Mappings of finite distortion: Sharp Sobolev assumptions for the $K_I$-inequality

Joint work with Stanislav Hencl

Let $Ω \subset \mathbb{R}^n$, $n \geq 2$, be a domain. Recall that mapping $f \in W^{1,1}_{\text{loc}}(Ω, \mathbb{R}^n)$ is called a mapping of finite distortion if

1. $J_f \in L^1_{\text{loc}}(Ω),$
2. $J_f(x) \geq 0$ for almost every $x \in Ω$, and
3. the matrix $Df(x)$ vanishes almost everywhere in the zero set of the Jacobian $J_f(x)$.

With these mappings we associate two distortion functions, $K_O$ (outer) and $K_I$ (inner), defined as

$$K_O(x) = \begin{cases} \frac{|Df(x)|^n}{J_f(x)} & \text{if } J_f(x) > 0 \\ 1, & \text{otherwise} \end{cases} \quad \text{and} \quad K_I(x) = \begin{cases} \frac{|Df(x)|^n}{J_f(x)^{n-1}} & \text{if } J_f(x) > 0 \\ 1, & \text{otherwise}. \end{cases}$$

We say that a continuous and open mapping $f \in W^{1,1}_{\text{loc}}(Ω, \mathbb{R}^n)$ of finite distortion with locally integrable inner distortion satisfies $K_I$-inequality if the weighted capacity inequality

$$\text{cap}(f(Ω), f(E)) \leq C \text{ cap}_{K_I}(Ω, E)$$

holds for all sufficiently nice condensers $(Ω, E)$ in $Ω$, where the constant $C > 0$ is depending only on $n$ and on the maximum multiplicity $m_G := \sup_{y \in \mathbb{R}^n} \text{ card } f^{-1}(y) \cap G$.

It is possible to show, for example, that mappings which satisfy the inequality (A) are differentiable almost everywhere.

It is known that inequality similar to (A), with some additional assumptions, holds for homeomorphisms of finite distortion in the Sobolev space $W^{1,n-1}$. We will now show that the inequality (A) might fail if we go below the space $W^{1,n-1}$. In the proof we will apply some probabilistic tools which have not been used to study mappings of finite distortion before. We will also use the fact that for Sobolev $W^{1,n-1-\epsilon}$-homeomorphisms the Lusin’s condition $(N)$ might fail on every hyperplane with respect to $(n-1)$-dimensional Hausdorff measure.