ON THE POROSITY OF FREE BOUNDARIES IN DEGENERATE VARIATIONAL INEQUALITIES

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Abstract. In this note we consider a certain degenerate variational problem with zero constraint. The exact growth of the solution near the free boundary is established. A consequence of this is that the free boundary is porous and therefore its Hausdorff dimension is less than N and hence it is of Lebesgue measure zero.

1. Preliminaries and the main result

In this paper we consider the obstacle problem for the nonhomogeneous p-Laplace equation $(1 < p < \infty)$

$$\operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x)) = f(x),$$

with zero obstacle. Given a bounded open subset Ω of $\mathbb{R}^N$, $N \geq 2$, and θ in $W^{1,p}(\Omega) \cap L^\infty(\Omega)$, we define

$$\mathcal{K}_\theta = \{v \in W^{1,p}(\Omega) : \ v - \theta \in W_0^{1,p}(\Omega), \ v \geq 0 \text{ a.e in } \Omega\}.$$

A function u in $\mathcal{K}_\theta$ is a *solution to the obstacle problem* if

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \cdot (\nabla v - \nabla u) + f(x)(v-u))dx \geq 0 \tag{1.1}$$

whenever $v \in \mathcal{K}_\theta$. According to a result of Choe and Lewis [CL] (see also [MZ]), the solution u to (1.1) lies in $W^{1,p}(\Omega) \cap C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$, provided $f \in L^q(\Omega)$ for some $q > N$. We will assume that $f \in L^\infty(\Omega)$.

The solution u to the obstacle problem satisfies

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f\chi_{\overline{\Omega}_+} - \mu, \tag{1.2}$$

weakly in Ω, where

$$\Omega_+ = \{x \in \Omega : u(x) > 0\}$$

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and μ is a nonnegative Radon measure with $\operatorname{supp}\mu \subset \partial\Omega_+$.

Plugging in (1.1) a test function $v = u + \eta$ with $\eta \in C_0^\infty(\Omega)$, $\eta \geq 0$, we see that $f - \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is a nonnegative distribution, hence a Radon measure. Since u vanishes outside $\overline{\Omega}_+$, this measure coincides with f there. To complete the proof of (1.2) we observe that if $\eta \in C_0^\infty(\Omega_+)$, then $u \geq \delta > 0$ in the support of η. Thus $v = u \pm \varphi$ with

$$\varphi = \delta \frac{\eta}{||\eta||_\infty},$$

are competing functions in $\mathcal{K}_\theta$. We conclude that $f - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ in Ω_+, and (1.2) is established.

As an opposite to (1.2) we have the following lemma.

Lemma 1.1. *Suppose that $u \in W^{1,p}(\Omega)$ is a nonnegative continuous function with*

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = g \quad \text{in } \Omega_+ = \{u > 0\},$$

where g is a signed Radon measure, living in Ω_+. Then there is a nonnegative Radon measure ν, supported on $\partial\Omega_+$ such that

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = g + \nu \quad \text{in } \Omega.$$

Proof. Let $\eta \in C_0^\infty(\Omega)$, $\eta \geq 0$. For $\varepsilon > 0$ define

$$\eta_\varepsilon = \eta\chi_\varepsilon,$$

where

$$\chi_\varepsilon = \begin{cases} 1 & \text{if } u(x) \geq 2\varepsilon \\ \frac{u(x)}{\varepsilon} - 1 & \text{if } \varepsilon < u(x) < 2\varepsilon \\ 0 & \text{if } u(x) \leq \varepsilon \end{cases}.$$

Then

$$\begin{aligned} -\langle \eta_\varepsilon, g\rangle &= \int_{\Omega_+} |\nabla u|^{p-2}\nabla u \cdot \nabla\eta_\varepsilon \, dx \\ &= \int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla\eta)\chi_\varepsilon \, dx + \frac{1}{\varepsilon}\int_{\varepsilon<u<2\varepsilon} |\nabla u|^p \eta \, dx \\ &\geq \int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla\eta)\chi_\varepsilon \, dx. \end{aligned}$$

Passing to the limit as $\varepsilon \to 0$, which is legitimate since $0 \leq \eta_\varepsilon \leq \eta$ and

$$\int_\Omega |\nabla u|^{p-1}|\nabla\eta| \, dx < \infty,$$

we obtain

$$-\langle \eta, g\rangle \geq \int_{\Omega_+} |\nabla u|^{p-2}\nabla u \cdot \nabla\eta \, dx = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\eta \, dx.$$

We have used that $\nabla u = 0$ a.e. on $\Omega \setminus \Omega_+$. The last inequality is equivalent to the statement of the lemma and the proof is completed. □

Lemma 1.2. *Suppose that u is a solution to the obstacle problem (1.1) in $\mathcal{K}_\theta$ with $f \in L^q(\Omega)$ for some $q > N$. Then u is continuous and*

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = h \tag{1.3}$$

weakly in Ω with $h \in L^q(\Omega)$ satisfying

$$f\chi_{\Omega_+} \leq h \leq f\chi_{\overline{\Omega}_+} \quad \text{a.e. in } \Omega. \tag{1.4}$$

If, in addition, $f \geq 0$ a.e. in Ω then

$$0 \leq u \leq \|\theta\|_{\infty,\Omega} \quad \text{in } \Omega. \tag{1.5}$$

Proof. As noted before u is even $C^{1,\alpha}$ regular; see [CL], [MZ]. Let h be a distribution defined by (1.3). From (1.2) and Lemma 1.1 with $g = f\chi_{\Omega_+}$ it follows that

$$h = f\chi_{\overline{\Omega}_+} - \mu = f\chi_{\Omega_+} + \nu \tag{1.6}$$

where both μ and ν are *nonnegative* Radon measures, supported on $\partial\Omega_+$. Further, (1.6) implies

$$\mu + \nu = f\chi_{\partial\Omega_+}.$$

In particular, since $f \in L^q(\Omega)$, it follows that $\mu, \nu \in L^q(\Omega)$ and therefore also $h \in L^q(\Omega)$. Inequality (1.4) follows now from (1.6).

To prove (1.5), we set $v = \min\{u, \|\theta\|_{\infty,\Omega}\} \in \mathcal{K}_\theta$ in (1.1), and use the assumption $f \geq 0$ to obtain $v = u$. Hence (1.5) follows.

The lemma is proved. □

To formulate the main result of this paper, we recall that a set E in $\mathbb{R}^N$ is called *porous* with *porosity constant* δ if there is an $r_0 > 0$ such that for each $x \in E$ and $0 < r < r_0$ there is a point y such that $B_{\delta r}(y) \subset B_r(x) \setminus E$. A porous set has Hausdorff dimension not exceeding $N - C\delta^N$, where $C = C(N) > 0$ is some constant (see e.g. Martio and Vuorinen [MV]). Consequently a porous set has Lebesgue measure zero.

Theorem 1.3. *Let u be a solution to the obstacle problem (1.1) in $\mathcal{K}_\theta$ with f satisfying*

$$0 < \lambda_0 \leq f \leq \Lambda_0 \quad \text{a.e. in } \Omega. \tag{1.7}$$

Then for every compact set $K \subset \Omega$ the intersection $\partial\Omega_+ \cap K$ is porous with porosity constant $\delta = \delta(\|\theta\|_{\infty,\Omega}, \lambda_0, \Lambda_0, \operatorname{dist}(K, \partial\Omega), p, N) > 0$.

We prove this theorem in section 3.

2. On a class of functions in the unit ball

The proof of Theorem 1.3 is based on the study of the following class of functions. We say that a function u in $W^{1,p}(B_1)$, where $B_1 = B_1(0)$ is the unit ball in $\mathbb{R}^N$, belongs to the class $\mathcal{G} = \mathcal{G}(p)$ $(1 < p < \infty)$ if

$$\|\operatorname{div}(|\nabla u|^{p-2}\nabla u)\|_\infty \leq 1; \tag{2.1}$$

$$0 \leq u \leq 1 \quad \text{a.e. in } B_1; \tag{2.2}$$

$$u(0) = 0. \tag{2.3}$$

Condition (2.1) is understood in the weak sense, i.e. $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = h$ weakly for $h \in L^\infty(B_1)$ with $\|h\|_\infty \leq 1$. Condition (2.3) makes sense since (2.1) and (2.2) provide that $u \in C^{1,\alpha}(B_1)$ for some $\alpha \in (0,1)$; (see e.g. [CL], [MZ]).

Theorem 2.1. *There is a positive constant $K = K(p, N)$ such that for every $u \in \mathcal{G}$, there holds*

$$|u(x)| \le K|x|^{p/(p-1)} \qquad \forall x \in B_1.$$

For a given nonnegative bounded function u, set

$$S(r, u, z) = \sup_{x \in B_r(z)} u(x), \qquad S(r, u) = S(r, u, 0)$$

and for u in $\mathcal{G}$ define $\mathbb{M}(u)$ to be the set of all nonnegative integers j such that the following doubling condition holds

$$2^{p/(p-1)} S(2^{-j-1}, u) \ge S(2^{-j}, u). \tag{2.4}$$

Lemma 2.2. *There exists a constant $K = K(p, N)$ such that*

$$S(2^{-j-1}, u) \le K \left(2^{-j}\right)^{p/(p-1)},$$

for all $u \in \mathcal{G}$, and $j \in \mathbb{M}(u)$.

Proof. We argue by contradiction. Thus we assume that for every $k \in \mathbb{N}$, there are $u_k \in \mathcal{G}$ and $j_k \in \mathbb{M}(u_k)$ such that

$$S(2^{-j_k-1}, u_k) \ge k \left(2^{-j_k}\right)^{p/(p-1)}. \tag{2.5}$$

Define now

$$\tilde{u}_k(x) := \frac{u_k(2^{-j_k}x)}{S(2^{-j_k-1}, u_k)} \qquad \text{in } B_1. \tag{2.6}$$

Then it follows from the definition of $\mathbb{M}(u)$ and $\mathcal{G}$ that

$$0 \le \tilde{u}_k \le 2^{p/(p-1)} \qquad \text{(by (2.1))} \tag{2.7}$$

$$\sup\nolimits_{B_{(1/2)}} |\tilde{u}_k| = 1 \qquad \text{(by (2.6))} \tag{2.8}$$

$$\tilde{u}_k(0) = 0 \qquad \text{(by (2.3))} \tag{2.9}$$

Now we have by (2.1) and (2.5) that

$$\|\mathrm{div}(|\nabla \tilde{u}_k|^{p-2} \nabla \tilde{u}_k)\|_\infty \le k^{1-p}. \tag{2.10}$$

Invoking Harnack inequalities and Hölder estimates of solutions (see e.g. [Se]) we infer that a subsequence of $\tilde{u}_k$ converges locally uniformly in B_1 to a function u. Moreover, the limit function $u \not\equiv 0$, by (2.8), and it satisfies by (2.9) and (2.10)

$$\mathrm{div}(|\nabla u|^{p-2} \nabla u) = 0, \qquad u(0) = 0, \qquad u \ge 0,$$

in B_1. This, however, contradicts the strict minimum principle (see [HKM, 7.12]) and the lemma follows. □

Proof of Theorem 2.1. We first claim that

$$S(2^{-j}, u) \le K(2^{-j+1})^{p/(p-1)} \tag{2.11}$$

for all $j \in \mathbb{N}$, where K is the constant in Lemma 2.2. Without loss of generality we may assume that $K \ge 1$. Thus (2.11) holds for $j = 0$. Next, let (2.11) hold for some $j \in \mathbb{N}$. Then it holds also for $j + 1$. Indeed, if $j \in \mathbb{M}(u)$ then this follows from Lemma 2.2. Otherwise, (2.4) fails and we obtain

$$S(2^{-j-1}, u) \le 2^{-p/(p-1)} S(2^{-j}, u) \le 2^{-p/(p-1)} K(2^{-j+1})^{p/(p-1)} = K(2^{-j})^{p/(p-1)}.$$

Thus (2.11) is established.

To complete the proof, let $2^{-j-1} \le r \le 2^{-j}$. Then by (2.11)

$$S(r, u) \le S(2^{-j}, u) \le K \left(2^{-j-1}\right)^{p/(p-1)} \le K r^{p/(p-1)},$$

and the theorem is proved. □

3. Proof of Theorem 1.3

The next lemma shows that Theorem 2.1 gives, in a sense, the exact growth of the solution to the obstacle problem (1.1) near the free boundary $\partial\Omega_+$. The lemma originates from the paper of Caffarelli [Ca].

Lemma 3.1. *Suppose that* $u \in W^{1,p}(\Omega)$ *is a nonnegative continuous function satisfying*

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f$$

weakly in $\Omega_+ = \{u > 0\}$ *with* f *as in (1.7). Then for every* $z \in \overline{\Omega}_+$ *and* $r > 0$ *with* $B_r(z) \subset \Omega$

$$S(r, u, z) \geq C_0 r^{p/(p-1)} + u(z),$$

where $C_0 = (1 - 1/p)\,(\lambda_0/N)^{1/(p-1)}$ *.*

Proof. First suppose that $z \in \Omega_+$, and for small $\varepsilon > 0$ set

$$w_\varepsilon(x) = u(x) - u(z)(1-\varepsilon), \qquad v(x) = C_0|x - z|^{p/(p-1)}.$$

Then $\operatorname{div}(|\nabla v|^{p-2}\nabla v) = \lambda_0$ and therefore

$$\operatorname{div}(|\nabla w_\varepsilon|^{p-2}\nabla w_\varepsilon) = \operatorname{div}(|\nabla u|^{p-2}\nabla u) \geq \operatorname{div}(|\nabla v|^{p-2}\nabla v)$$

in $\Omega_+ \cap B_r(z)$, and $w_\varepsilon \leq v$ on $\partial\Omega_+ \cap B_r(z)$. If also $w_\varepsilon \leq v$ on $\partial B_r(z) \cap \Omega$, then we may apply the comparison principle to obtain $w_\varepsilon \leq v$ in $B_r(z) \cap \Omega_+$, which contradicts to the fact that $w_\varepsilon(z) = \varepsilon u(z) > 0 = v(z)$. Hence

$$\sup_{\partial B_r(z)} w_\varepsilon \geq \sup_{\partial B_r(z)} v = C_0 r^{p/(p-1)}.$$

Letting $\varepsilon \to 0$, we obtain the desired result, for all $z \in \Omega_+$, and by continuity for all $z \in \overline{\Omega}_+$. The proof is completed. □

Proof of Theorem 1.3. Without loss of generality we may assume that the compact K in Theorem 1.3 is the closed unit ball $\overline{B}_1$, and moreover that $\overline{B}_2 \subset \Omega$.

For $x \in \Omega_+ \cap \overline{B}_1$ define

$$d(x) = \operatorname{dist}(x, \overline{B}_1 \setminus \Omega_+)$$

and take $z_x \in \partial\Omega_+ \cap \overline{B}_1$ with $|x - z_x| = d(x)$. Let

$$\tilde{u}(y) = u(z_x + y) \quad \text{for } y \in B_1.$$

Then, using Lemma 1.2 and condition (1.7), we see that

$$\|\operatorname{div}(|\nabla\tilde{u}|^{p-2}\nabla\tilde{u})\|_\infty \leq \Lambda_0, \qquad 0 \leq \tilde{u} \leq \|\theta\|_{\infty,\Omega}, \qquad \tilde{u}(0) = 0.$$

Therefore if $M = \max\{\Lambda_0^{1/(p-1)}, \|\theta\|_{\infty,\Omega}\}$, then $\tilde{u}/M$ is in $\mathcal{G}$ and we infer by Theorem 2.1 that

$$u(x) = \tilde{u}(x - z_x) \leq MK|x - z_x|^{p/(p-1)} = MKd(x)^{p/(p-1)}. \tag{3.1}$$

Next, let $z \in \partial\Omega_+ \cap \overline{B}_1$. Then for $0 < r < 1$, according to Lemma 3.1, there exists $x_z \in \partial B_r(z)$, such that

$$u(x_z) \geq C_0 r^{p/(p-1)}.$$

Then by (3.1)

$$C_0 r^{p/(p-1)} \leq u(x_z) \leq MKd(x_z)^{p/(p-1)},$$

which implies that

$$d(x_z) \geq \delta r, \qquad \delta = \left(\frac{C_0}{MK}\right)^{(p-1)/p},$$

or equivalently,

$$B_{\delta r}(x_z) \cap B_r(z) \subset \Omega_+.$$

Note that $\delta \leq 1$. Since $x_z \in \partial B_r(z)$, there is a ball

$$B_{(\delta/2)r}(y) \subset B_{\delta r}(x) \cap B_r(z) \subset B_r(z) \setminus \partial\Omega_+.$$

This shows that $\partial\Omega_+ \cap \overline{B}_1$ is porous with the porosity constant $\delta/2$. The theorem is proved. □

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