REMOVABLE SETS FOR CONTINUOUS SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS

TERO KILPELÄINEN AND XIAO ZHONG

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Abstract. We show that sets of $n - p + \alpha(p - 1)$ Hausdorff measure zero are removable for $\alpha$-Hölder continuous solutions to quasilinear elliptic equations similar to the $p$-Laplacian. The result is optimal. We also treat larger sets in terms of a growth condition. In particular, our results apply to quasiregular mappings.

1. Introduction

Throughout this paper we let $\Omega$ be an open set in $\mathbb{R}^n$ and $1 < p < \infty$ a fixed number. Continuous solutions $u \in W^{1,p}_{\text{loc}}(\Omega)$ of the equation

\begin{equation}
- \text{div}(A(x, \nabla u)) = 0
\end{equation}

are called $A$-harmonic in $\Omega$; here $A: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is assumed to verify for some constants $0 < \lambda \leq \Lambda < \infty$:

\begin{align*}
\text{(1.2) } & \quad \text{the function } x \mapsto A(x, \xi) \text{ is measurable for all } \xi \in \mathbb{R}^n, \text{ and} \\
& \quad \text{the function } \xi \mapsto A(x, \xi) \text{ is continuous for a.e. } x \in \mathbb{R}^n; \\
\text{for all } & \quad \xi \in \mathbb{R}^n \text{ and a.e. } x \in \mathbb{R}^n \\
\text{(1.3) } & \quad A(x, \xi) \cdot \xi \geq \lambda |\xi|^p, \\
\text{(1.4) } & \quad |A(x, \xi)| \leq \Lambda |\xi|^{p-1}, \\
\text{(1.5) } & \quad (A(x, \xi) - A(x, \zeta)) \cdot (\xi - \zeta) > 0
\end{align*}

whenever $\xi \neq \zeta$. A prime example of the operators is the $p$-Laplacian

\[-\Delta_p u = - \text{div}(|\nabla u|^{p-2}\nabla u),\]

in this case, the continuous solutions of (1.1) are called $p$-harmonic functions. The main result in this paper is the following theorem.
1.6. Theorem. Let \( E \subset \Omega \) be closed and \( s > 0 \). Suppose that \( u \) is a continuous function in \( \Omega \), \( A \)-harmonic in \( \Omega \setminus E \) such that
\[
|u(x_0) - u(y)| \leq C|x_0 - y|^{(s+p-n)/(p-1)}
\]
for all \( y \in \Omega \) and \( x_0 \in E \). If \( E \) is of \( s \)-Hausdorff measure zero, then \( u \) is \( A \)-harmonic in \( \Omega \).

Since sets of \( p \)-capacity zero are removable for bounded \( A \)-harmonic functions, Theorem 1.6 is interesting for \( s > n - p \) only. Kilpeläinen, Koskela, and Martio [KKM] had a special version of Theorem 1.6, where \( u \) was assumed to be flat on \( E \) and Hausdorff measure was replaced by a Minkowski content type condition.

1.8. Corollary. Suppose that \( u \in C^{0,\alpha}_0(\Omega) \), \( 0 < \alpha \leq 1 \), is \( A \)-harmonic in \( \Omega \setminus E \). If \( E \) is a closed set of \( n - p + \alpha(p-1) \) Hausdorff measure\(^1\) zero, then \( u \) is \( A \)-harmonic in \( \Omega \).

The following theorem shows that Corollary 1.8 is optimal. Before stating the theorem, we recall that there is a constant \( \kappa \), \( 0 < \kappa = \kappa(n, p, \lambda, \Lambda) \leq 1 \), such that every \( A \)-harmonic function \( h \) in \( \Omega \) verifies the local H"older continuity estimate
\[
\text{osc}(h, B(x, r)) \leq c\left(\frac{r}{R}\right)^\kappa \text{osc}(h, B(x, R))
\]
for each \( 0 < r < R \) and \( B(x, R) \subset \Omega \) [HKM, 6.6]. For smooth \( A \), in particular for the \( p \)-Laplacian, we may choose \( \kappa = 1 \) (see e.g. [K, 2.3]).

1.10. Theorem. Let \( \kappa \) be as above and \( 0 < \alpha < \kappa \). Suppose that \( E \subset \Omega \) is a closed set with positive \( n - p + \alpha(p-1) \) Hausdorff measure\(^1\). Then there is \( u \in C^{0,\alpha}(\Omega) \) which is \( A \)-harmonic in \( \Omega \setminus E \), but does not have an \( A \)-harmonic extension to \( \Omega \).

For the \( p \)-Laplacian we have the following sharp result.

1.11. Corollary. Let \( 0 < \alpha < 1 \). A closed set \( E \) is removable for \( \alpha \)-H"older continuous \( p \)-harmonic functions if and only if \( E \) is of \( n - p + \alpha(p-1) \) Hausdorff measure\(^1\) zero.

Carleson [C] proved Corollary 1.11 for the Laplacian \( (p = 2) \). As to the quasilinear case, Heinonen and Kilpeläinen [HK, 4.5] proved Corollary 1.8 with \( \alpha = 1 \), and Trudinger and Wang [TW] proved it under the assumption that \( u \) has an \( A \)-superharmonic extension to \( \Omega \), which assumption can be dispensed with for small \( \alpha \). However, in the general situation the growth condition of Theorem 1.6 yields a more useful result, since \( A \)-harmonic functions are not in general in \( C^{0,\alpha} \) for \( \alpha \) close to 1. Koskela and Martio [KM2] proved a weaker version of Corollary 1.13 and 1.8, where Minkowski content is used in place of Hausdorff measure. Buckley and Koskela [BK] also established very special cases of Corollary 1.8. In [K] there is a weaker version of Theorem 1.10.

A mapping \( f : \Omega \to \mathbb{R}^n \) is called quasiregular if \( f \in W^{1,n}_{\text{loc}}(\Omega) \) and there is a constant \( K \) such that
\[
|f'(x)|^n \leq K J_f(x)
\]
for a.e. \( x \in \Omega \); here \( J_f(x) \) is the Jacobian determinant of \( f \) at \( x \). The coordinate functions of a quasiregular map \( f \) satisfy an equation of type (1.1) with \( p = n \) (cf. [HKM, Ch. 14], whence we have:

\(^1\) Assume, of course that \( \alpha \geq (p - n)/(p-1) \).
1.12. Corollary. Let $E \subset \Omega$ be a closed set of $s$-Hausdorff measure zero, $0 < s \leq n$. Suppose that $f : \Omega \to \mathbb{R}^n$ is a continuous mapping quasiregular in $\Omega \setminus E$. If
$$|f(x_0) - f(y)| \leq C|x_0 - y|^{s/(n - 1)}$$
for all $y \in \Omega$ and $x_0 \in E$, then $f$ is quasiregular in $\Omega$.

1.13. Corollary. Suppose that $f \in C^{0, \alpha}(\Omega)$ is quasiregular in $\Omega \setminus E$. If $E$ is a closed set of $\alpha(n - 1)$-Hausdorff measure zero, then $f$ is quasiregular in $\Omega$.

Koskela and Martio [KM1] showed that sets whose Minkowski dimension is less than $\alpha n$ are removable for $\alpha$-Hölder continuous quasiregular mappings provided that $\alpha < 1 - 1/n$, and the same for sets of $\alpha n$-Hausdorff measure zero if $\alpha \leq 1/n$.

Our method of proof combines some ideas from [K], [L], and [TW]. We use solutions of equations
$$- \text{div} A(x, \nabla u) = \mu,$$
where $\mu$ is a nonnegative Radon measure from $W^{-1, p'}_{\text{loc}}(\Omega)$, i.e. $u \in W^{1, p}_{\text{loc}}(\Omega)$ and
$$\int_\Omega A(x, \nabla u) \cdot \nabla \varphi \, dx = \int_\Omega \varphi \, d\mu$$
for all $\varphi \in C^\infty_0(\Omega)$. In particular, we prove the following theorem that improves the main theorem in [K].

1.14. Theorem. Let $\kappa$ be the number given by (1.9). Suppose that $u \in W^{1, p}_{\text{loc}}(\Omega)$ is a solution of
$$- \text{div} A(x, \nabla u) = \mu,$$
where $\mu$ is a nonnegative Radon measure such that there are constants $M > 0$ and $0 < \alpha < \kappa$ with

$$(1.15) \quad \mu(B(x, r)) \leq Mr^{n-p+\alpha(p-1)}$$
whenever $B(x, 3r) \subset \Omega$. Then $u \in C^{0, \alpha}(\Omega)$. Moreover, $\kappa(n, p, 1, 1) = 1$, that is, in the case of the $p$-Laplacian any $\alpha < 1$ will do.

Theorem 1.14 is the best possible (see [KM, 4.18], [K, 2.7]).

Finally, we remark here that Corollary 1.11 is not true when $\alpha = 1$. The problem for which sets are removable for Lipschitz continuous $p$-harmonic functions is more delicate. David and Mattila [DM] treated the case $n = p = 2$: a compact set $E$ of finite 1-Hausdorff measure is removable for Lipschitz continuous harmonic functions if and only if $E$ is purely unrectifiable. The other cases remain open.

2. Proof of Theorem 1.6

We need a potential theoretic version of the obstacle problem. Suppose that $\psi$ is a continuous function on $\Omega$ and let the balayage $\hat{R}^\psi = \hat{R}^\psi(\Omega)$ be the pointwise infimum of all supersolutions\footnote{i.e. $u \in W^{1, p}_{\text{loc}}(\Omega)$ and $- \text{div} A(x, \nabla u) \geq 0$ in $\Omega$} $u$ to (1.1) that lie above $\psi$ in $\Omega$. Similarly, let $\check{R}^\psi = \check{R}^\psi(\Omega)$ be the pointwise supremum of all subsolutions that lie below $\psi$ in $\Omega$. Then $\hat{R}^\psi \geq \psi$ is a continuous supersolution in $\Omega$ and $A$-harmonic in $\{ \check{R}^\psi > \psi \}$; similar statements hold for $\hat{R}^\psi$. For a more thorough discussion see [HKM, Ch. 9].

Next we show the following estimate for the balayage; see [L] for a related result.
2.1. Lemma. Let $K \subset \Omega$ be compact. Suppose that $\psi$ is a continuous function with
\[ |\psi(x) - \psi(y)| \leq M|x - y|^\alpha \] for all $x \in K$ and $y \in \Omega,$
where $M > 0$ and $\alpha > 0$. Let $u = \hat{R}^\psi$ and
\[ \mu = - \div A(x, \nabla u). \]
Then
\[ \mu(B(x, r)) \leq cr^{n-p+\alpha(p-1)} \]
for all $r < r_0 = \frac{1}{64} \dist(K, \partial \Omega)$ and $x \in K$, here $c = c(n, p, \lambda, \Lambda, M, \alpha) > 0$.

Proof. Write
\[ I = \{ x \in \Omega : \psi(x) = u(x) \} \]
for the contact set.

First, let $x_0 \in I$. We assume, as we may, that $u(x_0) = 0 = \psi(x_0)$. If $r \leq \frac{1}{8} \dist(x_0, \partial \Omega)$ and
\[ \gamma_0 = \osc(\psi, B(x_0, 8r)), \]
then $(u - \gamma_0)^+$ is a subsolution and $u + \gamma_0$ a nonnegative supersolution in $B(x_0, 8r)$. Hence we deduce from the weak Harnack inequalities [HKM, 3.34 and 3.59] that
\[
\sup_{B(x_0, r)} (u - \gamma_0) \leq c \left( \int_{B(x_0, 2r)} |(u - \gamma_0)^+|^{p-1} \, dx \right)^{1/(p-1)} \\
\leq c \left( \int_{B(x_0, 2r)} (u + \gamma_0)^{p-1} \, dx \right)^{1/(p-1)} \\
\leq c \inf_{B(x_0, 2r)} (u + \gamma_0) \\
\leq c \gamma_0. \]
Keeping in mind that $u \geq \psi \geq -\gamma_0$ we conclude
\[ (2.2) \quad \osc(u, B(x_0, r)) \leq c \gamma_0 = c \osc(\psi, B(x_0, 8r)). \]

Let $r \leq \frac{1}{12} \dist(x_0, \partial \Omega)$ and let $\eta \in C_0^\infty(B(x_0, 2r))$ be a usual nonnegative cut-off function with $\eta = 1$ in $B(x_0, r)$ and $|\nabla \eta| \leq 2/r$. Then we obtain by applying the Caccioppoli estimate [HKM, 3.29] to $u - \sup_{B(x_0, 4r)} u$ and (2.2) that
\[
\mu(B(x_0, r)) = \int_{B(x_0, 2r)} \eta^p \, d\mu = p \int_{B(x_0, 2r)} \eta^{p-1} A(x, \nabla u) \cdot \nabla \eta \, dx \\
\leq c \left( \int_{B(x_0, 2r)} |\nabla u|^p \eta^p \, dx \right)^{(p-1)/p} \left( \int_{B(x_0, 2r)} |\nabla \eta|^p \, dx \right)^{1/p} \\
\leq cr^{n-p} \osc(u, B(x_0, 2r))^{p-1} \\
\leq cr^{n-p} \osc(\psi, B(x_0, 16r))^{p-1}. \]
Now if $x_0 \in I$ is such that
\[ \dist(x_0, K) \leq r \leq 2r_0, \]
we have the estimate

\( \mu(B(x_0, r)) \leq cr^{n-p+\alpha(p-1)} \),

where \( c = c(n, p, M) > 0 \).

Finally, for \( x_0 \in K \) and \( r < r_0 \), there are two alternatives. Either \( B(x_0, r) \cap I = \emptyset \) and thus \( \mu(B(x_0, r)) = 0 \), or there is \( x \in B(x_0, r) \cap I \). In this latter case

\[ \mu(B(x_0, r)) \leq \mu(B(x, 2r)) \leq cr^{n-p+\alpha(p-1)} \]

by (2.3). The lemma is proven.

\[ \square \]

**Remark.** Using (1.9) and (2.2), one can easily prove that if \( \psi \in C^{0,\alpha}_0(\Omega) \), then \( \hat{R} \psi \in C^{0,\beta}_0(\Omega) \), where \( \beta = \min(\alpha, \kappa) \) and \( \kappa > 0 \) is the constant such that (1.9) holds. (see e.g. [HKM, 6.47]).

**Proof of Theorem 1.6.** Fix a regular set \( D \subset \subset \Omega \), for instance a ball. Let \( v = \hat{R} u = \hat{R}^u(D) \) and

\[ \mu = -\text{div} \mathcal{A}(x, \nabla v). \]

Let \( K \subset E \) be compact. Since sets of \( n-p \) Hausdorff measure zero \( (p \leq n) \) are known to be removable for bounded \( \mathcal{A} \)-harmonic functions (see e.g. [HKM]), we need only consider the case, where \( \alpha = (s+p-n)/(p-1) > 0 \). Since \( s = n-p+\alpha(p-1) \) we infer from (1.7) and Lemma 2.1 that

\[ \mu(B(x, r)) \leq cr^s \]

for all \( r \leq r_0 \) and \( x \in K \). Because \( \mathcal{H}^s(K) = 0 \), we may cover \( K \) by balls \( B(x_j, r_j) \) so that

\[ \mu(K) \leq \sum_j \mu(B(x_j, r_j)) \leq c \sum_j r_j^s < \varepsilon, \]

where \( \varepsilon > 0 \) is given. Consequently, \( \mu(E) = 0 \) and therefore \( \mu = 0 \), which means that \( v \) is \( \mathcal{A} \)-harmonic in \( D \) [M, 3.19].

Next let \( w = \hat{R}^u(D) \). We similarly find that \( w \) is \( \mathcal{A} \)-harmonic in \( D \). Since \( v = u = w \) on \( \partial D \) by [HKM, 9.26], we have that \( v = w \) in \( D \) by the uniqueness of \( \mathcal{A} \)-harmonic functions. Since

\[ w \leq u \leq v = w, \]

\( u \) is \( \mathcal{A} \)-harmonic in \( D \) and the theorem follows.

\[ \square \]

3. **Proof of Theorems 1.14 and 1.10**

We recall that \( \kappa \) is the constant such that (1.9) holds for every \( \mathcal{A} \)-harmonic function \( h \) in \( \Omega \). Then

\[ \int_{B(x, r)} |\nabla h|^p \, dx \leq c \left( \frac{r}{R} \right)^{n-p+p\kappa} \int_{B(x, R)} |\nabla h|^p \, dx, \]

for each \( 0 < r < R \) with \( B(x, R) \subset \Omega \); here \( c = c(n, p, \lambda, \Lambda) > 0 \) (see e.g. [K, 2.1]).

The following lemma provides the key estimate.
3.2. Lemma. Let $u \in W^{1,p}(B(x_0, R))$ be a solution of
\[- \text{div} \, A(x, \nabla u) = \mu ,\]
where $\mu$ is a nonnegative Radon measure such that
\[\mu(B(x_0, r)) \leq c_0 \, r^{n-p+\alpha(p-1)}\]
for all $0 < r \leq R$. Then for each $0 < r < R$ and $\varepsilon > 0$ we have
\[\int_{B(x_0, r)} |\nabla u|^p \, dx \leq c_1 \left( \left( \frac{r}{R} \right)^{n-p+\alpha(p-1)} + \varepsilon \right) \int_{B(x_0, R)} |\nabla u|^p \, dx + c_2 \, R^{n-p+\alpha(p-1)},\]
where $c_1 = c_1(n, p, \lambda, \Lambda) > 0$ and $c_2 = c_2(n, p, \lambda, \alpha, c_0, \varepsilon) > 0$.

Proof. There is no loss of generality in assuming that $r < R/2$. Let $h$ be the $A$-harmonic function in $B(x_0, R)$ with $u - h \in W^{1,p}_0(B(x_0, R))$. Then
\[
\lambda \int_{B(x_0, r)} |\nabla u|^p \, dx \leq \int_{B(x_0, r)} A(x, \nabla u) \cdot \nabla u \, dx
\]
\[= \int_{B(x_0, r)} (A(x, \nabla u) - A(x, \nabla h)) \cdot (\nabla u - \nabla h) \, dx
\]
\[+ \int_{B(x_0, R)} A(x, \nabla h) \cdot (\nabla u - \nabla h) \, dx + \int_{B(x_0, r)} A(x, \nabla u) \cdot \nabla h \, dx
\]
\[\leq \int_{B(x_0, R)} (A(x, \nabla u) - A(x, \nabla h)) \cdot (\nabla u - \nabla h) \, dx
\]
\[+ \lambda \int_{B(x_0, r)} |\nabla h|^{p-1} |\nabla u| + |\nabla h| |\nabla u|^{p-1} \, dx
\]
where we used the structural assumptions (1.3)-(1.5). Since $h$ is $A$-harmonic with $h - u \in W^{1,p}_0(B(x_0, R))$ and thus quasiminimizes the $p$-Dirichlet integral, we have by using Adams’ inequality (see [AH, Thm 7.2.2] or [Z, Thm 4.7.2]) that
\[
\int_{B(x_0, R)} (A(x, \nabla u) - A(x, \nabla h)) \cdot (\nabla u - \nabla h) \, dx = \int_{B(x_0, R)} (u - h) \, d\mu
\]
\[\leq c \, R^{(p-1)(n-p+\alpha p)/p} \left( \int_{B(x_0, R)} |\nabla u - \nabla h|^p \, dx \right)^{1/p}
\]
\[\leq c \, R^{n-p+\alpha p} + \frac{\lambda}{2} \varepsilon \int_{B(x_0, R)} |\nabla u|^p \, dx ,
\]
where we also used Young’s inequality. The remaining integrals on the right of (3.3) do not exceed
\[
\frac{\lambda}{2} \int_{B(x_0, r)} |\nabla u|^p \, dx + c \int_{B(x_0, r)} |\nabla h|^p \, dx
\]
\[\leq \frac{\lambda}{2} \int_{B(x_0, r)} |\nabla u|^p \, dx + c \left( \frac{r}{R} \right)^{n-p+\alpha p} \int_{B(x_0, R)} |\nabla h|^p \, dx
\]
\[\leq \frac{\lambda}{2} \int_{B(x_0, r)} |\nabla u|^p \, dx + c \left( \frac{r}{R} \right)^{n-p+\alpha p} \int_{B(x_0, R)} |\nabla u|^p \, dx ,
\]
where we also employed (3.1) and the quasiminimizing property of $A$-harmonic functions. Plugging these estimates in (3.3) we arrive at

$$\int_{B(x_0, r)} |\nabla u|^p \, dx \leq c R^{n-p+\alpha p} + \varepsilon \int_{B(x_0, R)} |\nabla u|^p \, dx + c \left( \frac{r}{R} \right)^{n-p+\alpha p} \int_{B(x_0, R)} |\nabla u|^p \, dx .$$

The lemma follows.

**Proof of Theorem 1.14.** If $B(x_0, 4R) \subset \Omega$, then by appealing to [G, Lemma III.2.1, p. 86] Lemma 3.2 yields

$$\int_{B(x_0, r)} |\nabla u|^p \, dx \leq c \left( \frac{r}{R} \right)^{n-p+\alpha p}$$

for $r < R$. Thus $u \in C^{0,\alpha}(\Omega)$ by the Dirichlet growth theorem [G, Theorem III.1.1, p. 64].

**Proof of Theorem 1.10.** Let $\kappa$ be the number as in Theorem 1.14. Let $K \subset E$ be compact with $H^{n-p+\alpha(p-1)}(K) > 0$. Frostman’s lemma ([AH, 5.1.12], [C]) gives us a nonnegative Radon measure $\mu$ living on $K$ with $\mu(K) > 0$ and $\mu(B(x, r)) \leq r^{n-p+\alpha(p-1)}$. Any solution $u \in W^{1,p}_{\text{loc}}(\Omega)$ to

$$-\text{div} A(x, \nabla u) = \mu$$

is $A$-harmonic in $\Omega \setminus E$ [M, 3.19] and $u \in C^{0,\alpha}(\Omega)$ by Theorem 1.14, but $u$ fails to have an $A$-harmonic extension to $\Omega$, since $\mu(K) > 0$. 

**References**


University of Jyväskylä
Department of Mathematics
P.O. Box 35
40351 Jyväskylä, Finland

e-mail: terok@math.jyu.fi, zhong@math.jyu.fi