ON THE POROSITY OF FREE BOUNDARIES IN DEGENERATE VARIATIONAL INEQUALITIES

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Abstract. In this note we consider a certain degenerate variational problem with zero constraint. The exact growth of the solution near the free boundary is established. A consequence of this is that the free boundary is porous and therefore its Hausdorff dimension is less than $N$ and hence it is of Lebesgue measure zero.

1. Preliminaries and the main result

In this paper we consider the obstacle problem for the nonhomogeneous $p$-Laplace equation ($1 < p < \infty$)

$$
\text{div}(|\nabla u(x)|^{p-2}\nabla u(x)) = f(x),
$$

with zero obstacle. Given a bounded open subset $\Omega$ of $\mathbb{R}^N$, $N \geq 2$, and $\theta$ in $W^{1,p}(\Omega) \cap L^\infty(\Omega)$, we define

$$
K_\theta = \{ v \in W^{1,p}(\Omega) : v - \theta \in W^{1,p}_0(\Omega), v \geq 0 \text{ a.e in } \Omega \}.
$$

A function $u$ in $K_\theta$ is a solution to the obstacle problem if

$$
(1.1) \quad \int_\Omega (|\nabla u|^{p-2}\nabla u \cdot (\nabla v - \nabla u) + f(x)(v - u))dx \geq 0
$$

whenever $v \in K_\theta$. According to a result of Choe and Lewis [CL] (see also [MZ]), the solution $u$ to (1.1) lies in $W^{1,p}(\Omega) \cap C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$, provided $f \in L^q(\Omega)$ for some $q > N$. We will assume that $f \in L^\infty(\Omega)$.

The solution $u$ to the obstacle problem satisfies

$$
(1.2) \quad \text{div}(|\nabla u|^{p-2}\nabla u) = f\chi_{\Omega^+} - \mu,
$$

weakly in $\Omega$, where

$$
\Omega^+ = \{ x \in \Omega : u(x) > 0 \}
$$

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and \( \mu \) is a nonnegative Radon measure with \( \text{supp}\mu \subset \partial\Omega_+ \).

Plugging in (1.1) a test function \( v = u + \eta \) with \( \eta \in C_0^\infty(\Omega) \), \( \eta \geq 0 \), we see that \( f - \text{div}(|\nabla u|^{p-2}\nabla u) \) is a nonnegative distribution, hence a Radon measure. Since \( u \) vanishes outside \( \Omega_+ \), this measure coincides with \( f \) there. To complete the proof of (1.2) we observe that if \( \eta \in C_0^\infty(\Omega_+) \), then \( u \geq \delta > 0 \) in the support of \( \eta \). Thus 
\[
v = u \pm \varphi \text{ with } \varphi = \delta \frac{\eta}{||\eta||_\infty},
\]
are competing functions in \( K_\theta \). We conclude that \( f - \text{div}(|\nabla u|^{p-2}\nabla u) = 0 \) in \( \Omega_+ \), and (1.2) is established.

As an opposite to (1.2) we have the following lemma.

**Lemma 1.1.** Suppose that \( u \in W^{1,p}(\Omega) \) is a nonnegative continuous function with 
\[
\text{div}(|\nabla u|^{p-2}\nabla u) = g \text{ in } \Omega_+ = \{ u > 0 \},
\]
where \( g \) is a signed Radon measure, living in \( \Omega_+ \). Then there is a nonnegative Radon measure \( \nu \), supported on \( \partial\Omega_+ \) such that 
\[
\text{div}(|\nabla u|^{p-2}\nabla u) = g + \nu \text{ in } \Omega.
\]

**Proof.** Let \( \eta \in C_0^\infty(\Omega) \), \( \eta \geq 0 \). For \( \varepsilon > 0 \) define 
\[
\eta_\varepsilon = \eta \chi_\varepsilon,
\]
where
\[
\chi_\varepsilon = \begin{cases} 
1 & \text{if } u(x) \geq 2\varepsilon \\
\frac{u(x)}{\varepsilon} - 1 & \text{if } \varepsilon < u(x) < 2\varepsilon \\
0 & \text{if } u(x) \leq \varepsilon
\end{cases}
\]
Then
\[
-\langle \eta_\varepsilon, g \rangle = \int_{\Omega_+} |\nabla u|^{p-2}\nabla u \cdot \nabla \eta_\varepsilon \, dx
\]
\[
= \int_{\Omega} (|\nabla u|^{p-2}\nabla u \cdot \nabla \eta) \chi_\varepsilon \, dx + \frac{1}{\varepsilon} \int_{\varepsilon < u < 2\varepsilon} |\nabla u|^{p-1} \eta \, dx
\]
\[
\geq \int_{\Omega} (|\nabla u|^{p-2}\nabla u \cdot \nabla \eta) \chi_\varepsilon \, dx.
\]
Passing to the limit as \( \varepsilon \to 0 \), which is legitimate since \( 0 \leq \eta_\varepsilon \leq \eta \) and 
\[
\int_{\Omega} |\nabla u|^{p-1} |\nabla \eta| \, dx < \infty,
\]
we obtain
\[
-\langle \eta, g \rangle \geq \int_{\Omega_+} |\nabla u|^{p-2}\nabla u \cdot \nabla \eta \, dx = \int_{\Omega} |\nabla u|^{p-2}\nabla u \cdot \nabla \eta \, dx.
\]
We have used that \( \nabla u = 0 \) a.e. on \( \Omega \setminus \Omega_+ \). The last inequality is equivalent to the statement of the lemma and the proof is completed. \( \square \)
Lemma 1.2. Suppose that $u$ is a solution to the obstacle problem (1.1) in $K_\theta$ with $f \in L^q(\Omega)$ for some $q > N$. Then $u$ is continuous and

$$\text{(1.3)} \quad \text{div}(|\nabla u|^{p-2}\nabla u) = h$$

weakly in $\Omega$ with $h \in L^q(\Omega)$ satisfying

$$\text{(1.4)} \quad f \chi_{\Omega_+} \leq h \leq f \chi_{\Omega_+} \quad \text{a.e. in } \Omega.$$

If, in addition, $f \geq 0$ a.e. in $\Omega$ then

$$\text{(1.5)} \quad 0 \leq u \leq \|\theta\|_{\infty, \Omega} \quad \text{in } \Omega.$$

Proof. As noted before $u$ is even $C^{1,\alpha}$ regular; see [CL], [MZ]. Let $h$ be a distribution defined by (1.3). From (1.2) and Lemma 1.1 with $g = f \chi_{\Omega_+}$ it follows that

$$\text{(1.6)} \quad h = f \chi_{\Omega_+} - \mu = f \chi_{\Omega_+} + \nu$$

where both $\mu$ and $\nu$ are nonnegative Radon measures, supported on $\partial \Omega_+$. Further, (1.6) implies

$$\mu + \nu = f \chi_{\partial \Omega_+}.$$

In particular, since $f \in L^q(\Omega)$, it follows that $\mu, \nu \in L^q(\Omega)$ and therefore also $h \in L^q(\Omega)$. Inequality (1.4) follows now from (1.6).

To prove (1.5), we set $v = \min\{u, \|\theta\|_{\infty, \Omega}\} \in K_\theta$ in (1.1), and use the assumption $f \geq 0$ to obtain $v = u$. Hence (1.5) follows.

The lemma is proved. □

To formulate the main result of this paper, we recall that a set $E$ in $\mathbb{R}^N$ is called porous with porosity constant $\delta$ if there is an $r_0 > 0$ such that for each $x \in E$ and $0 < r < r_0$ there is a point $y$ such that $B_{\delta r}(y) \subset B_r(x) \setminus E$. A porous set has Hausdorff dimension not exceeding $N - C\delta^N$, where $C = C(N) > 0$ is some constant (see e.g. Martio and Vuorinen [MV]). Consequently a porous set has Lebesgue measure zero.

Theorem 1.3. Let $u$ be a solution to the obstacle problem (1.1) in $K_\theta$ with $f$ satisfying

$$\text{(1.7)} \quad 0 < \lambda_0 \leq f \leq \Lambda_0 \quad \text{a.e. in } \Omega.$$

Then for every compact set $K \subset \Omega$ the intersection $\partial \Omega_+ \cap K$ is porous with porosity constant $\delta = \delta(\|\theta\|_{\infty, \Omega}, \lambda_0, \Lambda_0, \text{dist}(K, \partial \Omega), p, N) > 0$.

We prove this theorem in section 3.

2. On a class of functions in the unit ball

The proof of Theorem 1.3 is based on the study of the following class of functions. We say that a function $u$ in $W^{1,p}(B_1)$, where $B_1 = B_1(0)$ is the unit ball in $\mathbb{R}^N$, belongs to the class $G = G(p)$ ($1 < p < \infty$) if

$$\text{(2.1)} \quad \|\text{div}(|\nabla u|^{p-2}\nabla u)\|_{\infty} \leq 1;$$

$$\text{(2.2)} \quad 0 \leq u \leq 1 \quad \text{a.e. in } B_1;$$

$$\text{(2.3)} \quad u(0) = 0.$$

Condition (2.1) is understood in the weak sense, i.e. $\text{div}(|\nabla u|^{p-2}\nabla u) = h$ weakly for $h \in L^\infty(B_1)$ with $\|h\|_{\infty} \leq 1$. Condition (2.3) makes sense since (2.1) and (2.2) provide that $u \in C^{1,\alpha}(B_1)$ for some $\alpha \in (0, 1)$; (see e.g. [CL], [MZ]).
Theorem 2.1. There is a positive constant $K = K(p, N)$ such that for every $u \in G$, there holds
\[ |u(x)| \leq K |x|^{p/(p-1)} \quad \forall x \in B_1. \]

For a given nonnegative bounded function $u$, set
\[ S(r, u, z) = \sup_{x \in B_r(z)} u(x), \quad S(r, u) = S(r, u, 0) \]
and for $u$ in $G$ define $M(u)$ to be the set of all nonnegative integers $j$ such that the following doubling condition holds
\[ 2^{p/(p-1)} S(2^{-j-1}, u) \geq S(2^{-j}, u). \]

Lemma 2.2. There exists a constant $K = K(p, N)$ such that
\[ S(2^{-j-1}, u) \leq K (2^{-j})^{p/(p-1)}, \]
for all $u \in G$, and $j \in M(u)$.

Proof. We argue by contradiction. Thus we assume that for every $k \in \mathbb{N}$, there are $u_k \in G$ and $j_k \in M(u_k)$ such that
\[ S(2^{-j_k-1}, u_k) \geq k (2^{-j_k})^{p/(p-1)}. \]

Define now
\[ \tilde{u}_k(x) := \frac{u_k(2^{-j_k}x)}{S(2^{-j_k-1}, u_k)} \quad \text{in} \ B_1. \]

Then it follows from the definition of $M(u)$ and $G$ that
\[ 0 \leq \tilde{u}_k \leq 2^{p/(p-1)} \quad \text{(by (2.1))}, \]
\[ \sup_{B_{1/2}} |\tilde{u}_k| = 1 \quad \text{(by (2.6))}, \]
\[ \tilde{u}_k(0) = 0 \quad \text{(by (2.3))}. \]

Now we have by (2.1) and (2.5) that
\[ ||\text{div}(|\nabla \tilde{u}_k|^{p-2} \nabla \tilde{u}_k)||_{\infty} \leq k^{1-p}. \]

Invoking Harnack inequalities and Hölder estimates of solutions (see e.g. [Se]) we infer that a subsequence of $\tilde{u}_k$ converges locally uniformly in $B_1$ to a function $u$. Moreover, the limit function $u \neq 0$, by (2.8), and it satisfies by (2.9) and (2.10)
\[ \text{div}(|\nabla u|^{p-2} \nabla u) = 0, \quad u(0) = 0, \quad u \geq 0, \]
in $B_1$. This, however, contradicts the strict minimum principle (see [HKM, 7.12]) and the lemma follows. $\Box$

Proof of Theorem 2.1. We first claim that
\[ S(2^{-j}, u) \leq K (2^{-j+1})^{p/(p-1)} \]
for all $j \in \mathbb{N}$, where $K$ is the constant in Lemma 2.2. Without loss of generality we may assume that $K \geq 1$. Thus (2.11) holds for $j = 0$. Next, let (2.11) hold for some $j \in \mathbb{N}$. Then it holds also for $j + 1$. Indeed, if $j \in M(u)$ then this follows from Lemma 2.2. Otherwise, (2.4) fails and we obtain
\[ S(2^{-j-1}, u) \leq 2^{-p/(p-1)} S(2^{-j}, u) \leq 2^{-p/(p-1)} K (2^{-j+1})^{p/(p-1)} = K (2^{-j})^{p/(p-1)}. \]

Thus (2.11) is established.

To complete the proof, let $2^{-j-1} \leq r \leq 2^{-j}$. Then by (2.11)
\[ S(r, u) \leq S(2^{-j}, u) \leq K (2^{-j-1})^{p/(p-1)} \leq K r^{p/(p-1)}, \]
and the theorem is proved. $\Box$
The next lemma shows that Theorem 2.1 gives, in a sense, the exact growth of the solution to the obstacle problem (1.1) near the free boundary \( \partial \Omega_+ \). The lemma originates from the paper of Caffarelli [Ca].

**Lemma 3.1.** Suppose that \( u \in W^{1,p}(\Omega) \) is a nonnegative continuous function satisfying
\[
\text{div}(|\nabla u|^{p-2} \nabla u) = f
\]
weakly in \( \Omega_+ = \{ u > 0 \} \) with \( f \) as in (1.7). Then for every \( z \in \overline{\Omega}_+ \) and \( r > 0 \) with \( B_r(z) \subset \Omega \)
\[
S(r, u, z) \geq C_0 r^{p/(p-1)} + u(z),
\]
where \( C_0 = (1 - 1/p) (\lambda_0/N)^{1/(p-1)} \).

**Proof.** First suppose that \( z \in \Omega_+ \), and for small \( \varepsilon > 0 \) set
\[
w_\varepsilon(x) = u(x) - u(z)(1 - \varepsilon), \quad v(x) = C_0 |x - z|^{p/(p-1)}.
\]
Then \( \text{div}(|\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon) = \lambda_0 \) and therefore
\[
\text{div}(|\nabla w_\varepsilon|^{p-2} \nabla w_\varepsilon) = \text{div}(|\nabla u|^{p-2} \nabla u) \geq \text{div}(|\nabla v|^{p-2} \nabla v)
\]
in \( \Omega_+ \cap B_r(z) \), and \( w_\varepsilon \leq v \) on \( \partial \Omega_+ \cap B_r(z) \). If also \( w_\varepsilon \leq v \) on \( \partial B_r(z) \cap \Omega_+ \), then we may apply the comparison principle to obtain \( w_\varepsilon \leq v \) in \( B_r(z) \cap \Omega_+ \), which contradicts to the fact that \( w_\varepsilon(z) = \varepsilon u(z) > 0 = v(z) \). Hence
\[
\sup_{\partial B_r(z)} w_\varepsilon \geq \sup_{\partial B_r(z)} v = C_0 r^{p/(p-1)}.
\]
Letting \( \varepsilon \to 0 \), we obtain the desired result, for all \( z \in \Omega_+ \), and by continuity for all \( z \in \overline{\Omega}_+ \). The proof is completed. \( \square \)

**Proof of Theorem 1.3.** Without loss of generality we may assume that the compact \( K \) in Theorem 1.3 is the closed unit ball \( B_1 \), and moreover that \( B_2 \subset \Omega \).

For \( x \in \Omega_+ \cap \overline{B_1} \) define
\[
d(x) = \text{dist}(x, \overline{B}_1 \setminus \Omega_+)
\]
and take \( z_x \in \partial \Omega_+ \cap \overline{B}_1 \) with \( |x - z_x| = d(x) \). Let
\[
\tilde{u}(y) = u(z_x + y) \quad \text{for } y \in B_1.
\]
Then, using Lemma 1.2 and condition (1.7), we see that
\[
\|\text{div}(|\nabla \tilde{u}|^{p-2} \nabla \tilde{u})\|_{\infty} \leq \Lambda_0, \quad 0 \leq \tilde{u} \leq \|\theta\|_{\infty, \Omega}, \quad \tilde{u}(0) = 0.
\]
Therefore if \( M = \max\{\Lambda_0^{1/(p-1)} \|\theta\|_{\infty, \Omega}\} \), then \( \tilde{u}/M \) is in \( G \) and we infer by Theorem 2.1 that
\[
u(x) = \tilde{u}(x - z_x) \leq M K |x - z_x|^{p/(p-1)} = M K d(x)^{p/(p-1)}.
\]
Next, let $z \in \partial \Omega_+ \cap \overline{B}_1$. Then for $0 < r < 1$, according to Lemma 3.1, there exists $x_z \in \partial B_r(z)$, such that

$$u(x_z) \geq C_0 r^{p/(p-1)}.$$  

Then by (3.1)

$$C_0 r^{p/(p-1)} \leq u(x_z) \leq M K d(x_z)^{p/(p-1)},$$

which implies that

$$d(x_z) \geq \delta r, \quad \delta = \left( \frac{C_0 M}{K} \right)^{(p-1)/p},$$

or equivalently,

$$B_{\delta r}(x_z) \cap B_r(z) \subset \Omega_+.$$

Note that $\delta \leq 1$. Since $x_z \in \partial B_r(z)$, there is a ball

$$B_{(\delta/2)r}(y) \subset B_{\delta r}(x) \cap B_r(z) \subset B_r(z) \setminus \partial \Omega_+.$$

This shows that $\partial \Omega_+ \cap B_1$ is porous with the porosity constant $\delta/2$. The theorem is proved. □

References


