

# An Iterative Method for Pricing American Options under Jump-Diffusion Models

Santtu Salmi\*      Jari Toivanen<sup>†</sup>

## Abstract

We propose an iterative method for pricing American options under jump-diffusion models. A finite difference discretization is performed on the partial integro-differential equation, and the American option pricing problem is formulated as a linear complementarity problem (LCP). Jump-diffusion models include an integral term, which causes the resulting system to be dense. We propose an iteration to solve the LCPs efficiently and prove its convergence. Numerical examples with Kou's and Merton's jump-diffusion models show that the resulting iteration converges rapidly.

## 1 Introduction

It is widely recognized that the classic option pricing model proposed in 1973 by Black and Scholes in [5] and Merton in [18], does not ideally fit observed empirical market data. Two identified empirical features have been under much attention: (1) skewed distribution with higher peak and heavier tails (i.e. leptokurtic behavior) of the return distribution and (2) the volatility smile [3].

Many studies have been undergone to propose modifications to the Black-Scholes model to explain these phenomena. Here we focus on jump-diffusion models proposed by Kou in [16], and by Merton in [19]. These models have finite jump activity, unlike the more general approach with possibly infinite jump activity proposed by Carr, Geman, Madan and Yor in [7]. Another approach is to consider stochastic volatility models with jumps. The model proposed by Bates in [4] with jumps only in the value of the underlying asset is an example of such an approach. More general jump-diffusion models with stochastic volatility are considered in [10], for example.

Jumps can have a large impact on the price of an option, especially when near expiry. Merton's jump-diffusion model with the log-normal distribution correctly

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\*Department of Mathematical Information Technology, P.O. Box 35 (Agora), FI-40014 University of Jyväskylä, Finland. Email: ssalmi@jyu.fi

<sup>†</sup>Institute for Computational and Mathematical Engineering, Stanford University, Stanford, CA 94305, USA. Email: toivanen@stanford.edu

produces the volatility smile phenomenon. However, the use of the log-double-exponential distribution instead of the log-normal distribution allows the introduction of asymmetric leptokurtic features. Kou argues in [16] that this model better matches empirical data without adding much complexity to the model. The downside of moving to the log-double-exponential distribution is that it uses more parameters than the log-normal distribution. Moreover, many theoretical results are only valid for the log-normal distribution [25].

A solution to a jump-diffusion model can be obtained by solving a partial integro-differential equation (PIDE). Due to the integral term the discretization leads to a full matrix. Direct solution methods are usually too expensive with a full matrix, and therefore other numerical methods should be considered. Several methods have been proposed to approximate the linear complementarity problems resulting from American option pricing. These include a penalty method presented by d'Halluin, Forsyth and Labahn in [8] and an operator splitting method presented by Ikonen and Toivanen in [13, 15]. One alternative approach to PIDEs is to employ a risk-neutral valuation formula and evaluate it quickly using FFT (Fast Fourier Transform) [17].

Tavella and Randall in [22] described a stationary iterative method for pricing European options. Here, we propose a generalization of this iterative method to price American options. Using this approach, each iteration requires the solution of LCP with a banded matrix instead of the full matrix. Under Kou's and Merton's models the banded matrix is tridiagonal. Brennan and Schwartz algorithm [6] can be used to solve these LCPs. While we do not consider any jump-diffusion model with stochastic volatility, the iteration is also applicable for such a model.

The outline of this paper is the following. In Section 2, Merton's and Kou's jump-diffusion models are introduced. In Section 3, finite difference discretization for these models is presented. Section 4 describes the iterative methods used to solve the resulting systems of linear equations and linear complementarity problems for European and American options respectively. Also, the Brennan and Schwartz algorithm is briefly introduced. Numerical experiments are given in Section 5, and finally Section 6 contains conclusions. The main contributions of this paper are the iterative method proposed in Section 4.1, and numerical experiments in Section 5.

## 2 Models for European and American options

The value of the underlying asset  $x$  under the classic Black-Scholes model [5] is given by

$$\frac{dx(t)}{x(t-)} = \mu dt + \sigma dW(t), \quad (2.1)$$

where  $\mu$  is the drift rate,  $\sigma$  is the volatility and  $W(t)$  is a standard Brownian motion. In general, a finite activity jump term is introduced into the model as follows

$$\frac{dx(t)}{x(t-)} = \mu dt + \sigma dW(t) + d \left( \sum_{j=1}^{N(t)} V_j \right), \quad (2.2)$$

where  $N(t)$  is a Poisson process with rate  $\lambda$  and the set  $\{V_j\}$  is a sequence of independent identically distributed random variables. Under Merton's jump-diffusion model the set  $\{V_j\}$  is from the log-normal distribution with density

$$f_{ln}(y) := \frac{1}{y\delta\sqrt{2\pi}} e^{-\frac{(\log y - \gamma)^2}{2\delta^2}}, \quad (2.3)$$

whereas under Kou's jump-diffusion model the set  $\{V_j\}$  is from a distribution with the log-double-exponential density

$$f_{ld}(y) := \begin{cases} q\alpha_2 y^{\alpha_2-1}, & y < 1 \\ p\alpha_1 y^{-\alpha_1-1}, & y \geq 1, \end{cases} \quad (2.4)$$

where  $p, q, \alpha_1 > 1$ , and  $\alpha_2$  are positive constants such that  $p + q = 1$ .

Under the assumptions of the general jump-diffusion model (2.2), the value  $v$  of a European option can be obtained by solving a final value problem defined by a backward PIDE

$$v_t = Lv = -\frac{1}{2}\sigma^2 x^2 v_{xx} + (r - \lambda\kappa)xv_x + (r + \lambda)v - \lambda \left( \int_0^\infty v(t, xy)f(y)dy \right), \quad (2.5)$$

for all  $(t, x) \in [0, T] \times [0, \infty)$ . Above  $r$  is the (continuously compounded) risk free interest rate,  $f$  is the density function, and  $\kappa$  is the expected relative jump size. The final value of  $v$  is given by

$$v(T, x) = g(x), \quad x \in \mathbb{R}_+ \quad (2.6)$$

where  $g(x)$  is the payoff function of the option contract. For a put option, it is

$$g(x) = \max\{K - x, 0\}, \quad (2.7)$$

where  $K$  is the strike price. The boundary conditions for a European put option are given by

$$\begin{aligned} v(t, 0) &= Ke^{-r(T-t)} \\ \lim_{x \rightarrow \infty} v(t, x) &= 0, \quad t \in [0, T]. \end{aligned} \quad (2.8)$$

American options can be exercised at any time before expiry. Due to this, an additional constraint has to be introduced to the model to avoid arbitrage opportunities. The value  $v$  of an American option can be obtained by solving an LCP

$$\begin{cases} (v_t - Lv) \geq 0, & v \geq g, \\ (v_t - Lv)(v - g) = 0. \end{cases} \quad (2.9)$$

For American put options the behavior on the boundaries is given by

$$\begin{aligned} v(t, 0) &= K, \\ \lim_{x \rightarrow \infty} v(t, x) &= 0, \quad t \in [0, T]; \end{aligned} \quad (2.10)$$

see [12] for more information.

### 3 Discretization

#### 3.1 Discretization of spatial derivatives

We use finite differences to obtain an approximate solution. The infinite space domain is truncated to  $[0, X]$  with a sufficiently large  $X$  to avoid an unacceptably large truncation error. The value of  $v$  at  $X$  is set to be  $g(x)$ . An  $n$  nodes grid

$$0 = x_1 < x_2 < \dots < x_n = X \quad (3.1)$$

is used. The space derivatives of equation (2.5) are approximated with central-differences

$$v_x(t, x_i) \approx \frac{v_{i+1}(t) - v_{i-1}(t)}{\Delta x_{i-1} + \Delta x_i} \quad (3.2)$$

and

$$v_{xx}(t, x_i) \approx \frac{2[\Delta x_{i-1}v_{i+1}(t) - (\Delta x_{i-1} + \Delta x_i)v_i(t) + \Delta x_i v_{i-1}(t)]}{\Delta x_{i-1}\Delta x_i(\Delta x_{i-1} + \Delta x_i)}, \quad (3.3)$$

where  $v_i(t) = v(t, x_i)$  and  $\Delta x_i = x_{i+1} - x_i$ . We apply these approximations to equation (2.5). This leads to a set of semi-discrete equations having a matrix form

$$\mathbf{v}_t + \mathbf{A}\mathbf{v} = \mathbf{0}, \quad \mathbf{A} = \mathbf{D} - \mathbf{R} \quad (3.4)$$

where  $\mathbf{D}$  is a tridiagonal matrix and  $\mathbf{R}$  is a full matrix resulting from the integral term. The matrix  $\mathbf{D}$  has the following structure

$$\mathbf{D} = \begin{pmatrix} r & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \mathbf{D}_{2,1} & \mathbf{D}_{2,2} & \mathbf{D}_{2,3} & 0 & \dots & 0 & 0 & 0 \\ 0 & \mathbf{D}_{3,2} & \mathbf{D}_{3,3} & \mathbf{D}_{3,4} & 0 & \dots & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \dots & 0 \\ 0 & \dots & 0 & \mathbf{D}_{i,i-1} & \mathbf{D}_{i,i} & \mathbf{D}_{i,i+1} & 0 & \vdots \\ 0 & 0 & \dots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 0 & \mathbf{D}_{n-1,n-2} & \mathbf{D}_{n-1,n-1} & \mathbf{D}_{n-1,n} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}, \quad (3.5)$$

where the first and last row of the matrix are enforcing the boundary conditions. We use a modified volatility  $\hat{\sigma}$  instead of  $\sigma$  defined by

$$\hat{\sigma}^2 = \max \left\{ \sigma^2, (r - \lambda\kappa) \frac{\Delta x_i}{x_i}, -(r - \lambda\kappa) \frac{\Delta x_{i-1}}{x_i} \right\}. \quad (3.6)$$

This artificial volatility ensures that all off-diagonal elements are non positive [23]. The off-diagonal elements of  $D$  are given by

$$D_{i,i-1} = \frac{-\hat{\sigma}^2 x_i^2 + (r - \lambda\kappa)x_i \Delta x_i}{\Delta x_{i-1}(\Delta x_{i-1} + \Delta x_i)}, \quad (3.7)$$

and

$$D_{i,i+1} = \frac{-\hat{\sigma}^2 x_i^2 - (r - \lambda\kappa)x_i \Delta x_{i-1}}{\Delta x_i(\Delta x_{i-1} + \Delta x_i)}. \quad (3.8)$$

The diagonal elements are given by

$$D_{i,i} = r + \lambda - D_{i,i-1} - D_{i,i+1}. \quad (3.9)$$

Non negative off-diagonal elements together with the strict diagonal dominance under the condition  $r + \lambda > 0$  makes  $D$  an M-matrix.

### 3.2 Approximation of the integral term

The integral term

$$I = \int_0^\infty v(t, xy) f(y) dy \quad (3.10)$$

of (2.5) can be discretized by using the linear interpolation for  $v$  between grid points. First, performing a change of variable  $y = z/x$ , we get

$$I = \int_0^\infty v(t, z) f(z/x) / x dz. \quad (3.11)$$

Now, by using linear interpolation we get an approximation of  $I$

$$I_i \approx A_i = \sum_{j=1}^{n-1} A_{i,j} \quad (3.12)$$

at each grid point  $x_i, i = 2, \dots, n-1$ , where

$$A_{i,j} = \int_{x_j}^{x_{j+1}} \left( \frac{x_{j+1} - z}{\Delta x_j} v(t, x_j) + \frac{z - x_j}{\Delta x_j} v(t, x_{j+1}) \right) f(z/x_i) / x_i dz. \quad (3.13)$$

In case of Merton's model, the log-normal distribution  $f(y) = f_{ln}(y)$  is used, and thus we have

$$A_{i,j} = \frac{1}{\delta\sqrt{2\pi}} \int_{x_j}^{x_{j+1}} \left( \frac{x_{j+1} - z}{\Delta x_j} v(t, x_j) + \frac{z - x_j}{\Delta x_j} v(t, x_{j+1}) \right) e^{-\frac{(\log \frac{z}{x_i} - \gamma)^2}{2\delta^2}} / z dz. \quad (3.14)$$

By performing the integration, we obtain

$$A_{i,j} = \frac{1}{2\Delta x_j} x_i \left[ \left( \operatorname{erf} \left( \frac{\gamma - \log \frac{x_{j+1}}{x_i}}{\delta\sqrt{2}} \right) - \operatorname{erf} \left( \frac{\gamma - \log \frac{x_j}{x_i}}{\delta\sqrt{2}} \right) \right) \alpha_j + \left( \operatorname{erf} \left( \frac{\gamma + \delta^2 - \log \frac{x_{j+1}}{x_i}}{\delta\sqrt{2}} \right) - \operatorname{erf} \left( \frac{\gamma + \delta^2 - \log \frac{x_j}{x_i}}{\delta\sqrt{2}} \right) \right) x_i \beta_j \right], \quad (3.15)$$

where  $\text{erf}(\cdot)$  is the error function and

$$\alpha_j = v(t, x_{j+1})x_j - v(t, x_j)x_{j+1}, \quad \beta_j = (v(t, x_j) - v(t, x_{j+1}))e^{\gamma + \delta^2/2}. \quad (3.16)$$

In case of Kou's model, we have the log-double-exponential distribution  $f(y) = f_{ld}(y)$ . We decompose the integral as  $I = I^- + I^+$  and its approximation as  $A_i = (A_i^- + A_i^+)$ ,  $i = 2, \dots, n-1$ , where

$$I^- = \int_0^x v(t, z)f(z/x)_{ld}/xdz = q\alpha_2 x^{-\alpha_2} \int_0^x v(t, z)z^{\alpha_2-1}dz \quad (3.17)$$

and

$$I^+ = \int_x^\infty v(t, z)f(z/x)_{ld}/xdz = p\alpha_1 x^{\alpha_1} \int_x^\infty v(t, z)z^{-\alpha_1-1}dz. \quad (3.18)$$

The approximations  $A_i^-$  and  $A_i^+$  are given by

$$I_i^- \approx A_i^- = \sum_{j=1}^{i-1} A_{i,j}^- \quad \text{and} \quad I_i^+ \approx A_i^+ = \sum_{j=i}^{n-1} A_{i,j}^+, \quad (3.19)$$

where

$$A_{i,j}^- = \frac{qx_i^{-\alpha_2}}{(\alpha_2 + 1)\Delta x_j} [(x_{j+1}^{\alpha_2+1} - (x_{j+1} + \alpha_2\Delta x_j)x_j^{\alpha_2})v(t, x_j) + (x_j^{\alpha_2+1} - (x_j - \alpha_2\Delta x_j)x_{j+1}^{\alpha_2})v(t, x_{j+1})] \quad (3.20)$$

for  $j = 1, \dots, i-1$ , and

$$A_{i,j}^+ = \frac{px_i^{\alpha_1}}{(\alpha_1 - 1)\Delta x_j} [(x_{j+1}^{1-\alpha_1} - (x_{j+1} + \alpha_1\Delta x_j)x_j^{-\alpha_1})v(t, x_j) + (x_j^{1-\alpha_1} - (x_j + \alpha_1\Delta x_j)x_{j+1}^{-\alpha_1})v(t, x_{j+1})] \quad (3.21)$$

for  $j = i, \dots, n-1$  are given by performing the integration in (3.13) with  $f(y) = f_{ld}(y)$ .

The matrix  $\mathbf{R}$  resulting from approximating the integral has the form

$$\mathbf{R} = \lambda \begin{pmatrix} 0 & 0 & \dots & 0 \\ A_{2,1} & A_{2,2} & \dots & A_{2,n} \\ A_{3,1} & A_{3,2} & \dots & A_{3,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n-1,1} & A_{n-1,2} & \dots & A_{n-1,n} \\ 0 & 0 & \dots & 0 \end{pmatrix}. \quad (3.22)$$

In case of Kou's model, the actual computation of  $A_{i,j}^-$ s and  $A_{i,j}^+$ s is performed by using recursion formulas given by Toivanen in [23]. With this approach, only  $O(n)$  operations are required to calculate the integral approximation. However, this approach is only applicable when a log-double-exponential distribution is used.

### 3.3 Time discretization

Now for European options we have a semi-discrete linear problem

$$\mathbf{v}_t + \mathbf{A}\mathbf{v} = \mathbf{0}. \quad (3.23)$$

For American options we obtain a semi-discrete LCP

$$\begin{cases} (\mathbf{v}_t - \mathbf{A}\mathbf{v}) \geq \mathbf{0}, & \mathbf{v} \geq \mathbf{g}, \\ (\mathbf{v}_t - \mathbf{A}\mathbf{v})(\mathbf{v} - \mathbf{g}) = 0. \end{cases} \quad (3.24)$$

We use the Rannacher scheme [20] for time-discretization; see [11] for convergence analysis. This scheme performs a few first time steps using the implicit Euler scheme, and after that the remaining steps are performed with the Crank-Nicolson method. The implicit Euler steps are taken to avoid possible oscillations due to the nonsmooth final value (payoff function). We perform the first four steps with the implicit Euler method. The time domain  $[0, T]$  is split equally with time step  $\Delta t = T/m$ , with the exception of the implicit Euler steps that are set to have the length  $\Delta t/2$ .

Applying Rannacher time-stepping scheme to the semi-discrete problem (3.23) gives us

$$\mathbf{B}^{(k)}\mathbf{v}^{(k)} = \mathbf{b}^{(k)}, \quad k = m-1, \dots, 1, \quad (3.25)$$

where

$$\mathbf{B}^{(k)} = \mathbf{I} + \theta_k \Delta t_k \mathbf{A} \quad \text{and} \quad \mathbf{b}^{(k)} = (\mathbf{I} - (1 - \theta_k) \Delta t_k \mathbf{A}) \mathbf{v}^{(k+1)}. \quad (3.26)$$

The parameter  $\theta_k$  is defined by

$$\theta_k = \begin{cases} 1, & m-4 \leq k \leq m-1, \\ \frac{1}{2}, & k = m-5, \dots, 1. \end{cases} \quad (3.27)$$

For the semi-discrete LCP (3.24) the time-stepping gives

$$\begin{cases} (\mathbf{B}^{(k)}\mathbf{v}^{(k)} - \mathbf{b}^{(k)}) \geq \mathbf{0}, & \mathbf{v}^{(k)} \geq \mathbf{g}, \\ (\mathbf{B}^{(k)}\mathbf{v}^{(k)} - \mathbf{b}^{(k)})^T (\mathbf{v}^{(k)} - \mathbf{g}) = 0, \end{cases} \quad k = m-1, \dots, 1. \quad (3.28)$$

## 4 Solution methods

### 4.1 Iterative method

To solve the resulting system of linear equations for European options, we use a stationary iterative method proposed by Tavella and Randall in [22] and analyzed by d'Halluin, Forsyth, and Vetzal in [9]. We adopt the formulation presented by Almendral and Oosterlee in [2], where the matrix  $\mathbf{B}^{(k)}$  is split with a regular splitting

$$\mathbf{B}^{(k)} = \mathbf{T} - \mathbf{J}, \quad \text{where} \quad \mathbf{T} = \mathbf{I} + \theta_k \Delta t_k \mathbf{D} \quad \text{and} \quad \mathbf{J} = -\theta_k \Delta t_k \mathbf{R}. \quad (4.1)$$

Now the iterative method reads

$$\mathbf{v}^{l+1} = \mathbf{T}^{-1}(\mathbf{b}^{(k)} + \mathbf{J}\mathbf{v}^l), \quad l = 0, 1, \dots, \quad (4.2)$$

where the initial guess  $\mathbf{v}^0$  is taken to be  $\mathbf{v}^{(k+1)}$ . Each iteration requires a solution with the tridiagonal  $\mathbf{T}$ , and the multiplication of a vector by  $\mathbf{J}$ .

Let us consider an iterative method for American options. We denote an LCP

$$\mathbf{B}\mathbf{x} \geq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{g}, \quad (\mathbf{B}\mathbf{x} - \mathbf{b})^T(\mathbf{x} - \mathbf{g}) = 0 \quad (4.3)$$

by  $\text{LCP}(\mathbf{B}, \mathbf{x}, \mathbf{b}, \mathbf{g})$ .

**Theorem 1** Let  $\mathbf{B}$  be a strictly diagonally dominant square matrix with positive diagonal entries. Let a matrix splitting

$$\mathbf{B} = \mathbf{T} - \mathbf{J} \quad (4.4)$$

be such that the inequality

$$\mathbf{T}_{ii} - \sum_{j \neq i} |\mathbf{T}_{ij}| - \sum_j |\mathbf{J}_{ij}| > 0 \quad (4.5)$$

holds for all  $i$ . Then the vectors  $\mathbf{x}^{k+1}$ ,  $k = 1, 2, 3, \dots$ , defined by the iteration

$$\text{LCP}(\mathbf{T}, \mathbf{x}^{k+1}, \mathbf{J}\mathbf{x}^k + \mathbf{b}, \mathbf{g}), \quad (4.6)$$

and an initial guess  $\mathbf{x}^0$  converges to the solution  $\mathbf{x}$  of the LCP (4.3). Furthermore, for the error  $\mathbf{e}^k = \mathbf{x}^k - \mathbf{x}$  the norm inequality

$$\|\mathbf{e}^{k+1}\|_\infty \leq \left( \max_i \frac{\sum_j |\mathbf{J}_{ij}|}{\mathbf{T}_{ii} - \sum_{j \neq i} |\mathbf{T}_{ij}|} \right) \|\mathbf{e}^k\|_\infty \quad (4.7)$$

holds.

*Proof.* Substituting  $\mathbf{x}^{k+1} = \mathbf{x} + \mathbf{e}^{k+1}$  and  $\mathbf{x}^k = \mathbf{x} + \mathbf{e}^k$  to (4.6) gives us for the error  $\mathbf{e}^{k+1}$  an LCP

$$\begin{aligned} \mathbf{T}\mathbf{e}^{k+1} &\geq \mathbf{J}\mathbf{e}^k - \mathbf{B}\mathbf{x} + \mathbf{b}, & \mathbf{e}^{k+1} &\geq \mathbf{g} - \mathbf{x}, \\ (\mathbf{T}\mathbf{e}^{k+1} - \mathbf{J}\mathbf{e}^k + \mathbf{B}\mathbf{x} - \mathbf{b})^T &(\mathbf{e}^{k+1} - \mathbf{g} + \mathbf{x}) = 0. \end{aligned} \quad (4.8)$$

Let the  $i$ th component of  $\mathbf{e}^{k+1}$  denoted by  $e_i^{k+1}$  have the largest absolute value, that is,  $\|\mathbf{e}^{k+1}\|_\infty = |e_i^{k+1}| \geq |e_j^{k+1}|$  for all  $j$ . In the following, we need an estimate of  $|(\mathbf{J}\mathbf{e}^k)_i|$  from above given by

$$|(\mathbf{J}\mathbf{e}^k)_i| = \left| \sum_j \mathbf{J}_{ij} e_j^k \right| \leq \sum_j |\mathbf{J}_{ij}| |e_j^k| \leq \left( \sum_j |\mathbf{J}_{ij}| \right) \|\mathbf{e}^k\|_\infty. \quad (4.9)$$

Let us consider the  $i$ th inequalities and equations of the LCPs (4.3) and (4.8). We have the following four possibilities:

1.  $(\mathbf{B}\mathbf{x} - \mathbf{b})_i = 0$  and  $(\mathbf{T}\mathbf{e}^{k+1} - \mathbf{J}\mathbf{e}^k + \mathbf{B}\mathbf{x} - \mathbf{b})_i = 0$ : We have  $(\mathbf{T}\mathbf{e}^{k+1})_i = (\mathbf{J}\mathbf{e}^k)_i$  and also

$$|(\mathbf{T}\mathbf{e}^{k+1})_i| = |(\mathbf{J}\mathbf{e}^k)_i|. \quad (4.10)$$

We estimate  $|(\mathbf{T}\mathbf{e}^{k+1})_i|$  from below as follows:

$$\begin{aligned} |(\mathbf{T}\mathbf{e}^{k+1})_i| &= \left| \mathbf{T}_{ii}\mathbf{e}_i^{k+1} + \sum_{j \neq i} \mathbf{T}_{ij}\mathbf{e}_j^{k+1} \right| \geq |\mathbf{T}_{ii}\mathbf{e}_i^{k+1}| - \left| \sum_{j \neq i} \mathbf{T}_{ij}\mathbf{e}_j^{k+1} \right| \\ &\geq \mathbf{T}_{ii}|\mathbf{e}_i^{k+1}| - \sum_{j \neq i} |\mathbf{T}_{ij}||\mathbf{e}_j^{k+1}| \geq \left( \mathbf{T}_{ii} - \sum_{j \neq i} |\mathbf{T}_{ij}| \right) |\mathbf{e}_i^{k+1}|. \end{aligned} \quad (4.11)$$

By combining the estimates (4.9) and (4.11) with the equation (4.10), we obtain

$$\|\mathbf{e}^{k+1}\|_\infty \leq \frac{\sum_j |\mathbf{J}_{ij}|}{\mathbf{T}_{ii} - \sum_{j \neq i} |\mathbf{T}_{ij}|} \|\mathbf{e}^k\|_\infty. \quad (4.12)$$

2.  $(\mathbf{B}\mathbf{x} - \mathbf{b})_i = 0$  and  $\mathbf{e}_i^{k+1} - \mathbf{g}_i + \mathbf{x}_i = 0$ : We have

$$-(\mathbf{T}\mathbf{e}^{k+1})_i \leq -(\mathbf{J}\mathbf{e}^k - \mathbf{B}\mathbf{x} + \mathbf{b})_i = -(\mathbf{J}\mathbf{e}^k)_i \leq |(\mathbf{J}\mathbf{e}^k)_i|. \quad (4.13)$$

We note that  $\mathbf{e}_i^{k+1} = \mathbf{g}_i - \mathbf{x}_i \leq 0$ . We estimate  $-(\mathbf{T}\mathbf{e}^{k+1})_i$  from below as follows:

$$\begin{aligned} -(\mathbf{T}\mathbf{e}^{k+1})_i &= -\mathbf{T}_{ii}\mathbf{e}_i^{k+1} - \sum_{j \neq i} \mathbf{T}_{ij}\mathbf{e}_j^{k+1} \geq \mathbf{T}_{ii}\mathbf{e}_i^{k+1} - \left| \sum_{j \neq i} \mathbf{T}_{ij}\mathbf{e}_j^{k+1} \right| \\ &\geq \mathbf{T}_{ii}|\mathbf{e}_i^{k+1}| - \sum_{j \neq i} |\mathbf{T}_{ij}||\mathbf{e}_j^{k+1}| \geq \left( \mathbf{T}_{ii} - \sum_{j \neq i} |\mathbf{T}_{ij}| \right) |\mathbf{e}_i^{k+1}|. \end{aligned} \quad (4.14)$$

By combining the estimates (4.9) and (4.14) with the inequality (4.13), we obtain again the inequality (4.12).

3.  $\mathbf{x}_i - \mathbf{g}_i = 0$  and  $(\mathbf{T}\mathbf{e}^{k+1} - \mathbf{J}\mathbf{e}^k + \mathbf{B}\mathbf{x} - \mathbf{b})_i = 0$ : Using the inequality  $(\mathbf{B}\mathbf{x} - \mathbf{b})_i \geq 0$ , we obtain

$$(\mathbf{T}\mathbf{e}^{k+1})_i = (\mathbf{J}\mathbf{e}^k - \mathbf{B}\mathbf{x} + \mathbf{b})_i = (\mathbf{J}\mathbf{e}^k)_i - (\mathbf{B}\mathbf{x} - \mathbf{b})_i \leq (\mathbf{J}\mathbf{e}^k)_i \leq |(\mathbf{J}\mathbf{e}^k)_i|. \quad (4.15)$$

We note that  $\mathbf{e}_i^{k+1} = \mathbf{e}_i^{k+1} - \mathbf{g}_i + \mathbf{x}_i \geq 0$ . We estimate  $(\mathbf{T}\mathbf{e}^{k+1})_i$  from below as follows:

$$\begin{aligned} (\mathbf{T}\mathbf{e}^{k+1})_i &= \mathbf{T}_{ii}\mathbf{e}_i^{k+1} + \sum_{j \neq i} \mathbf{T}_{ij}\mathbf{e}_j^{k+1} \geq \mathbf{T}_{ii}\mathbf{e}_i^{k+1} - \left| \sum_{j \neq i} \mathbf{T}_{ij}\mathbf{e}_j^{k+1} \right| \\ &\geq \mathbf{T}_{ii}|\mathbf{e}_i^{k+1}| - \sum_{j \neq i} |\mathbf{T}_{ij}||\mathbf{e}_j^{k+1}| \geq \left( \mathbf{T}_{ii} - \sum_{j \neq i} |\mathbf{T}_{ij}| \right) |\mathbf{e}_i^{k+1}|. \end{aligned} \quad (4.16)$$

By combining the estimates (4.9) and (4.16) with the inequality (4.15), we obtain again the inequality (4.12).

4.  $x_i - g_i = 0$  and  $e_i^{k+1} - g_i + x_i = 0$ : We have  $\|e^{k+1}\|_\infty = |e_i^{k+1}| = 0$ .

Thus, in all four possible cases the norm inequality (4.12) holds for the error  $e^{k+1}$ . Taking maximum of the factor in the right-hand side of (4.12) over  $i$  gives us theorem's norm inequality (4.7).

From the assumption (4.5) for the matrix splitting (4.4) it follows that the factor in the right-hand side of (4.7) is strictly less than one. Thus, the error  $e^{k+1}$  converges to zero and  $x^{k+1}$  converges to  $x$ .  $\square$

From (3.26) it is clear that small time steps increase diagonal dominance of  $B$ , and therefore the assumption (4.5) will be satisfied with a sufficiently small time step.

## 4.2 Brennan and Schwartz algorithm

To solve the resulting LCP (4.6) with a tridiagonal  $T$  in each iteration, we use the Brennan and Schwartz algorithm. The original algorithm presented by Brennan and Schwartz in [6], is based on Gaussian elimination; see also [1, 14]. Consider a linear complementarity problem  $LCP(T, v, b, g)$ . We form an  $UL$ -decomposition of  $T$

$$UL = T, \quad (4.17)$$

and select the diagonal of  $L$  to consist of ones. The algorithm now reads

$$\begin{aligned} y &= U^{-1}b \\ v_1 &= \max\{y_1, g_1\} \\ \text{DO } i &= 2, \dots, n \\ v_i &= \max\{y_i - l_{ii-1}v_{i-1}, g_i\} \\ \text{END DO} \end{aligned} \quad (4.18)$$

where  $l_{ij}$  are components of  $L$ . Note that an American call option can be priced with the same approach by reversing the order of the Brennan and Schwartz algorithm, assuming the underlying asset pays dividends continuously; see [14] for the algorithm in reverse order.

## 5 Numerical experiments

### 5.1 European put option using Kou's model

First, we price European put options with Kou's model using the following model parameters:

$$\begin{aligned} \sigma &= 0.15, & r &= 0.05, & T &= 0.25, & K &= 100, \\ \lambda &= 0.1, & \alpha_1 &= 3.0465, & \alpha_2 &= 3.0775, & p &= 0.3445. \end{aligned} \quad (5.1)$$

These parameters are also used by d'Halluin, Forsyth and Vetzal in [9] and Toivanen in [23]. We use the reference prices described in [23]. Table 1 gives a complete list

Table 1: Reference prices used for numerical experiments.

model and type	value at 90	value at 100	value at 110
Kou, European Put [23]	9.430457	2.731259	0.552363
Kou, American Put [23]	10.005071	2.807879	0.561876
Merton, European Call [9]	0.527638	4.391246	12.643406
Merton, American Put	10.003815	3.241215	1.419796

of reference prices used in this section. A uniform space grid between  $[0, X]$  is used with  $n$  nodes and  $X = 400$ . We continue the iteration until  $\|v^l - v^{l+1}\|_2$  is less than  $10^{-8}$ . This stopping criterion is used in all the examples.

The results of the pricing are given in Table 2, which reports pricing errors, the number of total iterations and execution times. Also the ratio of the consecutive errors in the  $l_2$ -norm at  $x = 90$ ,  $x = 100$ , and  $x = 110$ . The computations were performed on a PC with a 1.8 GHz Intel Core 2 Duo processor. The codes were run on Matlab, however time consuming parts were implemented as mex files using C language. Second-order convergence is evident from Table 2 as the ratio is close to 4.

## 5.2 American put option using Kou's model

In this second example, we price American put options with the same parameters as in the previous example. This problem has been considered in [23, 24] and the reference prices are taken from [23]. The Brennan and Schwartz algorithm is used to solve the resulting LCPs.

Results are listed in Table 3. Errors, execution times and iteration counts are similar to the previous example with European options. The iteration converges rapidly with typically two iterations on each time step. Accuracy is similar to the European option, which could be further increased with a nonuniform grid. Note that we described the discretization (3.2) and (3.3) also for nonuniform grids. Execution times

Table 2: European put option pricing errors, execution times, total iterations and the ratios of errors with Kou's model.

n	m	error at 90	error at 100	error at 110	ratio	iter.	time (ms)
50	20	3.061e-1	4.459e-1	1.766e-1		66	1.1
100	40	4.513e-2	-1.116e-1	1.047e-2		126	2.5
200	80	-2.441e-3	-2.648e-2	-5.386e-3	3.79	246	7.3
400	160	-6.598e-4	-6.550e-3	-1.377e-3	4.03	324	19.6
800	320	-1.678e-4	-1.634e-3	-3.462e-4	4.01	644	74.5
1600	640	-4.199e-5	-4.084e-4	-8.686e-5	4.00	1284	290.6

Table 3: American put option pricing errors, execution times, total iterations and the ratios of errors with Kou’s model.

n	m	error at 90	error at 100	error at 110	ratio	iter.	time (ms)
50	20	2.436e-1	4.279e-1	1.757e-1		66	1.1
100	40	2.987e-2	-1.201e-1	1.064e-2		126	2.6
200	80	-5.071e-3	-2.989e-2	-5.767e-3	3.55	193	6.4
400	160	-4.263e-3	-7.623e-3	-1.547e-3	3.28	324	20.5
800	320	-3.123e-4	-1.964e-3	-4.126e-4	4.11	644	78.1
1600	640	-1.003e-4	-5.090e-4	-1.106e-4	3.83	1284	305.2

are only slightly higher when compared to the previous example. A second-order convergence is obtained.

### 5.3 European call option using Merton’s model

In the third example, we price European call options using Merton’s model. We use the following parameters:

$$\begin{aligned} \sigma &= 0.15, & r &= 0.05, & T &= 0.25, & K &= 100, \\ \lambda &= 0.1, & \gamma &= -0.9, & \delta &= 0.45. \end{aligned} \quad (5.2)$$

These parameters are also used by d’Halluin, Forsyth, and Vetzal in [9]. They are equal to the ones used in the previous examples with the exception of the distribution parameters  $\gamma$  and  $\delta$ . The same reference prices were used as in [9].

Instead of pricing a call option directly, we compute the price of a put option with the same strike price, and then use the put-call parity

$$v_c(t, x) = v_p(t, x) + x - Ke^{-t(T-t)} \quad (5.3)$$

to obtain the price of the call option. Above  $v_c$  is the price of the call option and  $v_p$  is the price of the put option.

Table 4: European call option pricing errors, execution times, total iterations and the ratios of errors with Merton’s model.

n	m	error at 90	error at 100	error at 110	ratio	iter.	time (ms)
50	20	3.845e-1	4.025e-1	1.060e-1		66	26.2
100	40	6.102e-2	-1.172e-1	7.445e-4		126	119.1
200	80	-1.801e-3	-2.778e-2	-5.746e-3	3.89	246	566.3
400	160	-5.144e-4	-6.873e-3	-1.464e-3	4.04	482	2800.8
800	320	-1.325e-4	-1.714e-3	-3.677e-4	4.01	644	35631.6
1600	640	-3.336e-5	-4.285e-4	-9.215e-5	4.00	1284	76711.3

Table 5: American put option pricing errors, execution times, total iterations and the ratios of errors with Merton’s model.

n	m	error at 90	error at 100	error at 110	ratio	iter.	time (ms)
50	20	2.846e-1	3.722e-1	1.043e-1		66	26.3
100	40	3.492e-2	-1.286e-1	7.774e-4		126	119.7
200	80	-3.815e-3	-3.211e-2	-6.144e-3	3.69	246	574.3
400	160	-3.815e-3	-8.166e-3	-1.626e-3	3.42	324	2536.8
800	320	-8.542e-4	-2.067e-3	-4.208e-4	4.02	644	35641.6
1600	640	-2.841e-4	-5.064e-4	-1.052e-4	3.83	1284	76725.4

Results are listed in Table 4. Again, second-order convergence is observed, and similar errors and iteration counts as in previous examples. However, this time execution times are significantly higher. This is to be expected, and is due to vector-matrix multiplication with a full  $n \times n$  matrix that is required to compute the integral approximation. A much faster implementation can be done with FFT. With this approach the number of required operations to compute the integral approximation can be reduced from  $O(n^2)$  to  $O(n \log n)$ ; see [9, 21], for example. This is almost as good as the  $O(n)$  operations required by the recursion formulas used in the first two examples.

#### 5.4 American put option using Merton’s model

In this example, we price American put options with Merton’s model. The same parameters are used as in the previous example. The reference prices listed in Table 5 were computed numerically using a fine grid of  $n = 6400$  and  $m = 2560$ . Results are reported in Table 5. Again errors, the ratios of errors, iteration counts and execution times are similar to the European counterpart.

## 6 Conclusions

We described an efficient iteration to solve LCPs resulting from the implicit finite difference discretization of PIDEs for pricing American options under jump-diffusion models. It is a generalization of the stationary iterative method for European options presented by Tavella and Randall in [22]. The iteration requires solving LCPs with a banded matrix instead of the full matrix resulting from the discretization of the jump model. We proved that the iteration converges and gave the convergence rate in  $l_\infty$ -norm.

We considered Kou’s model [16] and Merton’s model [19] for pricing American and European options. The resulting LCPs in the iteration for American options are tridiagonal and they were solved with the Brennan Schwartz algorithm [6]. European options were priced with the iteration described by Tavella and Randall in

[22]. For both American and European options, the iterations converged rapidly with typically two iterations per time step.

A reasonably accurate solution can be computed in a few milliseconds, assuming the integral approximation can be efficiently computed. For Kou's model, recursion formulas introduced by Toivanen in [23] were used to achieve good performance. Under both models, second-order convergence was observed.

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