

REMOVABLE SETS FOR CONTINUOUS SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS

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ABSTRACT. We show that sets of $n - p + \alpha(p - 1)$ Hausdorff measure zero are removable for α -Hölder continuous solutions to quasilinear elliptic equations similar to the p -Laplacian. The result is optimal. We also treat larger sets in terms of a growth condition. In particular, our results apply to quasiregular mappings.

1. Introduction

Throughout this paper we let Ω be an open set in \mathbf{R}^n and $1 < p < \infty$ a fixed number. Continuous solutions $u \in W_{\text{loc}}^{1,p}(\Omega)$ of the equation

$$(1.1) \quad -\operatorname{div} \mathcal{A}(x, \nabla u) = 0$$

are called \mathcal{A} -harmonic in Ω ; here $\mathcal{A}: \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ is assumed to verify for some constants $0 < \lambda \leq \Lambda < \infty$:

$$(1.2) \quad \begin{aligned} &\text{the function } x \mapsto \mathcal{A}(x, \xi) \text{ is measurable for all } \xi \in \mathbf{R}^n, \text{ and} \\ &\text{the function } \xi \mapsto \mathcal{A}(x, \xi) \text{ is continuous for a.e. } x \in \mathbf{R}^n; \end{aligned}$$

for all $\xi \in \mathbf{R}^n$ and a.e. $x \in \mathbf{R}^n$

$$(1.3) \quad \mathcal{A}(x, \xi) \cdot \xi \geq \lambda |\xi|^p,$$

$$(1.4) \quad |\mathcal{A}(x, \xi)| \leq \Lambda |\xi|^{p-1},$$

$$(1.5) \quad (\mathcal{A}(x, \xi) - \mathcal{A}(x, \zeta)) \cdot (\xi - \zeta) > 0$$

whenever $\xi \neq \zeta$. A prime example of the operators is the p -Laplacian

$$-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2} \nabla u),$$

in this case, the continuous solutions of (1.1) are called p -harmonic functions. The main result in this paper is the following theorem.

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1.6. Theorem. *Let $E \subset \Omega$ be closed and $s > 0$. Suppose that u is a continuous function in Ω , \mathcal{A} -harmonic in $\Omega \setminus E$ such that*

$$(1.7) \quad |u(x_0) - u(y)| \leq C|x_0 - y|^{(s+p-n)/(p-1)}$$

for all $y \in \Omega$ and $x_0 \in E$. If E is of s -Hausdorff measure zero, then u is \mathcal{A} -harmonic in Ω .

Since sets of p -capacity zero are removable for bounded \mathcal{A} -harmonic functions, Theorem 1.6 is interesting for $s > n - p$ only. Kilpeläinen, Koskela, and Martio [KKM] had a special version of Theorem 1.6, where u was assumed to be flat on E and Hausdorff measure was replaced by a Minkowski content type condition.

1.8. Corollary. *Suppose that $u \in C^{0,\alpha}(\Omega)$, $0 < \alpha \leq 1$, is \mathcal{A} -harmonic in $\Omega \setminus E$. If E is a closed set of $n - p + \alpha(p - 1)$ Hausdorff measure¹ zero, then u is \mathcal{A} -harmonic in Ω .*

The following theorem shows that Corollary 1.8 is optimal. Before stating the theorem, we recall that there is a constant κ , $0 < \kappa = \kappa(n, p, \lambda, \Lambda) \leq 1$, such that every \mathcal{A} -harmonic function h in Ω verifies the local Hölder continuity estimate

$$(1.9) \quad \text{osc}(h, B(x, r)) \leq c\left(\frac{r}{R}\right)^\kappa \text{osc}(h, B(x, R))$$

for each $0 < r < R$ and $B(x, R) \subset \Omega$ [HKM, 6.6]. For smooth \mathcal{A} , in particular for the p -Laplacian, we may choose $\kappa = 1$ (see e.g. [K, 2.3]).

1.10. Theorem. *Let κ be as above and $0 < \alpha < \kappa$. Suppose that $E \subset \Omega$ is a closed set with positive $n - p + \alpha(p - 1)$ Hausdorff measure¹. Then there is $u \in C^{0,\alpha}(\Omega)$ which is \mathcal{A} -harmonic in $\Omega \setminus E$, but does not have an \mathcal{A} -harmonic extension to Ω .*

For the p -Laplacian we have the following sharp result.

1.11. Corollary. *Let $0 < \alpha < 1$. A closed set E is removable for α -Hölder continuous p -harmonic functions if and only if E is of $n - p + \alpha(p - 1)$ Hausdorff measure¹ zero.*

Carleson [C] proved Corollary 1.11 for the Laplacian ($p = 2$). As to the quasilinear case, Heinonen and Kilpeläinen [HK, 4.5] proved Corollary 1.8 with $\alpha = 1$, and Trudinger and Wang [TW] proved it under the assumption that u has an \mathcal{A} -superharmonic extension to Ω , which assumption can be dispensed with for small α . However, in the general situation the growth condition of Theorem 1.6 yields a more useful result, since \mathcal{A} -harmonic functions are not in general in $C^{0,\alpha}$ for α close to 1. Koskela and Martio [KM2] proved a weaker version of Corollary 1.13 and 1.8, where Minkowski content is used in place of Hausdorff measure. Buckley and Koskela [BK] also established very special cases of Corollary 1.8. In [K] there is a weaker version of Theorem 1.10.

A mapping $f: \Omega \rightarrow \mathbf{R}^n$ is called *quasiregular* if $f \in W_{\text{loc}}^{1,n}(\Omega)$ and there is a constant K such that

$$|f'(x)|^n \leq K J_f(x)$$

for a.e. $x \in \Omega$; here $J_f(x)$ is the Jacobian determinant of f at x . The coordinate functions of a quasiregular map f satisfy an equation of type (1.1) with $p = n$ (cf. [HKM, Ch. 14], whence we have:

¹Assume, of course that $\alpha \geq (p - n)/(p - 1)$.

1.12. Corollary. *Let $E \subset \Omega$ be a closed set of s -Hausdorff measure zero, $0 < s \leq n$. Suppose that $f: \Omega \rightarrow \mathbf{R}^n$ is a continuous mapping quasiregular in $\Omega \setminus E$. If*

$$|f(x_0) - f(y)| \leq C|x_0 - y|^{s/(n-1)}$$

for all $y \in \Omega$ and $x_0 \in E$, then f is quasiregular in Ω .

1.13. Corollary. *Suppose that $f \in C^{0,\alpha}(\Omega)$ is quasiregular in $\Omega \setminus E$. If E is a closed set of $\alpha(n-1)$ -Hausdorff measure zero, then f is quasiregular in Ω .*

Koskela and Martio [KM1] showed that sets whose Minkowski dimension is less than αn are removable for α -Hölder continuous quasiregular mappings provided that $\alpha < 1 - 1/n$, and the same for sets of αn -Hausdorff measure zero if $\alpha \leq 1/n$.

Our method of proof combines some ideas from [K], [L], and [TW]. We use solutions of equations

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = \mu,$$

where μ is a nonnegative Radon measure from $W_{\text{loc}}^{-1,p'}(\Omega)$, i.e. $u \in W_{\text{loc}}^{1,p}(\Omega)$ and

$$\int_{\Omega} \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx = \int_{\Omega} \varphi \, d\mu$$

for all $\varphi \in C_0^\infty(\Omega)$. In particular, we prove the following theorem that improves the main theorem in [K].

1.14. Theorem. *Let κ be the number given by (1.9). Suppose that $u \in W_{\text{loc}}^{1,p}(\Omega)$ is a solution of*

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = \mu,$$

where μ is a nonnegative Radon measure such that there are constants $M > 0$ and $0 < \alpha < \kappa$ with

$$(1.15) \quad \mu(B(x, r)) \leq Mr^{n-p+\alpha(p-1)}$$

whenever $B(x, 3r) \subset \Omega$. Then $u \in C^{0,\alpha}(\Omega)$. Moreover, $\kappa(n, p, 1, 1) = 1$, that is, in the case of the p -Laplacian any $\alpha < 1$ will do.

Theorem 1.14 is the best possible (see [KM, 4.18], [K, 2.7]).

Finally, we remark here that Corollary 1.11 is not true when $\alpha = 1$. The problem for which sets are removable for Lipschitz continuous p -harmonic functions is more delicate. David and Mattila [DM] treated the case $n = p = 2$: a compact set E of finite 1-Hausdorff measure is removable for Lipschitz continuous harmonic functions if and only if E is purely unrectifiable. The other cases remain open.

2. Proof of Theorem 1.6

We need a potential theoretic version of the obstacle problem. Suppose that ψ is a continuous function on Ω and let the *balayage* $\hat{R}^\psi = \hat{R}^\psi(\Omega)$ be the pointwise infimum of all supersolutions² u to (1.1) that lie above ψ in Ω . Similarly, let $\underline{\hat{R}}^\psi = \underline{\hat{R}}^\psi(\Omega)$ be the pointwise supremum of all subsolutions that lie below ψ in Ω . Then $\hat{R}^\psi \geq \psi$ is a continuous supersolution in Ω and \mathcal{A} -harmonic in $\{\hat{R}^\psi > \psi\}$; similar statements hold for $\underline{\hat{R}}^\psi$. For a more thorough discussion see [HKM, Ch. 9]. Next we show the following estimate for the balayage; see [L] for a related result.

²i.e. $u \in W_{\text{loc}}^{1,p}(\Omega)$ and $-\operatorname{div} \mathcal{A}(x, \nabla u) \geq 0$ in Ω

2.1. Lemma. *Let $K \subset \Omega$ be compact. Suppose that ψ is a continuous function with*

$$|\psi(x) - \psi(y)| \leq M|x - y|^\alpha \text{ for all } x \in K \text{ and } y \in \Omega,$$

where $M > 0$ and $\alpha > 0$. Let $u = \hat{R}^\psi$ and

$$\mu = -\operatorname{div} \mathcal{A}(x, \nabla u).$$

Then

$$\mu(B(x, r)) \leq cr^{n-p+\alpha(p-1)}$$

for all $r < r_0 = \frac{1}{64} \operatorname{dist}(K, \partial\Omega)$ and $x \in K$, here $c = c(n, p, \lambda, \Lambda, M, \alpha) > 0$.

Proof. Write

$$I = \{x \in \Omega : \psi(x) = u(x)\}$$

for the contact set.

First, let $x_0 \in I$. We assume, as we may, that $u(x_0) = 0 = \psi(x_0)$. If $r \leq \frac{1}{8} \operatorname{dist}(x_0, \partial\Omega)$ and

$$\gamma_0 = \operatorname{osc}(\psi, B(x_0, 8r)),$$

then $(u - \gamma_0)^+$ is a subsolution and $u + \gamma_0$ a nonnegative supersolution in $B(x_0, 8r)$. Hence we deduce from the weak Harnack inequalities [HKM, 3.34 and 3.59] that

$$\begin{aligned} \sup_{B(x_0, r)} (u - \gamma_0) &\leq c \left(\int_{B(x_0, 2r)} |(u - \gamma_0)^+|^{p-1} dx \right)^{1/(p-1)} \\ &\leq c \left(\int_{B(x_0, 2r)} (u + \gamma_0)^{p-1} dx \right)^{1/(p-1)} \\ &\leq c \inf_{B(x_0, 2r)} (u + \gamma_0) \\ &\leq c\gamma_0. \end{aligned}$$

Keeping in mind that $u \geq \psi \geq -\gamma_0$ we conclude

$$(2.2) \quad \operatorname{osc}(u, B(x_0, r)) \leq c\gamma_0 = c \operatorname{osc}(\psi, B(x_0, 8r)).$$

Let $r \leq \frac{1}{32} \operatorname{dist}(x_0, \partial\Omega)$ and let $\eta \in C_0^\infty(B(x_0, 2r))$ be a usual nonnegative cut-off function with $\eta = 1$ in $B(x_0, r)$ and $|\nabla\eta| \leq 2/r$. Then we obtain by applying the Caccioppoli estimate [HKM, 3.29] to $u - \sup_{B(x_0, 4r)} u$ and (2.2) that

$$\begin{aligned} \mu(B(x_0, r)) &\leq \int_{B(x_0, 2r)} \eta^p d\mu = p \int_{B(x_0, 2r)} \eta^{p-1} \mathcal{A}(x, \nabla u) \cdot \nabla \eta dx \\ &\leq c \left(\int_{B(x_0, 2r)} |\nabla u|^p \eta^p dx \right)^{(p-1)/p} \left(\int_{B(x_0, 2r)} |\nabla \eta|^p dx \right)^{1/p} \\ &\leq cr^{n-p} \operatorname{osc}(u, B(x_0, 2r))^{p-1} \\ &\leq cr^{n-p} \operatorname{osc}(\psi, B(x_0, 16r))^{p-1}. \end{aligned}$$

Now if $x_0 \in I$ is such that

$$\operatorname{dist}(x_0, K) \leq r \leq 2r_0,$$

we have the estimate

$$(2.3) \quad \mu(B(x_0, r)) \leq c r^{n-p+\alpha(p-1)},$$

where $c = c(n, p, M) > 0$.

Finally, for $x_0 \in K$ and $r < r_0$, there are two alternatives. Either $B(x_0, r) \cap I = \emptyset$ and thus $\mu(B(x_0, r)) = 0$, or there is $x \in B(x_0, r) \cap I$. In this latter case

$$\mu(B(x_0, r)) \leq \mu(B(x, 2r)) \leq c r^{n-p+\alpha(p-1)}$$

by (2.3). The lemma is proven. \square

Remark. Using (1.9) and (2.2), one can easily prove that if $\psi \in C^{0,\alpha}(\Omega)$, then $\hat{R}\psi \in C^{0,\beta}(\Omega)$, where $\beta = \min(\alpha, \kappa)$ and $\kappa > 0$ is the constant such that (1.9) holds. (see e.g. [HKM, 6.47]).

Proof of Theorem 1.6. Fix a regular set $D \subset\subset \Omega$, for instance a ball. Let $v = \hat{R}u = \hat{R}u(D)$ and

$$\mu = -\operatorname{div} \mathcal{A}(x, \nabla v).$$

Let $K \subset E$ be compact. Since sets of $n-p$ Hausdorff measure zero ($p \leq n$) are known to be removable for bounded \mathcal{A} -harmonic functions (see e.g. [HKM]), we need only consider the case, where $\alpha = (s+p-n)/(p-1) > 0$. Since $s = n-p+\alpha(p-1)$ we infer from (1.7) and Lemma 2.1 that

$$\mu(B(x, r)) \leq c r^s$$

for all $r \leq r_0$ and $x \in K$. Because $\mathcal{H}^s(K) = 0$, we may cover K by balls $B(x_j, r_j)$ so that

$$\mu(K) \leq \sum_j \mu(B(x_j, r_j)) \leq c \sum_j r_j^s < \varepsilon,$$

where $\varepsilon > 0$ is given. Consequently, $\mu(E) = 0$ and therefore $\mu = 0$, which means that v is \mathcal{A} -harmonic in D [M, 3.19].

Next let $w = \hat{R}u(D)$. We similarly find that w is \mathcal{A} -harmonic in D . Since $v = u = w$ on ∂D by [HKM, 9.26], we have that $v = w$ in D by the uniqueness of \mathcal{A} -harmonic functions. Since

$$w \leq u \leq v = w,$$

u is \mathcal{A} -harmonic in D and the theorem follows. \square

3. Proof of Theorems 1.14 and 1.10

We recall that κ is the constant such that (1.9) holds for every \mathcal{A} -harmonic function h in Ω . Then

$$(3.1) \quad \int_{B(x,r)} |\nabla h|^p dx \leq c \left(\frac{r}{R}\right)^{n-p+p\kappa} \int_{B(x,R)} |\nabla h|^p dx,$$

for each $0 < r < R$ with $B(x, R) \subset \Omega$; here $c = c(n, p, \lambda, \Lambda) > 0$ (see e.g. [K, 2.1]).

The following lemma provides the key estimate.

3.2. Lemma. *Let $u \in W^{1,p}(B(x_0, R))$ be a solution of*

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = \mu,$$

where μ is a nonnegative Radon measure such that

$$\mu(B(x_0, r)) \leq c_0 r^{n-p+\alpha(p-1)}$$

for all $0 < r \leq R$. Then for each $0 < r < R$ and $\varepsilon > 0$ we have

$$\int_{B(x_0, r)} |\nabla u|^p dx \leq c_1 \left(\left(\frac{r}{R} \right)^{n-p+p\kappa} + \varepsilon \right) \int_{B(x_0, R)} |\nabla u|^p dx + c_2 R^{n-p+p\alpha},$$

where $c_1 = c_1(n, p, \lambda, \Lambda) > 0$ and $c_2 = c_2(n, p, \lambda, \Lambda, \alpha, c_0, \varepsilon) > 0$.

Proof. There is no loss of generality in assuming that $r < R/2$. Let h be the \mathcal{A} -harmonic function in $B(x_0, R)$ with $u - h \in W_0^{1,p}(B(x_0, R))$. Then

$$\begin{aligned} \lambda \int_{B(x_0, r)} |\nabla u|^p dx &\leq \int_{B(x_0, r)} \mathcal{A}(x, \nabla u) \cdot \nabla u dx \\ &= \int_{B(x_0, r)} (\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla h)) \cdot (\nabla u - \nabla h) dx \\ (3.3) \quad &+ \int_{B(x_0, r)} \mathcal{A}(x, \nabla h) \cdot (\nabla u - \nabla h) dx + \int_{B(x_0, r)} \mathcal{A}(x, \nabla u) \cdot \nabla h dx \\ &\leq \int_{B(x_0, R)} (\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla h)) \cdot (\nabla u - \nabla h) dx \\ &\quad + \Lambda \int_{B(x_0, r)} |\nabla h|^{p-1} |\nabla u| + |\nabla h| |\nabla u|^{p-1} dx \end{aligned}$$

where we used the structural assumptions (1.3)-(1.5). Since h is \mathcal{A} -harmonic with $h - u \in W_0^{1,p}(B(x_0, R))$ and thus quasiminimizes the p -Dirichlet integral, we have by using Adams' inequality (see [AH, Thm 7.2.2] or [Z, Thm 4.7.2]) that

$$\begin{aligned} \int_{B(x_0, R)} (\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla h)) \cdot (\nabla u - \nabla h) dx &= \int_{B(x_0, R)} (u - h) d\mu \\ &\leq c R^{(p-1)(n-p+\alpha p)/p} \left(\int_{B(x_0, R)} |\nabla u - \nabla h|^p dx \right)^{1/p} \\ &\leq c R^{n-p+\alpha p} + \frac{\lambda}{2} \varepsilon \int_{B(x_0, R)} |\nabla u|^p dx, \end{aligned}$$

where we also used Young's inequality. The remaining integrals on the right of (3.3) do not exceed

$$\begin{aligned} &\frac{\lambda}{2} \int_{B(x_0, r)} |\nabla u|^p dx + c \int_{B(x_0, r)} |\nabla h|^p dx \\ &\leq \frac{\lambda}{2} \int_{B(x_0, r)} |\nabla u|^p dx + c \left(\frac{r}{R} \right)^{n-p+p\kappa} \int_{B(x_0, R)} |\nabla h|^p dx \\ &\leq \frac{\lambda}{2} \int_{B(x_0, r)} |\nabla u|^p dx + c \left(\frac{r}{R} \right)^{n-p+p\kappa} \int_{B(x_0, R)} |\nabla u|^p dx, \end{aligned}$$

where we also employed (3.1) and the quasiminimizing property of \mathcal{A} -harmonic functions. Plugging these estimates in (3.3) we arrive at

$$\begin{aligned} \int_{B(x_0, r)} |\nabla u|^p dx &\leq c R^{n-p+\alpha p} + \varepsilon \int_{B(x_0, R)} |\nabla u|^p dx \\ &\quad + c \left(\frac{r}{R}\right)^{n-p+p\kappa} \int_{B(x_0, R)} |\nabla u|^p dx. \end{aligned}$$

The lemma follows. \square

Proof of Theorem 1.14. If $B(x_0, 4R) \subset \Omega$, then by appealing to [G, Lemma III.2.1, p. 86] Lemma 3.2 yields

$$\int_{B(x_0, r)} |\nabla u|^p dx \leq c \left(\frac{r}{R}\right)^{n-p+p\alpha}$$

for $r < R$. Thus $u \in C^{0,\alpha}(\Omega)$ by the Dirichlet growth theorem [G, Theorem III.1.1, p. 64]. \square

Proof of Theorem 1.10. Let κ be the number as in Theorem 1.14. Let $K \subset E$ be compact with $\mathcal{H}^{n-p+\alpha(p-1)}(K) > 0$. Frostman's lemma ([AH, 5.1.12], [C]) gives us a nonnegative Radon measure μ living on K with $\mu(K) > 0$ and $\mu(B(x, r)) \leq r^{n-p+\alpha(p-1)}$. Any solution $u \in W_{\text{loc}}^{1,p}(\Omega)$ to

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = \mu$$

is \mathcal{A} -harmonic in $\Omega \setminus E$ [M, 3.19] and $u \in C^{0,\alpha}(\Omega)$ by Theorem 1.14, but u fails to have an \mathcal{A} -harmonic extension to Ω , since $\mu(K) > 0$. \square

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