K, \Omega) = \inf \int_{\Omega} |\nabla u|^p dx;$$

here u runs through all $u \in C_0^\infty(\Omega)$ with $u \geq 1$ on K .

2.14. Sweeping the measure. Let $K \subset \mathbf{R}^n$ be a closed set and let μ be a finite Radon measure. Our goal is to find a new Radon measure $\tilde{\mu}$, supported on K , such that the total mass of μ is preserved and the Wolff potential of $\tilde{\mu}$ is not essentially smaller than that of μ on K . More precisely, we prove:

2.15. Theorem. *Let $K \subset \mathbf{R}^n$ be a closed set and let μ be a finite Radon measure. Then there is Radon measure $\tilde{\mu}$ such that*

- i) *the support of $\tilde{\mu}$ is contained in K , $\mu(\mathbb{C}K) = 0$,*
- ii) *$\mu(\mathbf{R}^n) = \tilde{\mu}(\mathbf{R}^n)$, and*
- iii) *there is $c = c(n, p) > 0$ such that*

$$\mathbf{W}^{\tilde{\mu}}(x, 7t) \geq c\mathbf{W}^{\mu}(x, t)$$

for each $x \in K$ and $t > 0$.

Later we sometimes refer to the measure $\tilde{\mu}$ with the above properties as the *swept out measure of μ into K* .

Proof. Let be the Whitney decomposition of $\mathbb{C}K$, i.e. \mathcal{W} is a countable collection of pairwise disjoint cubes Q (with parts of the boundaries included) such that

$$\bigcup_{Q \in \mathcal{W}} Q = \mathbb{C}K$$

and

$$\text{diam}(Q) \leq \text{dist}(Q, K) \leq 4 \text{diam}(Q).$$

For each $Q \in \mathcal{W}$ choose a point $x_Q \in K$ with

$$\text{dist}(x_Q, Q) \leq 5 \text{diam}(Q).$$

Define

$$\tilde{\mu} = \mu_{\perp K} + \sum_{Q \in \mathcal{W}} \mu(Q) \delta_{x_Q},$$

where δ_y is the Dirac measure at y and $\mu_{\perp K}$ stands for the restriction to K of the measure μ , i.e.

$$\mu_{\perp K}(E) = \mu(E \cap K)$$

for $E \subset \mathbf{R}^n$. Then $\tilde{\mu}$ defines a finite Radon measure supported on K with $\tilde{\mu}(K) = \mu(\mathbf{R}^n)$. Moreover, we have the estimate

$$(2.16) \quad \tilde{\mu}(B(x, 7r)) \geq \mu(B(x, r))$$

for $x \in K$ and $r > 0$. Indeed, fix $x \in K$ and $r > 0$, and let

$$\mathcal{W}_r = \{Q \in \mathcal{W} : Q \cap B(x, r) \neq \emptyset\}.$$

If $Q \in \mathcal{W}_r$, choose $y \in Q \cap B(x, r)$. Then

$$|x_Q - y| \leq 6 \text{diam}(Q) \leq 6 \text{dist}(Q, K) \leq 6|x - y| < 6r,$$

and hence

$$|x_Q - x| \leq |x_Q - y| + |x - y| < 7r,$$

or $x_Q \in B(x, 7r)$. Consequently,

$$\begin{aligned} \mu(B(x, r)) &\leq \mu(B(x, r) \cap K) + \sum_{Q \in \mathcal{W}_r} \mu(Q) \\ &\leq \mu(B(x, 7r) \cap K) + \sum_{x_Q \in B(x, 7r)} \mu(Q) \\ &= \tilde{\mu}(B(x, 7r)). \end{aligned}$$

Next we write the estimate the Wolff potential: if $x \in K$ and $t > 0$, then by (2.16)

$$\begin{aligned} \mathbf{W}^{\tilde{\mu}}(x, 7t) &= \int_0^{7t} \left(\frac{\tilde{\mu}(B(x, r))}{r^{n-p}} \right)^{1/(p-1)} \frac{dr}{r} \\ &\geq \int_0^{7t} \left(\frac{\mu(B(x, r/7))}{r^{n-p}} \right)^{1/(p-1)} \frac{dr}{r} \\ &\geq 7^{(p-n)/(p-1)+1} \int_0^t \left(\frac{\mu(B(x, s))}{s^{n-p}} \right)^{1/(p-1)} \frac{ds}{s} \\ &= 7^{(2p-n-1)/(p-1)} \mathbf{W}^\mu(x, t), \end{aligned}$$

and the theorem follows. \square

3. The F_σ case

In this section we prove the following:

3.1. Theorem. *Suppose that E is a G_δ -set of p -capacity zero. If E is also an F_σ -set, then there is an \mathcal{A} -superharmonic function u in \mathbf{R}^n such that*

$$E = \{x : u(x) = \infty\}$$

and u is \mathcal{A} -harmonic in $\mathfrak{C}\bar{E}$.

Proof. Choose an increasing sequence K_j of compact sets and a decreasing sequence G_k of open sets such that

$$E = \bigcup_{j=1}^{\infty} K_j = \bigcap_{k=1}^{\infty} G_k.$$

Let u be an \mathcal{A} -superharmonic function in \mathbf{R}^n such that $u = \infty$ on E and write

$$\mu = -\operatorname{div} \mathcal{A}(x, \nabla u).$$

Sweep the measure μ onto K_j by Theorem 2.15: obtain a Radon measure μ_j supported on K_j such that $\mu_j(\mathbf{R}^n) \leq 1$ and

$$\mathbf{W}^{\mu_j}(x, 1) = \infty$$

for each $x \in K_j$. By multiplying μ_j with a positive constant ≤ 1 we may assume that

$$\mathbf{W}^{\mu_j}(x, 1) < 4^{-j}$$

whenever $x \notin G_j$; observe that $\text{dist}(K_j, \mathbb{C}G_j) > 0$ and $\text{supp } \mu_j \subset K_j$.

Let

$$\sigma = \sum_{j=1}^{\infty} 2^{-j} \mu_j.$$

Then σ is a finite Radon measure with $\sigma(\mathbb{C}E) = 0$. Moreover,

$$\mathbf{W}^{\sigma}(x, 1) \geq 2^{-j/(p-1)} \mathbf{W}^{\mu_j}(x, 1) = \infty$$

for $x \in K_j$ whence

$$\mathbf{W}^{\sigma}(x, 1) = \infty$$

for each $x \in E$.

We next show that

$$\mathbf{W}^{\sigma}(x, 1) < \infty$$

if $x \notin E$. For this we use the estimate

$$\sigma(A)^{1/(p-1)} \leq \begin{cases} \sum_{j=1}^{\infty} 2^{-j/(p-1)} \mu_j(A)^{1/(p-1)} & \text{if } p \geq 2 \\ \left(\sum_{j=1}^{\infty} 2^{-j/(2-p)} \right)^{(2-p)/(p-1)} \sum_{j=1}^{\infty} \mu_j(A)^{1/(p-1)} & \text{if } p < 2. \end{cases}$$

Hence

$$\mathbf{W}^{\sigma}(x, 1) \leq \begin{cases} \sum_{j=1}^{\infty} 2^{-j/(p-1)} \mathbf{W}^{\mu_j}(x, 1) & \text{if } p \geq 2 \\ \left(\sum_{j=1}^{\infty} 2^{-j/(2-p)} \right)^{(2-p)/(p-1)} \sum_{j=1}^{\infty} \mathbf{W}^{\mu_j}(x, 1) & \text{if } p < 2. \end{cases}$$

Next we observe that if $x \notin E$, then $x \notin G_j$ except possibly for finitely many, say k_x , j 's, and therefore $\mathbf{W}^{\sigma}(x, 1)$ does not exceed

$$\sum_{j=1}^{k_x} 2^{-j/(p-1)} \mathbf{W}^{\mu_j}(x, 1) + \sum_{j=k_x+1}^{\infty} 2^{-j/(p-1)} \mathbf{W}^{\mu_j}(x, 1)$$

if $p \geq 2$ and

$$\left(\sum_{j=1}^{\infty} 2^{-j/(2-p)} \right)^{(2-p)/(p-1)} \left(\sum_{j=1}^{k_x} \mathbf{W}^{\mu_j}(x, 1) + \sum_{j=k_x+1}^{\infty} \mathbf{W}^{\mu_j}(x, 1) \right)$$

if $p < 2$. Finally since

$$\mathbf{W}^{\mu_j}(x, 1) < \begin{cases} 4^{-j} & \text{if } x \notin G_j \text{ i.e. } j > k_x \\ \infty & \text{if } x \notin E \text{ i.e. } j \leq k_x, \end{cases}$$

we infer that

$$\mathbf{W}^\sigma(x, 1) < \infty$$

for all $x \notin E$, as desired.

Now we are in the position to conclude the proof: by Theorem 2.10 there is an \mathcal{A} -superharmonic function v on \mathbf{R}^n such that

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = \sigma.$$

The potential estimate 2.12 implies that $v(x) = \infty$ if and only if $x \in E$. Moreover, v is \mathcal{A} -harmonic in $\mathbb{C}E$, for $\sigma(\mathbb{C}E) = 0$ (see [M, 3.19]). \square

4. Polar set as the set of infinity of an \mathcal{A} -superharmonic function

In this section we prove Theorem 1.3. We start with a lemma whose proof is displeasingly technical.

4.1. Lemma. *Suppose that Ω is a bounded open set and $E \Subset \Omega$ is of p -capacity zero. Let $F \subset \mathbb{C}E$ be closed. Then there is an \mathcal{A} -superharmonic function u in Ω such that $u \in W_0^{1,p}(\Omega)$, $u = \infty$ on E , and $u \leq 1$ on $F \cap \Omega$. Moreover, u can be chosen so that $\mu(\Omega) \leq 1$, where $\mu = -\operatorname{div} \mathcal{A}(x, \nabla u)$.*

Proof. We assume, as we well may, that F contains a neighborhood of $\partial\Omega$. Then choose open sets $G_1 \ni G_2 \ni \dots \ni \bigcap_j G_j = F$. Let v be an \mathcal{A} -superharmonic function in \mathbf{R}^n such that $v = \infty$ on E and let $\sigma = -\operatorname{div} \mathcal{A}(x, \nabla v)$. Now sweep the measure σ into $\Omega \setminus G_j$ and let σ_j stand for the swept out measure (see Theorem 2.15). As in the proof of the F_σ case (Theorem 3.1) we find positive constants c_j such that for the measure

$$\mu = \sum_j c_j \sigma_j$$

it holds that $\mu(\mathbf{R}^n) \leq 1$,

$$\mathbf{W}_{1,p}^\mu(x, 1) \leq 1$$

for all $x \in F$, and

$$\mathbf{W}_{1,p}^\mu(x, 1) \geq \mathbf{W}_{1,p}^{c_j \sigma_j}(x, 1) = \infty$$

for all $x \in E \setminus G_j$, whence

$$\mathbf{W}_{1,p}^\mu(x, 1) = \infty$$

for all $x \in E$.

To complete the proof we let w be an \mathcal{A} -superharmonic solution of the problem

$$(4.2) \quad \begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla w) = \mu & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Now w is not quite that function we are looking for but close to it. To find the final function we choose an open set $D \Subset \Omega$ such that $E \Subset D$, $\partial D \subset F$, $\partial D = \partial \bar{D}$, and that all points on ∂D are regular points for the Dirichlet problem in $\Omega \setminus \bar{D}$ (for instance, D may be a polyhedron; see [HKM]). Since the distance between D and

$\partial\Omega$ is positive, there is $r_0 > 0$ such that $B(x, 3r_0) \subset \Omega$ whenever $x \in \bar{D}$. Now we infer from the estimate Lemma 2.13¹ that

$$\inf_{B(x, r_0)} w \leq c \left(\frac{\mu(\mathbf{R}^n)}{\text{cap}_p(B(x, r_0), \Omega)} \right)^{1/(p-1)} \leq C < \infty,$$

where C is independent of the point $x \in \bar{D}$. Hence the potential estimate 2.12 implies that $w \leq c_0$ in $F \cap \bar{D}$.

Next we observe that the function $\log(w + 1)$ is a positive \mathcal{A} -superharmonic function in Ω , uniformly bounded from above in $F \cap \bar{D}$, and $\in W_{\text{loc}}^{1,p}(\Omega)$ (see [HKM, 7.48]). If $h \in W^{1,p}(\Omega \setminus \bar{B})$ is the \mathcal{A} -harmonic function in $\Omega \setminus \bar{B}$ that agrees with $\log(w + 1)$ on ∂D and with 0 on $\partial\Omega$ (in the Sobolev sense), then the function

$$\tilde{u} = \begin{cases} \log(w + 1) & \text{in } \bar{D} \\ h & \text{in } \Omega \setminus D \end{cases}$$

is a positive \mathcal{A} -superharmonic function in Ω by the pasting lemma [HKM, 7.9], since $h = \min(h, \log(w + 1))$ in $\Omega \setminus D$. Moreover, $\tilde{u} \in W_0^{1,p}(\Omega)$, $\tilde{u} = \infty$ on E , \tilde{u} is uniformly bounded from above in $F \cap \Omega$, and its Riesz mass $-\text{div } \mathcal{A}(x, \nabla \tilde{u})$ is finite in Ω . In conclusion, we may choose a constant $\lambda > 0$ such that the function $u = \lambda \tilde{u}$ enjoys the desired properties of the lemma. \square

Proof of Theorem 1.3. Let $B_k = B(0, 2^k)$ and $E_k = E \cap B_{k-1}$. Suppose that G_j are open sets such that $G_1 \supset G_2 \supset \cdots \supset \bigcap_j G_j = E$. Fix k . Let u_j be an \mathcal{A} -superharmonic function in B_k with finite Riesz mass $-\text{div } \mathcal{A}(x, \nabla u_j)$ such that $u_j = \infty$ on E_k , $u_j \in W_0^{1,p}(B_k)$, and $u_j \leq 1$ on $B_k \setminus (G_j \cap B_{k-1})$ (Lemma 4.1). Let

$$v_j = \min(4^{j/(p-1)}, \lambda_j u_j),$$

where $0 < \lambda_j < 1$ is chosen so that $v_j \leq 2^{-j}$ on $B_k \setminus (G_j \cap B_{k-1})$ and $\mu_j(B_k) \leq 1$, $\mu_j = -\text{div } \mathcal{A}(x, \nabla v_j)$; for the last property observe that μ_j agrees with the Riesz measure of $\lambda_j^{p-1} u_j$ on $B_k \setminus \bar{B}_{k-1}$ and

$$\begin{aligned} \mu_j\left(\frac{3}{2}B_{k-1}\right) &\leq \int_{B_k} \varphi d\mu_j = \int_{B_k} \mathcal{A}(x, \nabla v_j) \cdot \nabla \varphi dx \\ &\leq c \left(\int_{B_k} |\nabla v_j|^p dx \right)^{(p-1)/p} \left(\int_{B_k} |\nabla \varphi|^p dx \right)^{1/p} \\ &\leq c \lambda_j^{(p-1)} \left(\int_{B_k} |\nabla u_j|^p dx \right)^{(p-1)/p}, \end{aligned}$$

where $\varphi \in C_0^\infty(B_k)$ is any nonnegative function with $\varphi = 1$ on $\frac{3}{2}B_{k-1}$.

¹Of course, $w \notin W_0^{1,p}(\Omega)$ contrary to the assumptions of Lemma 2.13. However, following the standard construction of \mathcal{A} -superharmonic solutions to (4.2) as done e.g. in [KM1] one is easily convinced that there is a solution w of (4.2) for which the estimate of Lemma 2.13 holds. Let us pick such a function w .

Now by the estimate Lemma 2.13 we have for $x \in B_{k-1}$ that

$$\inf_{B(x,1)} v_j \leq c \left(\frac{\mu_j(B_k)}{\text{cap}_p(B(x,1), B_k)} \right)^{1/(p-1)} \leq M,$$

where M is independent of j . Thus by the potential estimate 2.12

$$c \mathbf{W}_{1,p}^{\mu_j}(x, 2) \geq v_j(x) - c \inf_{B(x,1)} v_j \geq 3^{j/(p-1)}$$

if $x \in E_k$ and j is large enough.

Setting

$$\mu^{(k)} = \sum_j 2^{-j} \mu_j$$

we obtain a finite Radon measure on \mathbf{R}^n with the properties:

$$\begin{cases} \mathbf{W}_{1,p}^{\mu^{(k)}}(x, 1) < \infty & \text{for all } x \notin E_k \\ \mathbf{W}_{1,p}^{\mu^{(k)}}(x, 1) = \infty & \text{for all } x \in E_k. \end{cases}$$

Indeed, the finiteness of $\mathbf{W}_{1,p}^{\mu^{(k)}}(x, 1)$ outside E_k is proven similarly as in the F_σ case (see 3.1). Further, if $x \in E_k$, then for j large enough it holds that

$$\mathbf{W}_{1,p}^{\mu^{(k)}}(x, 2) \geq c 2^{-j/(p-1)} 3^{j/(p-1)} \geq \left(\frac{3}{2}\right)^{j/(p-1)},$$

whence $\mathbf{W}_{1,p}^{\mu^{(k)}}(x, 1) = \infty$.

To complete the proof we write

$$\mu = \sum_{k=1}^{\infty} \mu^{(k)} \llcorner_{(B_k \setminus B_{k-2})},$$

here $\mu^{(k)} \llcorner_{(B_k \setminus B_{k-2})}$ stands for the restriction to $B_k \setminus B_{k-2}$ of the measure $\mu^{(k)}$, and $B_0 = B_{-1} = \emptyset$. Then μ is a finite Radon measure in \mathbf{R}^n and

$$\mathbf{W}_{1,p}^{\mu}(x, 1) = \infty \text{ if and only if } x \in E.$$

An \mathcal{A} -superharmonic solution of $-\text{div } \mathcal{A}(x, \nabla u) = \mu$ on \mathbf{R}^n is the desired function u (see Theorems 2.10 and 2.12). \square

REFERENCES

- [BBGGPV] Bénylan, P., Boccardo, L., Gallouët, T., Gariépy, R., Pierre, M., and Vazquez, J.L., *An L^1 -theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa. Cl. Science, Ser. IV **22** (1995), 241–273.
- [C] Choquet, G., *Potentiels sur un ensemble de capacité nulle*, C.R. Acad. Sci. Paris **244** (1957), 1707–1710.
- [DHM] Dolzmann, G., Hungerbühler, N., and Müller, S., Preprint (1997).

- [E] Evans, G. C., *Potentials and positively infinite singularities of harmonic functions*, Monatsh. Math. Phys. **43** (1936), 419–424.
- [HK] Heinonen, J. and Kilpeläinen, T., *Polar sets for supersolutions of degenerate elliptic equations*, Math. Scand. **63** (1988), 136–150.
- [HKM] Heinonen, J., Kilpeläinen, T., and Martio, O., *Nonlinear potential theory of degenerate elliptic equations*, Oxford University Press, Oxford, 1993.
- [H] Holopainen, I., *Nonlinear potential theory and quasiregular mappings on Riemannian manifolds*, Ann. Acad. Sci. Fenn. Ser. A I. Math. Dissertationes **74** (1990), 1–45.
- [K] Kilpeläinen, T., *Potential theory for supersolutions of degenerate elliptic equations*, Indiana Univ. Math. J. **38** (1989), 253–275.
- [KM1] Kilpeläinen, T. and Malý, J., *Degenerate elliptic equations with measure data and nonlinear potentials*, Ann. Scuola Norm. Sup. Pisa. Cl. Science, Ser. IV **19** (1992), 591–613.
- [KM2] Kilpeläinen, T. and Malý, J., *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math. **172** (1994), 137–161.
- [LM] Lindqvist, P. and Martio, O., *Regularity and polar sets of supersolutions of certain degenerate elliptic equations*, J. Anal. Math. **50** (1988), 1–17.
- [M] Mikkonen, P., *On the Wolff potential and quasilinear elliptic equations involving measures*, Ann. Acad. Sci. Fenn. Ser. A I. Math. Dissertationes **104** (1996), 1–71.
- [R] Reshetnyak, Yu. G., *Space mappings with bounded distortion*, Translations of Mathematical Monographs 73. American Mathematical Society, Providence, RI, 1989.

Mailing address:

University of Jyväskylä
Department of Mathematics
P.O. Box 35
40351 Jyväskylä, Finland
e-mail: terok@math.jyu.fi