NONLINEAR GROUND STATES IN IRREGULAR DOMAINS

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Abstract. In this note we show that the Wiener criterion characterizes the boundary points for which the nonlinear ground state attains its limit value 0.

1. Introduction

The objective of our note is to complement the nonlinear potential theory with a result about the nature of irregular boundary points for the Dirichlet problem. The interesting phenomenon in this connection for equations like the one arising from torisional creep problems,

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + 1 = 0\,, \tag{1.1}$$

where $1 < p < \infty$, is that all irregular boundary points are detected even by the solution with the value zero prescribed on the whole boundary. (Such a phenomenon is out of the question for the related equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$!) The boundary point is classified as *irregular* if the requirement

$$\lim_{x\to\xi} u(x) = 0 \tag{1.2}$$

is violated for the solution u belonging to the Sobolev space $W_0^{1,p}(\Omega)$, where Ω denotes the underlying domain in the Euclidean n-space. The solution is interpreted in the weak sense. A similar property is exhibited by the ground state of the nonlinear eigenvalue problem

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0 \tag{1.3}$$

with zero Dirichlet boundary values.

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Before proceeding, let us recall some facts in the classical potential theory. The ground state of

$$\Delta u + \lambda u = 0$$

attains its prescribed boundary value 0 at a given point $\xi \in \partial\Omega$ if the celebrated Wiener integral diverges,

$$\int_0^1 \frac{\operatorname{cap}_2(E \cap B(\xi, r), B(\xi, 2r))}{r^{n-2}} \frac{dr}{r} = \infty \,. \tag{1.4}$$

The electrostatic capacity cap_2 of the part of the complement of Ω that lies in the ball $\{x\colon |\xi - x| < r\}$ is involved. Thus (1.4) implies (1.2) for the ground state. It is the converse implication that is crucial from our point of view: it follows from Bouligand's theorem (see [H]) that (1.2) implies (1.4) for the ground state. However, this latter implication can be false for the excited states (the higher eigenfunctions with prescribed boundary value 0). The decisive feature is that the ground state does not change its sign in the domain Ω and it can itself act as a weak barrier.

Let us return to the nonlinear eigenvalue problem (1.3). There is a corresponding version of the Wiener integral, due to Maz'ya [M]. It will follow from our main result (Theorem 3.1) that the convergence of this Wiener integral characterizes the boundary points where the nonlinear ground state does not attain its boundary value zero in the classical sense (1.2).

1.5. Theorem. *Suppose that $\xi \in \partial\Omega$ and let u be the p-ground state in Ω. Then*

$$\lim_{x \to \xi} u(x) = 0$$

if and only if

$$\int_0^1 \left(\frac{\operatorname{cap}_p(\complement\Omega \cap B(\xi, r), B(\xi, 2r))}{r^{n-p}} \right)^{1/(p-1)} \frac{dr}{r} = \infty \,. \tag{1.6}$$

The p-capacity involved is explained in Section 2. The sufficiency of the divergence of the integral above follows from [GZ], which extends the nonlinear Wiener criterion of [M] to a wider class of nonlinear equations. As a matter of fact, for many equations of the type

$$\operatorname{div} \mathcal{A}(x, \nabla u) = 0$$

(1.6) is sufficient to guarantee that *any* prescribed continuous boundary values are attained in the classical sense. Among them is the p-Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0 \,.$$

The necessity is a consequence of Theorem 3.1 below. Our proof is not longer than the linear one, but it relies upon the knowledge that (1.6) is equivalent to the existence of a strong barrier in Ω. The advanced proof of this fundamental fact is given in [KM]. The situation is geometrically less complicated when $n - 1 < p \le n$; see [LM] for a simpler proof in that case. When $p > n$ there are no irregular boundary points, since then any function in $W_0^{1,p}(\Omega)$ can be regarded as continuous in $\overline{\Omega}$ and zero on the boundary. Let us finally mention that the last pages of [M] contain sharp examples of inward cusps, spines, and wedges. The celebrated Lebesgue spine corresponds to the case $p = 2$, $n = 3$.

2. Preliminaries

In this section we give some definitions and state the auxiliary results for the reader's convenience. We let Ω denote a bounded domain in $\mathbf{R}^n$. The Sobolev space $W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the usual Sobolev norm

$$\left(\int_\Omega |\varphi|^p\,dx + \int_\Omega |\nabla\varphi|^p\,dx\right)^{1/p}.$$

For a subset E of Ω, we define the p-capacity of E with respect to Ω as the number

$$\operatorname{cap}_p(E,\Omega) = \inf_u \int_\Omega |\nabla u|^p\,dx\,,$$

where the infimum is taken over all $u \in W_0^{1,p}(\Omega)$ such that $u \ge 1$ on an open neighborhood of E. This concept of capacity is naturally connected with equations involving operators akin to the p-Laplacian $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ (cf. [HKM]), and for $p = 2$ we have the familiar electrostatic capacity.

A set $E \subset \mathbf{R}^n$ is called *p-thin* at the point ξ if

$$\int_0^1 \left(\frac{\operatorname{cap}_p(E\cap B(\xi,r), B(\xi,2r))}{r^{n-p}}\right)^{1/(p-1)} \frac{dr}{r} < \infty\,.$$

If E is not p-thin at ξ, it is *p-thick* at ξ. In the present context E will usually be the complement $\complement\Omega$ of the domain Ω and $\xi \in \partial\Omega$.

The solutions to the nonlinear eigenvalue problem are interpreted in the weak sense: u is a solution of (1.3) if $u \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx - \lambda\int_\Omega |u|^{p-2}u\varphi\,dx = 0$$

for all $\varphi \in C_0^\infty(\Omega)$. The *$p$-ground state* is a positive solution to (1.3).

It is easily seen that if u is a solution, then $\varphi = u$ can be used as a test function, whence

$$\lambda = \frac{\int_\Omega |\nabla u|^p\,dx}{\int_\Omega |u|^p\,dx} \tag{2.1}$$

provided $u \ne 0$. The existence and many properties of a ground state come from the fact that it minimizes the Rayleigh quotient (2.1) above among all functions in $W_0^{1,p}(\Omega)$. The p-ground state is unique in an arbitrary domain, up to a multiplication by a constant. See [L] for a proof.

3. Irregular boundary points

The main result in this paper is the following.

3.1. Theorem. *Suppose that $\xi \in \partial\Omega$. Then the following conditions are equivalent:*

i) *The complement $\complement\Omega$ is p-thick at ξ.*
ii) *There is a positive supersolution u of the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ in Ω such that*

$$\lim_{x\to\xi} u(x) = 0\,.$$

Recall that u is a supersolution of $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ in Ω if $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \varphi \, dx \geq 0$$

for all nonnegative $\varphi \in C_0^\infty(\Omega)$.

The following lemma is the heart of the proof for Theorem 3.1.

3.2. Lemma. *Suppose that $\xi \in \partial\Omega$ is such that $\complement\Omega$ is p-thin at ξ. Then there is an open set $D \subset \Omega$ such that*

$$\partial D \cap \partial \Omega = \{\xi\}$$

and also $\complement D$ is p-thin at ξ.

Proof. Since $\complement\Omega$ is p-thin at ξ, there is an open set G that is p-thin at ξ and contains $\complement\Omega \setminus \{\xi\}$ [HKM, 12.11]. Let

$$E = \{y \colon |y - x| \leq \frac{1}{2} \operatorname{dist}(x, \complement G) \text{ for some } x \in \complement\Omega\}\,.$$

Then E is a closed set, p-thin at ξ, since $E \subset G \cup \{\xi\}$. Moreover, since each point $x \in \complement\Omega$, $x \neq \xi$, belongs to the interior of E we have that $D = \complement E$ is the desired open set. □

Proof of Theorem 3.1. The implication i) ⇒ ii) follows from Maz'ya's fundamental work [M]. Thus the Theorem is established once we show that ii) implies i). To this end, let u be a positive supersolution in Ω with

$$\lim_{x \to \xi} u(x) = 0$$

and assume, on the contrary that $\complement\Omega$ is p-thin at ξ. Let $D \subset \Omega$ be the open set provided by Lemma 3.2. Now for $y \in \partial D$, $y \neq \xi$, we have that

$$\liminf_{x \to y} u(x) \geq u(y) > 0\,,$$

since $y \in \Omega$. Therefore u is a (strong) barrier in D at ξ, and thus by [HKM, 9.8] ξ is a regular point in the sense described by the Perron process. Now appeal to [KM, Thm 5.4] to conclude that the complement of D is p-thick at ξ. This contradiction completes the proof. □

Now, theorem 1.5 follows immediately since the p-ground state is a positive supersolution. We also have a similar result for the torsional creep problem.

3.3. Corollary. *Suppose that $\xi \in \partial\Omega$. Then the following statements are equivalent:*

i) *The complement $\complement\Omega$ is p-thick at ξ.*
ii) *If $u \in W_0^{1,p}(\Omega)$ satisfies $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + 1 = 0$, then*

$$\lim_{x \to \xi} u(x) = 0\,.$$

It is worth our while mentioning that the two conditions above are equivalent to the following one:

iii) *The solution of the equation* $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ *with any prescribed continuous boundary values will attain the right value at the point* ξ.

Here the solutions are obtained via Perron's procedure, which works also for nonlinear equations like this one. If the boundary values φ happen to have a smooth extension to Ω, then the Perron solution is the unique solution u with $u - \varphi \in W_0^{1,p}(\Omega)$.

3.4. Remark. For the sake of the simple exposition we have chosen to consider the p-Laplacian operator only. However, similar results with same proofs as given above can be written for a wider class of equations, even for those involving weights, considered in [HKM]. In particular, the irregular boundary points for the equation

$$\operatorname{div} \mathcal{A}(x, \nabla u) = 0$$

can be detected by the solution of the equation

$$\operatorname{div} \mathcal{A}(x, \nabla u) + 1 = 0$$

with zero boundary values.

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