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DIMENSION OF MEAN POROUS MEASURES

Abstract

This reports a joint work with D. Beliaev, E. Järvenpää, M. Järvenpää, A. Käenmäki, S. Smirnov and V. Suomala. We proved in [1] that the packing dimension of any mean porous Radon measure on $\mathbb{R}^d$ may be estimated from above by a function which depends on mean porosity. The upper bound tends to $d-p$ as mean porosity tends to its maximum value. Similar results are well known for porous sets in $\mathbb{R}^d$. This is due to the work of P. Mattila [8] and A. Salli [9].

It is natural to ask in what generality these type of results are valid. Together with E. Järvenpää, M. Järvenpää, A. Käenmäki, S. Rogovin and V. Suomala we established a connection between regularity and porosity of sets [5]. The obtained results are however relevant only for small porosity and quite naturally the above mentioned $d-1$ results for large porosity can not be generalized to regular metric spaces. This will be illustrated by an example.

1 Definitions

In this section we will define different kinds of porosities in $\mathbb{R}^d$. We will be dealing only with lower porosity. This means that we study sets and measures that have holes on large portion of scales. First we define porosity of sets. Let $A \subset \mathbb{R}^d$. For all $x \in \mathbb{R}^d$ and $r > 0$, we define

$$\mathrm{por}(A, x, r) = \sup\{\alpha \geq 0 \,:\, B(y, \alpha r) \subset B(x, r) \setminus A \text{ for some } y \in \mathbb{R}^d\}.$$

Here $B(x, r)$ is the closed ball with centre at x and radius r. Given $0 \leq \alpha \leq \frac{1}{2}$, the set A is said to be α-porous at x if

$$\liminf_{r \to 0} \mathrm{por}(A, x, r) \geq \alpha.$$

Moreover, A is α-porous if it is α-porous at every point $x \in A$.

Next we define porosity for measures. Let μ be a Radon measure on $\mathbb{R}^d$. For all $x \in \mathbb{R}^d$ and for all positive real numbers r and ε, set

$$\begin{aligned}\operatorname{por}(\mu,x,r,\varepsilon) = \sup\{\alpha \geq 0 \,:\, &\text{there is } z \in \mathbb{R}^d \text{ such that} \\ &B(z,\alpha r) \subset B(x,r) \text{ and } \mu(B(z,\alpha r)) \leq \varepsilon\mu(B(x,r))\}.\end{aligned}$$

Given $\alpha \geq 0$, the measure μ is α-porous at a point $x \in \mathbb{R}^d$ if

$$\lim_{\varepsilon\to 0}\liminf_{r\to 0} \operatorname{por}(\mu,x,r,\varepsilon) \geq \alpha.$$

Finally, the measure μ is α-porous if there is $A \subset \mathbb{R}^d$ with $\mu(A) > 0$ such that μ is α-porous at every point $x \in A$. It is not difficult to see that in this case $0 \leq \alpha \leq \frac{1}{2}$.

Larger classes of mean porous sets and measures are obtained by demanding that only a certain percentage of scales are porous. Given $\alpha \geq 0$ and a positive integer j, the set A is α-porous for scale j at a point $x \in \mathbb{R}^d$ whenever $\operatorname{por}(A,x,2^{-j}) \geq \alpha$. For $0 < p \leq 1$, the set A is called mean (α,p)-porous at a point $x \in \mathbb{R}^d$ if

$$\liminf_{i\to\infty} \frac{\#\{1 \leq j \leq i \,:\, \operatorname{por}(A,x,2^{-j}) \geq \alpha\}}{i} \geq p.$$

We say that A is mean (α,p)-porous if it is mean (α,p)-porous at every point $x \in A$. The measure μ, in turn, is mean (α,p)-porous at x if

$$\lim_{\varepsilon\to 0}\liminf_{i\to\infty} \frac{\#\{1 \leq j \leq i \,:\, \operatorname{por}(\mu,x,2^{-j},\varepsilon) \geq \alpha\}}{i} \geq p.$$

Finally, μ is mean (α,p)-porous if there is $A \subset \mathbb{R}^d$ with $\mu(A) > 0$ such that μ is mean (α,p)-porous at all points $x \in A$.

Another direction to generalize the concept of porosity is to look for more holes in orthogonal directions in our reference balls. This leads to k-porosity introduced in [7]. For $0 < k \leq d$, $x \in \mathbb{R}^d$, $A \subset \mathbb{R}^d$ and $r > 0$ we define

$$\begin{aligned}\operatorname{por}_k(A,x,r) = \sup\{\alpha \geq 0 \;:\; &B(y_i,\alpha r) \subset B(x,r) \setminus A \\ &\text{for all } i \text{ with some } y_1,\ldots,y_k \in \mathbb{R}^d \\ &\text{where } (y_i - x)\cdot(y_j - x) = 0 \text{ for } i \neq j\}.\end{aligned}$$

Similarly as before we say that A is α-k-porous at x if

$$\liminf_{r\to 0} \operatorname{por}_k(A,x,r) \geq \alpha.$$

We define k-porosity for measures in the same way.

2 Large porosity in $\mathbb{R}^d$

The study of large porosity and dimension was pioneered by P. Mattila. He showed in [8] that as the porosity of a set tends to its maximum value $\frac{1}{2}$, the dimension of the set must drop by nearly 1. This result was then improved by A. Salli in [9], where he proved an estimate with the correct asymptotic behaviour. Similar estimates have been later proved for different kind of porosities. A. Käenmäki and V. Suomala proved in [7] with similar argumentation as Mattila that the dimension of a very k-porous set is at most near $d-k$. Later in [6] E. Järvenpää, M. Järvenpää, A. Käenmäki and V. Suomala improved this result by proving assymptotically sharp dimension estimates with a similar error terms as in the estimate of Salli.

In [3] E. and M. Järvenpää claimed an analogue for Salli's estimate for packing dimension of porous measures but it later turned out that the argument works only for Hausdorff dimension as explained in [4]. Recall that $\dim_{\mathrm{H}} \leq \dim_{\mathrm{p}}$.

D. Beliaev and S. Smirnov proved a dimension estimate for mean porous sets in [2]. They stated in their paper that the estimate holds for mean porous measures, but their proof contained a serious error. They tried to estimate mean porous measures by mean porous sets, which is in fact impossible. We have shown this by an example in [1]. In the same paper we finally proved the estimate for mean porous measures, which is the following.

Theorem 2.1. *Let $0 \leq \alpha \leq \frac{1}{2}$ and $0 < p \leq 1$. There exists a constant C depending only on d such that for all mean (α, p)-porous Radon measures μ on $\mathbb{R}^d$ we have*

$$\dim_{\mathrm{p}} \mu \leq d - p + \frac{C}{\log(\frac{1}{1-2\alpha})}.$$

We will now walk thru the proof of the Theorem and emphasize some of its main ideas. For the readers convenience it is recommended to have the paper [1] in hand as we only refer to the lemmas in that paper without stating them. In the proof of Theorem 2.1 we first need something easily estimable which at the end yields our dimension estimate. This is the sum introduced in [1, Lemma 3.2.] This estimating is done by uniformly distributing mass on a suitable collection of dyadic cubes over and over again. These collections of cubes are essentially given by the geometric lemma [1, Lemma 3.4] which roughly states that a porous set is in a neighbourhood of some $d-1$ -dimensional set. This type of lemma can be used as a core to prove dimension estimates for porous and k-porous sets – thou in k-porous case the lemma is a bit more complicated.

Compared to the proof of porous sets there are still a couple of obstacles to go around. The first one is that in the proof we are using dyadic cubes whereas the porosity itself is defined using balls. This means that our reference balls might live partly outside the cube we are looking at. Moreover we do not know if the missing part has large or small mass. We can avoid this problem by simply forgetting the boundary area of our cube, since the boundary is after all $d-1$-dimensional. For technical reasons this is done by induction in [1, Lemma 3.6].

At this point also the second obstacle is crushed. So far we have been doing a proof for porous measures. At the end we want a proof for mean porous measures. Mean porosity is delt by adding different weights to the sums depending on whether the cube is porous or not at a desired scale. This hides the information of mean porosity and allows us to calculate as if the measure was porous at every scale. In [1, Lemma 3.7] we confirm that the used weights really work.

Question 2.2. *In the proof of Theorem 2.1, or more presicely in Lemma 3.6 of [1], we used the fact that the boundary of a cube is $d-1$-dimensional. This prevents the proof from directly working for mean k-porous measures and it is still an open question whether the similar estimate is true for them or not.*

3 Porosity and dimension in metric spaces

It is natural to ask if the type of estimate that is given by Theorem 2.1 holds in more general spaces at least for porous sets. Remember that X is called s-regular if there exists a measure μ and constants $r_\mu > 0$ and $\infty > b_\mu \geq a_\mu > 0$ such that for all $x \in X$ and $0 < r < r_\mu$ we have

$$a_\mu r^s \leq \mu(B(x,r)) \leq b_\mu r^s.$$

In [5] we proved that if X is s-regular and $A \subset X$ is ρ-porous, then

$$\dim_{\mathrm{p}}(A) \leq s - c\rho^s,$$

where c is a positive constant which is independent of ρ. This is an extension of a well known result in Euclidean spaces. However this estimate is relevant only for small porosity and the type of estimate given by Theorem 2.1 fails in regular spaces. One can somehow add one direction to a regular space without adding a whole dimension. Hence there is no hope for a drop of one in dimension for all the $\frac{1}{2}$-porous subsets in some regular spaces. This is shown in the next example by taking Sierpinski gaskets in $\mathbb{R}^n$, letting $n \to \infty$ and noticing that in these spaces we get the dimension of some $\frac{1}{2}$-porous sets arbitrarily close to the dimension of our space.

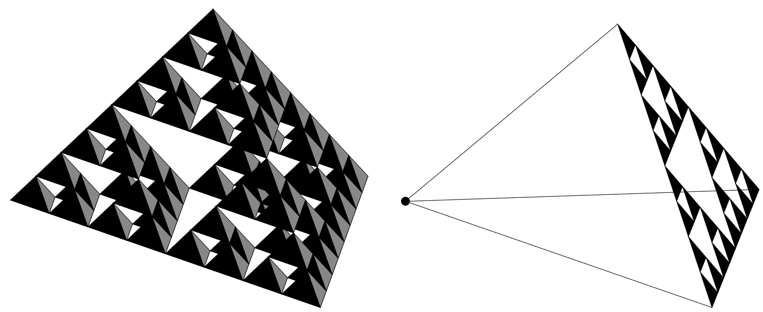

Figure 1: An illustration of space S_3 and S_2 as $\partial B_{S_3}((1,0,\dots,0),1)$.

Example 3.1. *For all $n \in \mathbb{N}$ we define a metric space (S_n, d_n). Here S_n is the attractor of function system*

$$f_i : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto \frac{1}{2}x + a_i,$$

where

$$a_i \in \{0, (\frac{1}{2}, 0, \dots, 0), (0, \frac{1}{2}, 0, \dots, 0), \dots, (0, \dots, 0, \frac{1}{2})\}.$$

Next we define our metric d_n as the path-metric induced by the maximum-metric in $\mathbb{R}^n$. In other words

$$d_n(x,y) = \inf\{\sum_{i=1}^{m} \max_{1 \le j \le n} |x_j^i - x_j^{i+1}| : m \in \mathbb{N}, x^k \in S_n \text{ and } x^1 = x, x^m = y\}.$$

Next some observations. (S_n, d_n) is a s-regular metric space, where s is the dimension of the space

$$\dim_{\mathrm{H}}(S_n) = \frac{\log(n+1)}{\log 2}.$$

Secondly by leaving one coordinate out and hence restricting our space (S_n, d_n) we get (S_{n-1}, d_{n-1}). Because of the definition of our metric we get also that

$$\partial B_{S_n}((1,0,\dots,0),1) = \{0\} \times S_{n-1}.$$

It is an easy remark that in a geodesic metric space the border of any ball is ρ-porous with some $\rho \ge \frac{1}{2}$. Next we note that when $n \to \infty$

$$\dim_{\mathrm{H}}(S_n) - \dim_{\mathrm{H}}(\partial B_{S_n}((1,0,\dots,0),1)) = \frac{\log(n+1) - \log(n)}{\log 2} \searrow 0.$$

Look at Figure 1 to see what S_3 looks like. Notice that in the picture we have a more symmetric Sierpinski gasket. This is the same space as in the example when we use the path-metric induced by the Euclidian one.

Question 3.2. *In what spaces does the dimension drop by one for sets of maximal porosity? How does the minimal drop in dimension for maximally porous sets depend on the structure of the space?*

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