Time-Periodic Solutions of Wave Equation via Controllability and Fictitious Domain Methods

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Abstract. We combine the controllability and fictitious domain methods to compute time-periodic solution for a wave equation describing scattering by an obstacle. The fictitious domain method uses distributed Lagrange multipliers to satisfy the Dirichlet boundary condition on the scatterer. The controllability technique leads to an optimization problem which we solve with a preconditioned conjugate gradient method. Numerical experiments are performed with a disc and a semi-open cavity as scatterers.

1 Introduction

We consider the computation of T-periodic solution of the wave equation

$$\varphi_{tt} - \Delta\varphi = 0, \quad \text{in } (\Pi \setminus \bar{\Omega}) \times (0, T), \tag{1}$$

$$\varphi = g, \quad \text{on } \partial\Omega \times (0, T), \tag{2}$$

$$\varphi_n + \varphi_t = 0, \quad \text{on } \partial\Pi \times (0, T), \tag{3}$$

where Ω is a scatterer and Π is a rectangular domain. The solution is T-periodic when the initial values $\varphi(0)$ and $\varphi_t(0)$ coincide with the final values, that is, $\varphi(0) = \varphi(T)$ and $\varphi_t(0) = \varphi_t(T)$.

Fictitious domain methods have been shown to be efficient numerical solution methods for flow and wave problems; see [1] and references therein. We consider the use of fictitious domain method to compute the solution of the wave problem. The Dirichlet boundary condition is enforced by using a distributed Lagrange multiplier. The controllability method with a least square integral measuring the periodicity of the solution gives us a convenient way to compute the periodic solution of the wave problem [2].

2 Controllability Technique with Fictitious Domain Method

We embed the domain $\Pi \setminus \bar{\Omega}$ to the rectangular domain Π and we enforce the boundary condition (2) by using a distributed Lagrange multiplier λ. The wave problem (1)-(3) can be formulated equivalently:

Find $\varphi \in H^1(\Pi)$ and $\lambda \in L^2(\Omega)$ such that

$$\int_{\Pi} \varphi_{tt} v\, dx + \int_{\Pi} \nabla\varphi \cdot \nabla v\, dx + \int_{\partial\Pi} \varphi_t v\, ds + \int_{\Omega} \lambda v\, dx = 0, \tag{4}$$

$$\int_{\Omega} \varphi\mu\, dx = \int_{\Omega} g\mu\, dx, \tag{5}$$

for all $v \in H^1(\Pi)$ and $\mu \in L^2(\Omega)$.

We introduce a least squares objective function

$$J\begin{pmatrix}\varphi(0)\\ \varphi_t(0)\end{pmatrix} = \frac{1}{2}\int_{\Pi}\left[|\nabla(\varphi(0)-\varphi(T))|^2 + |\varphi_t(0)-\varphi_t(T)|^2\right] dx, \tag{6}$$

where $\varphi(T)$ and $\varphi_t(T)$ are the solution of (4)-(5) and its time derivative, respectively, when $\varphi(0)$ and $\varphi_t(0)$ are used as the initial data. Then we obtain the T-periodic solution as the solution of the minimization problem:

Find $\varphi(0) \in H^1(\Pi)$ and $\varphi_t(0) \in L^2(\Pi)$ such that

$$J\begin{pmatrix}\varphi(0)\\ \varphi_t(0)\end{pmatrix} \leq J\begin{pmatrix}v(0)\\ v_t(0)\end{pmatrix} \quad \forall v(0) \in H^1(\Pi),\ v_t(0) \in L^2(\Pi). \tag{7}$$

3 Discretization

We use linear finite elements for the spatial discretization of the wave problem (4)-(5). A uniform rectangular mesh $\mathcal{T}_h(\Pi)$ is used for the variable φ. A quasi-uniform mesh $\mathcal{T}_h(\Omega)$ is constructed for the Lagrange multiplier λ in such a way that it coincides with the rectangular mesh in the interior of Ω. The mesh $\mathcal{T}_h(\Omega)$ should not be too fine near the boundary $\partial\Omega$ in order to satisfy the LBB condition. The meshes $\mathcal{T}_h(\Pi)$ and $\mathcal{T}_h(\Omega)$ are shown in Fig. 1 for a disc. The trapezoidal rule is used for numerical integrations leading to diagonal mass matrices.

For the temporal discretization, we divide the time interval $[0, T]$ into m subintervals of the size $\Delta t = T/m$. We denote by φ^k and the λ^k vectors containing the nodal values of the finite element functions $\varphi(k\Delta t)$ and $\lambda(k\Delta t)$, respectively. Similarly, we denote by g^k the vector associated with the integral in the right-hand side in (5). For the time derivatives in (4), we use central finite differences.

The discretization of (4)-(5) leads to systems of linear equations having the form

$$\begin{pmatrix}\frac{1}{(\Delta t)^2}M + \frac{1}{2\Delta t}B & C^T\\ C & 0\end{pmatrix}\begin{pmatrix}\varphi^{k+1}\\ \lambda^{k+1}\end{pmatrix} = \begin{pmatrix}f^{k+1}\\ g^{k+1}\end{pmatrix}, \tag{8}$$

where M is the mass matrix, B is the boundary mass matrix, and C is the discretized constraint operator in (5). The vector f^{k+1} is given by

$$f^{k+1} = \left(\frac{1}{(\Delta t)^2}M + A\right)\varphi^k - \left(\frac{1}{(\Delta t)^2}M - \frac{1}{2\Delta t}B\right)\varphi^{k-1}, \tag{9}$$

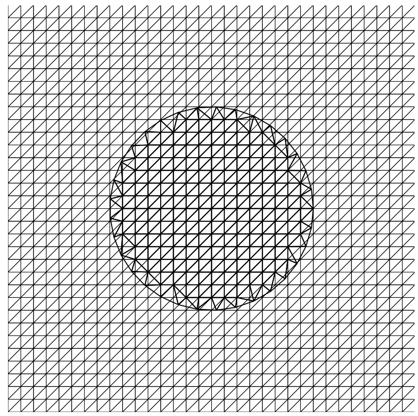

Fig. 1. The non matching finite element meshes for φ (thin line) and λ (thick line)

where A is the discretized Laplace operator. We solve the saddle point problems (8) by using the conjugate gradient accelerated Uzawa algorithm.

In order to compute the first vector φ^1, we eliminate the unknown vector φ^{-1} by using the central finite difference equation $\varphi_t^0 = \frac{1}{2\Delta t}(\varphi^1 - \varphi^{-1})$. We compute the time derivative φ_t^m at the time T by performing one extra time step and then by using the central finite difference.

4 Optimization

The solution vector containing the nodal values of the finite element function and its time derivative can be formally expressed in the form

$$\begin{pmatrix} \varphi^m \\ \varphi_t^m \end{pmatrix} = \mathcal{A} \begin{pmatrix} \varphi^0 \\ \varphi_t^0 \end{pmatrix} + \begin{pmatrix} u \\ u_t \end{pmatrix}, \tag{10}$$

where $\mathcal{A}$ is a matrix and the subvectors u and u_t are due to the heterogeneous Dirichlet boundary condition. Then the gradient of the objective function J is

$$\nabla J \begin{pmatrix} \varphi^0 \\ \varphi_t^0 \end{pmatrix} = \begin{pmatrix} \nabla_{\varphi^0} J \\ \nabla_{\varphi_t^0} J \end{pmatrix} + \mathcal{A}^T \begin{pmatrix} \nabla_{\varphi^m} J \\ \nabla_{\varphi_t^m} J \end{pmatrix}, \tag{11}$$

where $\mathcal{A}^T$ is the transpose of $\mathcal{A}$ and the subvectors are given by

$$(\nabla_{\varphi^0} J)_i = -(\nabla_{\varphi^m} J)_i = \int_{\Pi} \nabla(\varphi(0) - \varphi(T)) \nabla \psi_i \, dx \tag{12}$$

and

$$(\nabla_{\varphi_t^0} J)_i = -(\nabla_{\varphi_t^m} J)_i = \int_{\Pi} (\varphi_t(0) - \varphi_t(T)) \psi_i \, dx. \tag{13}$$

Here, ψ_i is the finite element basis function associated with the ith unknown.

We perform the minimization using the conjugate gradient method with a block diagonal preconditioner denoted by $\mathcal{B}$. The preconditioner block for φ^0 is the minus Laplace operator in $\Pi \setminus \Omega$. The solution with this block is performed by a fictitious domain method based on a distributed Lagrange multiplier [3]. We use the same discretization for the Lagrange multiplier as for the wave equation. For the time derivative φ_t^0 the preconditioner block is the mass matrix in Π.

We denote by $\mathbf{e}^k$ the vector containing the approximate T-periodic solutions φ^0 and φ_t^0 at the kth iteration. The conjugate gradient algorithm is as follows:

Set $k = 0$
Solve $\mathcal{B}\mathbf{g}^k = \nabla J(\mathbf{e}^k)$
Compute $\gamma^k = \int_\Pi (|\nabla g_0^k|^2 + |g_1^k|^2) dx$
Set $\mathbf{w}^k = \mathbf{g}^k$
Do While $\sqrt{\gamma^k} > \varepsilon \sqrt{\gamma^0}$
    Solve $\mathcal{B}\mathbf{x}^k = \nabla J(\mathbf{w}^k)$
    Compute $\rho^k = \gamma^k \Big/ \int_\Pi (\nabla x_0^k \cdot \nabla w_0^k + x_1^k w_1^k) dx$
    Set $k = k + 1$
    Set $\mathbf{e}^k = \mathbf{e}^{k-1} - \rho^{k-1}\mathbf{w}^{k-1}$
    Set $\mathbf{g}^k = \mathbf{g}^{k-1} - \rho^{k-1}\mathbf{x}^{k-1}$
    Compute $\gamma^k = \int_\Pi (|\nabla g_0^k|^2 + |g_1^k|^2) dx$
    Set $\mathbf{w}^k = \mathbf{g}^k + (\gamma^k/\gamma^{k-1})\mathbf{w}^{k-1}$
End Do

5 Numerical Experiments

In the first set of experiments, the scatterer Ω is a disc with the radius r which is placed to the middle of the domain $\Pi = (0, 4r)^2$. The length of the period T is one and the time step Δt is $h/2$, where h is the mesh step size of the uniform mesh $\mathcal{T}_h(\Pi)$. The Dirichlet boundary value g in (2) is given by the plane wave $-\sin(2\pi(x - t))$. In Table 1, we have given the number of conjugate gradient iterations with different numbers of nodes n in the mesh $\mathcal{T}_h(\Pi)$ and radii of the disc r. In the conjugate gradient algorithm, the stopping tolerance ε is 10^{-6}.

The second test problem is a semi-open cavity shown in Fig. 2. The internal dimensions of the cavity are 4 and 1 while the wall thickness is 0.2. The incoming plane wave has 30° angle of incidence and the boundary value is given by $g = -\cos(2\pi((\sqrt{3}x + y)/2 - t))$ in (2). The computational domain $\Pi = (-3, 3.2) \times (-1.7, 1.7)$ has been discretized with 125×69 mesh. The conjugate gradient method required 264 iterations to converge when the same stopping criterion was used as before. Contours of the solution φ have been plotted in Fig. 2.

Table 1. The number of conjugate gradient iterations

n	257^2	129^2	65^2	33^2
$r = 1/4$	85	62	49	32
$r = 1/2$	110	98	65	45
$r = 1$	146	157	110	
$r = 2$	198	188		
$r = 4$	275			

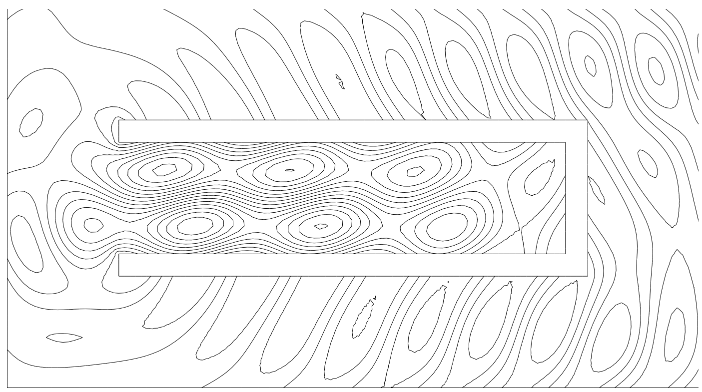

Fig. 2. Contours of the scattered wave by a semi-open cavity

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