

LOCAL HOMOGENEITY AND DIMENSIONS OF MEASURES IN DOUBLING METRIC SPACES

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ABSTRACT. We introduce two new concepts, local homogeneity and local L^q -spectrum, both of which are tools that can be used in studying the local structure of measures. The main emphasis is given to the study of local dimensions of measures in doubling metric spaces. As an application, we reach a new level of generality and obtain new estimates for conical densities, in multifractal analysis, and on the dimension of porous measures.

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1. INTRODUCTION

In geometric measure theory, it is common to encounter problems of the following type: given a measure μ and a set A of positive/full μ -measure, we have some local geometric information (on density, porosity, tangent measures, etc.) around all points of the set (or in a set of positive/full measure) and we want to gain some global information (on dimension, rectifiability, measure, etc.) from this. For example, if the set is porous in the sense that it contains large holes of fixed relative size around all of its points in all small scales, it is reasonable to estimate the dimension of the set from above using this information, see [54, 38, 49, 39, 32, 6, 30, 29, 45]. Thus, if we knew how the set (or a measure) is distributed in small balls, we would be able to bound its dimension. On the other hand, if μ is a measure of given dimension on an Euclidean space, it is a classical problem to estimate how it is distributed in different directions or cones, see [7, 8, 36, 20, 48, 17, 38, 39, 34, 52, 30, 31, 12].

In the study of fractals and dynamical systems, it is natural to analyse properties of measures using globally observable parameters arising from the asymptotic behaviour of the system, such as the Lyapunov exponents. The entropy and L^q -dimensions are concepts that measure the average distribution of the measure. In many cases, these global characteristics can then be related to the local regularity properties of the measure such as exact dimensionality and also to the values of the local dimension maps, see [55, 13, 41, 22, 4, 19].

In this article, the most important objects of interest are the upper and lower local dimensions of measures. Large part of the analysis on measures aims at estimating these dimensions. The essential suprema and infima of the local dimensions lead to the upper and lower Hausdorff and packing dimensions of the measure whereas investigating the level set structure of the local dimension maps leads to multifractal analysis. The purpose of this article is to introduce two new concepts, local homogeneity and local L^q -spectrum. Both of these concepts are tools that can be used in studying the local structure of measures.

When we want to estimate the dimension of a measure μ defined on an Euclidean space, it is very useful to inspect sums of the form $\sum_Q \mu(Q)^q$ over dyadic mesh cubes. This idea is visible in the definition of the local L^q -spectrum but we will also make extensive use of it in proving the local homogeneity estimate in §3. Although it is possible to define analogues of dyadic cubes in general metric spaces, see e.g. [11, 25], we will use slightly more flexible notions of δ -partitions and recursively defined packings of balls. In δ -partitions, we do not require the partition elements to have hierarchical structure. With these notions, we are able to overcome the technical problems ordinarily caused by the interplay between cubes and balls, see e.g. [5, 12, 3].

The local homogeneity and local L^q -spectrum are, however, of different nature since the order of taking limits in their definitions is different. In defining the local homogeneity, we first let the scale tend to zero and only after that increase the

resolution. This allows us to handle non-uniform properties, like porosity, with ease. On the other hand, the local L^q -spectrum sees some slight differences in the behaviour of the measure to which the local homogeneity is blind. This difference is made manifest in examples in §6.1.

We will next describe our main results. For notation and definitions of the basic concepts, we refer to §2 below. In Theorems 3.2 and 3.7, we will prove our main results concerning the local homogeneity of measures. We show that for any locally finite Borel regular measure μ , the upper local dimension $\overline{\dim}_{\text{loc}}(\mu, x)$ is bounded from above by the local homogeneity dimension $\dim_{\text{hom}}(\mu, x)$ at μ -almost all points. Here $\dim_{\text{hom}}(\mu, x)$ is the infimum of exponents s so that “large parts” of $B(x, r)$ in terms of μ can be covered by δ^{-s} balls of radius δr for all small $r, \delta > 0$, see (3.3) for a detailed definition. Using our results on the local homogeneity, we will obtain new estimates on the dimension of porous measures, see Theorems 5.2 and 5.6. In particular, these results settle problems left open in [28, 5]. As another application of the local homogeneity estimates, we obtain in Theorem 5.1 a new upper conical density result for measures with large packing dimension. This improves a result of [12] where a corresponding statement was proved for the Hausdorff dimension.

In §4, we introduce local versions of the classical L^q -spectra and dimensions. Using these concepts, we obtain a local metric space version of the results of [41, 22, 42, 19] on the relations between the Hausdorff, entropy, packing, and L^q -dimensions for measures in Euclidean spaces. We use the local L^q -spectrum as a tool to formulate a local multifractal formalism for measures defined on limit sets of Moran constructions in doubling metric spaces under the assumption that the measures satisfy a weak self-similarity condition, see Theorem 5.15.

Although the definitions of local homogeneity and local L^q -spectrum make sense in any metric space in which balls are totally bounded, we will restrict our analysis to the setting of doubling metric spaces since the doubling condition is needed in most of our proofs. As a consequence of the doubling property, all metric spaces that we consider are finite dimensional for any reasonable concept of dimension.

2. NOTATION AND PRELIMINARIES

We work on a general metric space (X, d) which is *doubling*: there is a constant $N \geq 2$ so that any closed ball $B(x, r) = \{y \in X : d(x, y) \leq r\}$ with centre $x \in X$ and radius $r > 0$ can be covered by N balls of radius $r/2$. We refer to the smallest N for which this holds as the doubling constant of X . Notice that even if $x \neq y$ or $r \neq t$, it may happen that $B(x, r) = B(y, t)$. For notational convenience, we henceforth assume that the centre and radius have been fixed whenever we talk about a ball $B \subset X$. This makes it possible to use notation such as $5B = B(x, 5r)$ without referring to the centre or radius of the ball $B = B(x, r)$.

We call any countable collection \mathcal{B} of pairwise disjoint closed balls a *packing*. It is called a *packing of A* for a subset $A \subset X$ if the centres of the balls of \mathcal{B} lie

in the set A , and it is a δ -packing for $\delta > 0$ if all of the balls in \mathcal{B} have radius δ . A δ -packing \mathcal{B} of A is termed *maximal* if for every $x \in A$ there is $B \in \mathcal{B}$ so that $B(x, \delta) \cap B \neq \emptyset$. Note that if \mathcal{B} is a maximal δ -packing of A , then $2\mathcal{B}$ covers A . Here $2\mathcal{B} = \{2B : B \in \mathcal{B}\}$.

For the following definition, we fix a constant $\Lambda \geq 2$. A countable partition \mathcal{Q} of a set $A \subset X$ to Borel sets is called a δ -partition for $\delta > 0$ if there exists a δ -packing \mathcal{B} of A with the same cardinality as \mathcal{Q} so that for each $B \in \mathcal{B}$ there exists exactly one $Q \in \mathcal{Q}$ so that $B \cap A \subset Q \subset \Lambda B$. Observe that doubling metric spaces are always separable. Hence for each $\delta > 0$ and $A \subset X$ there exists a maximal δ -packing of A and a δ -partition of A .

In writing down constants we often use notation such as $c = c(\dots)$ to emphasize that the constant depends only on the parameters listed inside the parentheses.

The following lemma is an easy consequence of the doubling property.

Lemma 2.1. (1) For every $0 < \lambda < 1$ there is a constant $M = M(N, \lambda) \in \mathbb{N}$, satisfying the following: If \mathcal{B} is a collection of closed balls of radius $\delta > 0$ so that $\lambda\mathcal{B}$ is pairwise disjoint, then there are δ -packings $\{\mathcal{B}_1, \dots, \mathcal{B}_M\}$ so that $\mathcal{B} = \bigcup_{i=1}^M \mathcal{B}_i$.

(2) If $1 < \gamma < \infty$ and \mathcal{B} is a (r/γ) -packing of a closed ball of radius r , then the cardinality of \mathcal{B} is at most $C(N, \gamma) = \gamma^{\log_2 N}$.

Proof. (1) We start by selecting a maximal disjoint subcollection \mathcal{B}_1 of \mathcal{B} . If $\mathcal{B}_1 \neq \mathcal{B}$, we continue by selecting a maximal disjoint subcollection \mathcal{B}_2 of $\mathcal{B} \setminus \mathcal{B}_1$. We proceed inductively by assuming that after $M - 1$ steps, $M \geq 2$, there is at least one ball $B \in \mathcal{B} \setminus \bigcup_{i=1}^{M-1} \mathcal{B}_i$ left. Then for each $i \in \{1, \dots, M - 1\}$ there is $B_i \in \mathcal{B}_i$ such that $B \cap B_i \neq \emptyset$. Since the balls $\lambda B_i \subset 3B$ are pairwise disjoint, at least M balls of radius $\lambda\delta/2$ are needed to cover the ball $3B$. Letting k to be the least integer with $2^{-k} \leq \lambda/6$, a repeated application of the doubling property implies that $M \leq N^k$. This shows that the process of selecting the subcollections \mathcal{B}_i must terminate after at most $M(N, \lambda) = N^k \leq (3/\lambda)^{\log_2 N}$ steps.

(2) Choose k to be the least integer with $2^{-k} \leq 1/(2\gamma)$. The doubling property implies that any ball of radius r can be covered by N^k balls of radius $r/(2\gamma)$ and thus no (r/γ) -packing of such ball can have more than $N^k \leq \gamma^{\log_2 N}$ elements. \square

We will exclusively work with nontrivial Borel regular (outer) measures defined on all subsets of X so that bounded sets have finite measure. In this article, we call such measures *Radon measures*. In complete doubling metric spaces this definition agrees with the one used, for example, in [39] and [20]. The support of a measure μ , denoted by $\text{spt}(\mu)$, is the smallest closed subset of X with full μ -measure. We say that a measure μ on X is *s-regular* (for $s > 0$) if there are constants $a_\mu, b_\mu, r_\mu > 0$ so that

$$a_\mu r^s \leq \mu(B(x, r)) \leq b_\mu r^s$$

for all $x \in \text{spt}(\mu)$ and $0 < r < r_\mu$. A metric space X is *s-regular* if it carries an *s-regular* measure μ with $\text{spt}(\mu) = X$. A simple volume argument shows that

an s -regular metric space is always doubling with a doubling constant at most $b_\mu 3^s / a_\mu$.

Denote the *upper and lower local dimensions of the measure μ at x* by

$$\begin{aligned}\overline{\dim}_{\text{loc}}(\mu, x) &= \limsup_{r \downarrow 0} \log \mu(B(x, r)) / \log r, \\ \underline{\dim}_{\text{loc}}(\mu, x) &= \liminf_{r \downarrow 0} \log \mu(B(x, r)) / \log r,\end{aligned}$$

respectively. If the upper and lower dimensions agree we call their mutual value the *local dimension of the measure μ at x* and write

$$\dim_{\text{loc}}(\mu, x) = \lim_{r \downarrow 0} \log \mu(B(x, r)) / \log r.$$

The local dimensions can also be defined using δ -partitions. The different definitions agree μ -almost everywhere, see Appendix B.

The *upper and lower Hausdorff dimensions of the measure μ* are then defined as

$$\begin{aligned}\overline{\dim}_{\text{H}}(\mu) &= \inf\{s \geq 0 : \underline{\dim}_{\text{loc}}(\mu, x) \leq s \text{ for } \mu\text{-almost all } x \in X\} \\ &= \inf\{\dim_{\text{H}}(A) : A \subset X \text{ is a Borel set with } \mu(X \setminus A) = 0\}, \\ \underline{\dim}_{\text{H}}(\mu) &= \sup\{s \geq 0 : \overline{\dim}_{\text{loc}}(\mu, x) \geq s \text{ for } \mu\text{-almost all } x \in X\} \\ &= \inf\{\dim_{\text{H}}(A) : A \subset X \text{ is a Borel set with } \mu(A) > 0\},\end{aligned}\tag{2.1}$$

respectively, where $\dim_{\text{H}}(A)$ is the *Hausdorff dimension of a set $A \subset X$* , see e.g. [18, Propositions 10.2 and 10.3] and [13, Theorem 2.3]. Similarly, the *upper and lower packing dimensions of the measure μ* are

$$\begin{aligned}\overline{\dim}_{\text{p}}(\mu) &= \inf\{s \geq 0 : \overline{\dim}_{\text{loc}}(\mu, x) \leq s \text{ for } \mu\text{-almost all } x \in X\} \\ &= \inf\{\dim_{\text{p}}(A) : A \subset X \text{ is a Borel set with } \mu(X \setminus A) = 0\}, \\ \underline{\dim}_{\text{p}}(\mu) &= \sup\{s \geq 0 : \underline{\dim}_{\text{loc}}(\mu, x) \geq s \text{ for } \mu\text{-almost all } x \in X\} \\ &= \inf\{\dim_{\text{p}}(A) : A \subset X \text{ is a Borel set with } \mu(A) > 0\},\end{aligned}\tag{2.2}$$

respectively. Here the *packing dimension $\dim_{\text{p}}(A)$ of a set $A \subset X$* is defined to be the modified upper Minkowski dimension, see [39, §5.9], or equivalently, the critical value of the radius based packing measure, see [18, Propositions 10.2 and 10.3] and [14]. The *upper Minkowski dimension $\overline{\dim}_{\text{M}}(A)$ of a set $A \subset X$* is defined in [39, §5.3].

If $\overline{\dim}_{\text{H}}(\mu) = \underline{\dim}_{\text{H}}(\mu)$, then we call their mutual value the *Hausdorff dimension of the measure μ* and denote it by $\dim_{\text{H}}(\mu)$. Similarly, if $\overline{\dim}_{\text{p}}(\mu) = \underline{\dim}_{\text{p}}(\mu)$, we have the *packing dimension of the measure μ* , denoted by $\dim_{\text{p}}(\mu)$.

The following lemma is obtained by an easy calculation, see Lemma 2.1(2) and [21, §10.13].

Lemma 2.2. *For every Radon measure μ on a doubling metric space X , it holds that $\overline{\dim}_p(\mu) \leq \dim_p(X) \leq \overline{\dim}_M(X) \leq C(N) = \log_2 N$, where N is the doubling constant of the space. If X is s -regular with an s -regular measure μ , then $\dim_H(X) = \dim_p(X) = \dim_H(\mu) = \dim_p(\mu) = s$.*

Finally, we say that a measure μ on X has the *density point property* if

$$\lim_{r \downarrow 0} \frac{\mu(A \cap B(x, r))}{\mu(B(x, r))} = 1 \quad (2.3)$$

for μ -almost all $x \in A$ whenever $A \subset X$ is μ -measurable. Recall that in Euclidean spaces, this property is always satisfied whereas in general doubling metric spaces this is not necessarily the case. For a detailed discussion on the density point property in doubling metric spaces, see Appendix A.

3. LOCAL HOMOGENEITY

In this section, we prove our main result concerning the local homogeneity of measures with large packing dimension. We start with the definitions. Let μ be a measure on X , $x \in X$, and $\delta, \varepsilon, r > 0$. Define

$$\begin{aligned} \text{hom}_{\delta, \varepsilon, r}(\mu, x) = \sup \{ \#\mathcal{B} : \mathcal{B} \text{ is a } (\delta r)\text{-packing of } B(x, r) \\ \text{so that } \mu(B) > \varepsilon \mu(B(x, 5r)) \text{ for all } B \in \mathcal{B} \} \end{aligned} \quad (3.1)$$

and from this let the *local δ -homogeneity of a measure μ at x* be

$$\text{hom}_\delta(\mu, x) = \lim_{\varepsilon \downarrow 0} \limsup_{r \downarrow 0} \text{hom}_{\delta, \varepsilon, r}(\mu, x). \quad (3.2)$$

The *local homogeneity dimension of a measure μ at x* is then defined as

$$\dim_{\text{hom}}(\mu, x) = \liminf_{\delta \downarrow 0} \frac{\log^+ \text{hom}_\delta(\mu, x)}{-\log \delta}, \quad (3.3)$$

where $\log^+(t) = \max\{0, \log t\}$ is used to ensure that $\dim_{\text{hom}}(\mu, x) \geq 0$.

Remark 3.1. (1) The limit in (3.2) exists as $\text{hom}_{\delta, \varepsilon_1, r}(\mu, x) \geq \text{hom}_{\delta, \varepsilon_2, r}(\mu, x)$ for all $0 < \varepsilon_1 < \varepsilon_2$.

(2) The definition of \dim_{hom} is quite technical. It may be helpful to compare it to the definition of the Assouad dimension (see [1, 35, 21]) and to observe that $\dim_{\text{hom}}(\mu, x)$ may be considered as a kind of local Assouad dimension for the measure μ around x : Roughly speaking, it is the least possible exponent s so that “large parts” of $B(x, r)$ in terms of μ can always be covered by δ^{-s} balls of radius δr for all small $r, \delta > 0$.

(3) We chose the constant 5 in the above definition for notational convenience and also since it is sufficiently large for all our applications. See also Proposition 6.16.

A fundamental property of the homogeneity dimension is stated in the following theorem. It is obtained as a corollary to a more quantitative result, Theorem 3.7, which will be essential in our applications in §5.

Theorem 3.2. *If μ is a Radon measure on a doubling metric space X , then*

$$\overline{\dim}_{\text{loc}}(\mu, x) \leq \dim_{\text{hom}}(\mu, x)$$

for μ -almost all $x \in X$.

Remark 3.3. (1) If μ is an s -regular measure on X , then

$$\dim_{\text{hom}}(\mu, x) = \dim_{\text{loc}}(\mu, x) = s$$

for μ -almost all $x \in X$. Indeed, according to Theorem 3.2 and Lemma 2.2, it suffices to show that $\dim_{\text{hom}}(\mu, x) \leq s$ for all $x \in \text{spt}(\mu)$. If $0 < \delta < 1$, then a simple volume argument gives $\text{hom}_{\delta}(\mu, x) \leq \sup\{\#\mathcal{B} : \mathcal{B} \text{ is a } (\delta r)\text{-packing of } B(x, r)\} \leq \frac{2^s b_{\mu}}{a_{\mu}} \delta^{-s}$ for all $x \in \text{spt}(\mu)$. Thus $\dim_{\text{hom}}(\mu, x) \leq s$ for all $x \in \text{spt}(\mu)$ finishing the proof.

(2) Let μ be a measure on X satisfying the density point property. Then, for every μ -measurable $A \subset X$, we have

$$\dim_{\text{hom}}(\mu|_A, x) = \dim_{\text{hom}}(\mu, x)$$

for μ -almost all $x \in A$. To see this, take a density point $x \in A$ and for each $\gamma > 0$ a radius $r_0 > 0$ so that $\mu(A \cap B(x, 5r)) > (1 - \gamma)\mu(B(x, 5r))$ for all $0 < r < r_0$. It follows that for every $\delta, \varepsilon > 0$ we have $\text{hom}_{\delta, \varepsilon, r}(\mu, x) \leq \text{hom}_{\delta, \varepsilon - \gamma, r}(\mu|_A, x)$ and $\text{hom}_{\delta, \varepsilon, r}(\mu, x) \geq \text{hom}_{\delta, \varepsilon/(1-\gamma), r}(\mu|_A, x)$ for all $0 < r < r_0$ and for μ -almost all $x \in X$.

Before we turn to Theorem 3.7, we exhibit some technical lemmas. The proof of Theorem 3.2 is postponed until the end of this section. Recall that the local dimension maps $x \mapsto \underline{\dim}_{\text{loc}}(\mu, x)$ and $x \mapsto \overline{\dim}_{\text{loc}}(\mu, x)$ are Borel functions, based on the fact that the mapping $x \mapsto \mu(B(x, r))$ is upper semicontinuous for each r . We provide a detailed proof of the fact that the local homogeneity dimension has the same property provided that the space is complete. See also Remark 6.15.

Lemma 3.4. *If μ is a Radon measure on a complete doubling metric space X , then $x \mapsto \dim_{\text{hom}}(\mu, x)$ defined on X is a Borel function.*

Proof. Given $0 < \delta, \varepsilon < 1$ and $0 < R < \infty$, consider the function $f: X \rightarrow \mathbb{N}$, $f(x) = \sup_{0 < r < R} \text{hom}_{\delta, \varepsilon, r}(\mu, x)$. We first show that f is lower semicontinuous. This is clearly the case in the set $f^{-1}(\{0\})$. If $x \in X$ is a point at which $f(x) = N > 0$, then we may find $0 < r < R$ so that $\text{hom}_{\delta, \varepsilon, r}(\mu, x) = N$. Hence there exists a (δr) -packing $\{B(x_1, \delta r), \dots, B(x_N, \delta r)\}$ of $B(x, r)$ so that $\mu(B(x_i, \delta r)) > \varepsilon \mu(B(x, 5r))$ for $i \in \{1, \dots, N\}$. Because the balls $B(x_i, \delta r)$ are distinct and closed, there is $\delta' > \delta$ so that also $\{B(x_1, \delta' r), \dots, B(x_N, \delta' r)\}$ is a packing of $B(x, r)$. Recall that a complete doubling metric space is locally compact. Now, as $\mu(B(x, 5r)) = \lim_{t \downarrow 5r} \mu(B(x, t))$, we may choose $0 < \varepsilon < \min\{R - r, \frac{\delta' - \delta}{\delta} r\}$ so that

$\mu(B(x_i, \delta r)) > \varepsilon \mu(B(x, 5r + 6\varepsilon))$ for all $i \in \{1, \dots, N\}$. If $y \in B(x, \varepsilon)$, it then follows that $\{B(x_1, \delta r + \delta\varepsilon), \dots, B(x_N, \delta r + \delta\varepsilon)\}$ is a $(\delta r + \delta\varepsilon)$ -packing of $B(y, r + \varepsilon)$ and that $\mu(B(x_i, \delta r + \delta\varepsilon)) \geq \mu(B(x_i, \delta r)) > \varepsilon \mu(B(x, 5r + 6\varepsilon)) \geq \mu(B(y, 5r + 5\varepsilon))$. Therefore $f(y) \geq N$ for all $y \in B(x, \varepsilon)$ giving the claim.

The rest of the proof runs along standard lines. First we choose sequences $0 < R_k \downarrow 0$ and $0 < \varepsilon_m \downarrow 0$ and observe that

$$\text{hom}_\delta(\mu, x) = \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \sup_{0 < r < R_k} \text{hom}_{\delta, \varepsilon_m, r}(\mu, x)$$

for all $\delta > 0$. Then we choose a sequence $0 < \delta_n \downarrow 0$ and obtain

$$\dim_{\text{hom}}(\mu, x) = \lim_{n \rightarrow \infty} \inf_{\substack{0 < \delta < \delta_n \\ \delta \in \mathbb{Q}}} \frac{\log \text{hom}_\delta(\mu, x)}{-\log \delta},$$

which finishes the proof. \square

Lemma 3.5. *Let μ be a Radon measure on a doubling metric space X , $0 < \delta < 1$, and $0 < t < s$. Then*

$$\inf\{r > 0 : \mu(B(x, 5r)) \leq (\delta r)^t \mu(B(x, \delta r))^{1-t/s}\} = 0$$

for μ -almost all $x \in X$ that satisfy $\overline{\dim}_{\text{loc}}(\mu, x) > s$.

Proof. Assume to the contrary that there are $0 < \delta < 1$, $0 < t < s < q$, $r_0 > 0$, a measure μ , and a Borel set $A \subset \{y \in X : \overline{\dim}_{\text{loc}}(\mu, y) > q\}$ with $\mu(A) > 0$ so that

$$\mu(B(x, 5r)) > (\delta r)^t \mu(B(x, \delta r))^{1-t/s} \quad (3.4)$$

for all $x \in A$ and $0 < r < r_0$. By Lemma 2.2, we may assume that $q \leq \log_2 N =: C$. Making $r_0 > 0$ smaller, if necessary, we may assume that

$$r^{t(q/s-1)} \leq 5^{-2C} \delta^{t+2C} \leq 25^{-q} \delta^{t+q} \quad (3.5)$$

for all $0 < r < r_0$. Since $\overline{\dim}_{\text{loc}}(\mu, x) > q$ implies $\liminf_{r \downarrow 0} \mu(B(x, 5r))/r^q = 0$, for each $x \in A$ there is a radius $0 < r_1 = r_1(x) < r_0$ so that $\mu(B(x, 5r_1)) < r_1^q$.

Given $x \in A$, let us show that

$$\mu(B(x, \delta r)) < r^q \quad (3.6)$$

for all $x \in A$ and $0 < r < r_1(x)/5$. If this is not the case, then there is $x \in A$ so that $r_2 := \sup\{0 < r < r_1/5 : \mu(B(x, \delta r)) \geq r^q\} > 0$, where $r_1 = r_1(x)$. Now, if $5r_2/\delta < r_1/5$, we have $\mu(B(x, 5r_2)) \leq (5r_2/\delta)^q$, and if $5r_2/\delta \geq r_1/5$, we have $\mu(B(x, 5r_2)) < r_1^q \leq (25r_2/\delta)^q$. Since $\mu(B(x, \delta r_2)) = r_2^q$, it follows that $\mu(B(x, 5r_2)) \leq (25/\delta)^q \mu(B(x, \delta r_2))$ in any case. Consequently, these estimates

and (3.5) imply

$$\begin{aligned}\mu(B(x, 5r_2)) &= \mu(B(x, 5r_2))^{t/s} \mu(B(x, 5r_2))^{1-t/s} \\ &\leq (25/\delta)^{qt/s} r_2^{qt/s} (25/\delta)^{q(1-t/s)} \mu(B(x, \delta r_2))^{1-t/s} \\ &\leq (\delta r_2)^t \mu(B(x, \delta r_2))^{1-t/s}.\end{aligned}$$

Since this contradicts with (3.4), we have shown (3.6).

Next we observe that for μ -almost every $x \in A$ there is $0 < r_3(x) < r_1(x)/5$ with $\mu(B(x, 5r_3)) < (5/\delta)^{2C} \mu(B(x, \delta r_3))$. Indeed, a simple iteration implies that $\overline{\dim}_{\text{loc}}(\mu, x) \geq 2C$ in points where this fails and by Lemma 2.2 this is possible only in a set of measure zero.

Now, using also (3.6) and (3.5), we get for almost all $x \in A$ that

$$\begin{aligned}\mu(B(x, 5r_3)) &\leq (5/\delta)^{2C} \mu(B(x, \delta r_3))^{t/s} \mu(B(x, \delta r_3))^{1-t/s} \\ &\leq (5/\delta)^{2C} r_3^{qt/s} \mu(B(x, \delta r_3))^{1-t/s} \leq (\delta r_3)^t \mu(B(x, \delta r_3))^{1-t/s}.\end{aligned}$$

Since this contradicts with (3.4), we have finished the proof. \square

In order to estimate the dimension of a measure in \mathbb{R}^d , it is very useful to consider sums of the form $\sum_Q \mu(Q)^q$ over mesh cubes of \mathbb{R}^d , see for instance [6] or [5, Lemma 3.2] (and also §4 below). The next lemma generalises this idea for collections of packings in metric spaces.

Lemma 3.6. *Suppose μ is a Radon measure on a doubling metric space X with bounded support and $A \subset X$ is a Borel set with $\mu(A) > 0$. If there are $0 < \delta < 1$, $0 < t < s$, and $k_0 \in \mathbb{N}$ such that for every integer $k \geq k_0$ and for every δ^k -packing \mathcal{B} of A there is an δ^{k-1} -packing \mathcal{B}' of A so that*

$$\delta^t \sum_{B \in \mathcal{B}} \mu(B)^{1-t/s} \leq \frac{1}{2} \sum_{B \in \mathcal{B}'} \mu(B)^{1-t/s}, \quad (3.7)$$

then $\mu(\{x \in A : \overline{\dim}_{\text{loc}}(\mu, x) \leq s\}) > 0$.

Proof. Suppose to the contrary that $\overline{\dim}_{\text{loc}}(\mu, x) > s$ for μ -almost all $x \in A$. By Lemma 3.5, for μ -almost all $x \in A$, there are arbitrary small radii $r > 0$ so that

$$\mu(B(x, 5r)) \leq (\delta r)^t \mu(B(x, \delta r))^{1-t/s}. \quad (3.8)$$

Observe also that for any δ^{k_0} -packing \mathcal{B} of A we have

$$\begin{aligned}\delta^{tk_0} \sum_{B \in \mathcal{B}} \mu(B)^{1-t/s} &\leq \delta^{tk_0} \#\mathcal{B} \left(\sum_{B \in \mathcal{B}} \mu(B) \right)^{1-t/s} \\ &\leq \delta^{tk_0} \left(\frac{\text{diam}(\text{spt}(\mu))}{\delta^{k_0}} \right)^{\log_2 N} \max\{1, \mu(X)\}\end{aligned} \quad (3.9)$$

using Lemma 2.1(2). Note that we may assume $A \subset \text{spt}(\mu)$. Let us denote the upper bound of (3.9) by $M = M(\mu, N, \delta, k_0, t)$. Therefore, (3.9) and a repeated application of (3.7) imply

$$\delta^{tk} \sum_{B \in \mathcal{B}_k} \mu(B)^{1-t/s} \leq M 2^{k_0-k} \quad (3.10)$$

for all δ^k -packings \mathcal{B}_k of A and integers $k \geq k_0$.

Let $k_1 \geq k_0$ and

$$\mathcal{B} = \{B(x, 5r) : x \in A \text{ and } 0 < r < \delta^{k_1} \text{ so that (3.8) holds}\}.$$

Using the $5r$ -covering theorem we find a sub-collection $\mathcal{B}' \subset \mathcal{B}$ so that \mathcal{B}' covers μ -almost all of A and $\frac{1}{5}\mathcal{B}'$ is a packing of A . For each $k \geq k_1$ we set $\mathcal{B}_k = \{B(x, \delta^k) : B(x, r) \in \mathcal{B}' \text{ for some } \delta^k \leq r < \delta^{k-1}\}$. Since (3.10) holds for \mathcal{B}_k we get

$$\begin{aligned} \mu(A) &\leq \sum_{B \in \mathcal{B}'} \mu(B) \leq \sum_{B(x, 5r) \in \mathcal{B}'} (\delta r)^t \mu(B(x, \delta r))^{1-t/s} \\ &\leq \sum_{k=k_1}^{\infty} \delta^{tk} \sum_{B \in \mathcal{B}_k} \mu(B)^{1-t/s} \leq M \sum_{k=k_1}^{\infty} 2^{k_0-k} = M 2^{k_0-k_1+1} \end{aligned}$$

by using the definition of \mathcal{B}' , (3.8), the definition of \mathcal{B}_k , and (3.10). Letting $k_1 \rightarrow \infty$ implies $\mu(A) = 0$, which is a contradiction. \square

The following theorem is our main quantitative result concerning local homogeneity of measures.

Theorem 3.7. *Suppose X is a doubling metric space with a doubling constant N . If $0 < m < s$, then there exists a constant $\delta_0 = \delta_0(m, s, N) > 0$ such that for every $0 < \delta < \delta_0$ there is $\varepsilon_0 = \varepsilon_0(m, s, N, \delta) > 0$ so that for every Radon measure μ on X we have*

$$\limsup_{r \downarrow 0} \text{hom}_{\delta, \varepsilon, r}(\mu, x) \geq \delta^{-m} \quad (3.11)$$

for all $0 < \varepsilon \leq \varepsilon_0$ and for μ -almost all $x \in X$ that satisfy $\overline{\dim}_{\text{loc}}(\mu, x) > s$.

Proof. To be able to use Lemma 3.4, we assume that X is complete. The non-complete case is discussed at the end of the proof. Set $t = (m + s)/2$ and define

$$\delta_0 = \delta_0(m, s, N) = (5^t/4M)^{1/(t-tm/s)}, \quad (3.12)$$

where $M = M(N, 1/10)$ is the constant of Lemma 2.1(1). Let $0 < \delta < \delta_0$, define $\varepsilon_0 = \delta^{-tm/(s-t)} N^{-l/(1-t/s)}$, where l is the least integer for which $2/\delta \leq 2^l$, and choose $0 < \varepsilon \leq \varepsilon_0$.

Assume on the contrary that there is a set $A \subset \{y \in X : \overline{\dim}_{\text{loc}}(\mu, y) > s\}$ with $\mu(A) > 0$ and $r_0 > 0$ such that $\text{hom}_{\delta, \varepsilon, r}(\mu, x) < \delta^{-m}$ for all $x \in A$ and $0 < r < r_0$. Recalling Lemma 3.4, we may assume that A is a Borel set. Choose $k_0 \in \mathbb{N}$ so

large that $\delta^{k_0-1}/5^{k_0} < r_0$ and fix $k \geq k_0$. Our aim is to obtain a contradiction by verifying the condition (3.7) of Lemma 3.6.

Let \mathcal{B} be a $(\delta/5)^k$ -packing of A and write $A_{\mathcal{B}}$ for the centres of the balls in \mathcal{B} . Choose a maximal $((\delta/5)^{k-1}/10)$ -packing \mathcal{B}_0 of the set $A_{\mathcal{B}}$ whence we clearly have $A_{\mathcal{B}} \subset \bigcup_{B \in 2\mathcal{B}_0} B$. For each ball $B \in 2\mathcal{B}_0$ we define $I_B = B \cap A_{\mathcal{B}}$ and $I'_B = \{x \in I_B : \mu(B(x, (\delta/5)^k)) > \varepsilon \mu(5B)\}$. Now the homogeneity assumption implies $\#I'_B \leq \delta^{-m}$ since the centre of B is contained in A . Using Hölder's inequality, we thus obtain

$$\begin{aligned} \sum_{x \in I'_B} \mu(B(x, (\delta/5)^k))^{1-t/s} &\leq \delta^{-tm/s} \left(\sum_{x \in I'_B} \mu(B(x, (\delta/5)^k)) \right)^{1-t/s} \\ &\leq \delta^{-tm/s} \mu(5B)^{1-t/s}. \end{aligned} \quad (3.13)$$

for all $B \in 2\mathcal{B}_0$. The doubling condition combined with the choice of l implies that $\#(I_B \setminus I'_B) \leq N^l$. Hence

$$\sum_{x \in I_B \setminus I'_B} \mu(B(x, (\delta/5)^k))^{1-t/s} \leq N^l \varepsilon^{1-t/s} \mu(5B)^{1-t/s} \leq \delta^{-tm/s} \mu(5B)^{1-t/s} \quad (3.14)$$

for all $B \in 2\mathcal{B}_0$ due to the definition of the constant ε_0 .

Using Lemma 2.1(1), we find packings $\mathcal{B}^1, \dots, \mathcal{B}^M \subset 10\mathcal{B}_0$ such that $\bigcup_{i=1}^M \mathcal{B}^i = 10\mathcal{B}_0$. Now there must be at least one $i \in \{1, \dots, M\}$ such that for $\mathcal{B}' = \mathcal{B}^i$ we have

$$\sum_{B \in \mathcal{B}'} \mu(B)^{1-t/s} \geq \frac{1}{M} \sum_{B \in 10\mathcal{B}_0} \mu(B)^{1-t/s}.$$

Combining this with (3.13) and (3.14), we have

$$\begin{aligned} \sum_{B \in \mathcal{B}} \mu(B)^{1-t/s} &\leq \sum_{B \in 2\mathcal{B}_0} \sum_{x \in I_B} \mu(B(x, (\delta/5)^k))^{1-t/s} \leq 2\delta^{-tm/s} \sum_{B \in 10\mathcal{B}_0} \mu(B)^{1-t/s} \\ &\leq 2M\delta^{t-tm/s} \delta^{-t} \sum_{B \in \mathcal{B}'} \mu(B)^{1-t/s} \leq \frac{1}{2}(\delta/5)^{-t} \sum_{B \in \mathcal{B}'} \mu(B)^{1-t/s} \end{aligned}$$

by recalling the choice of δ_0 . Thus we have verified the assumptions of Lemma 3.6 and so $\mu(\{x \in X : \overline{\dim}_{\text{loc}}(\mu, x) \leq s\} \cap \{x \in X : \overline{\dim}_{\text{loc}}(\mu, x) > s\}) > 0$ giving a contradiction. This finishes the proof provided that X is complete.

Let us now consider the non-complete case. Denote by \overline{X} the standard metric completion of X and let ν be a measure on \overline{X} defined by $\nu(A) = \mu(A \cap X)$ for $A \subset \overline{X}$. We consider X as a subset of \overline{X} in the natural way. Observe that \overline{X} is doubling with doubling constant N^2 . Let $M' = M(N^2, 1/2)$ be the constant of Lemma 2.1(1). The above proof now implies that for each $0 < \delta < \delta_0$ and each $0 < \varepsilon < \varepsilon_0$ we have

$$\limsup_{r \downarrow 0} \text{hom}_{\delta, \varepsilon, r}(\nu, x) \geq M' \delta^{-m} \quad (3.15)$$

for ν -almost all $x \in \overline{X}$. Compared to (3.11), there is an extra factor M' in (3.15), but this is easily handled by changing the constants $\delta_0 = \delta_0(s, m, N)$ and $\varepsilon_0 = \varepsilon_0(m, s, N, \delta)$. We show that (3.15) implies

$$\limsup_{r \downarrow 0} \text{hom}_{\delta, \varepsilon, r}(\mu, x) \geq \delta^{-m} \quad (3.16)$$

for μ -almost all $x \in X$. To see this, fix $x \in X$ and $\delta, \varepsilon, r > 0$ so that $\text{hom}_{\delta, \varepsilon, r}(\nu, x) \geq M' \delta^{-m}$. Let \mathcal{B} be a (δr) -packing of $B(x, r)$ in \overline{X} so that $\#\mathcal{B} \geq M' \delta^{-m}$ and $\nu(B) > \varepsilon \nu(B(x, 5r))$ for all $B \in \mathcal{B}$. Since $\lim_{R \downarrow r} \mu(B(x, 5R)) = \mu(B(x, 5r))$, we may choose $r < R < 3r/2$ so that also $\nu(B) > \varepsilon \nu(B(x, 5R))$ for all $B \in \mathcal{B}$. Using Lemma 2.1(1), we find a subcollection \mathcal{B}_1 of \mathcal{B} with $\#\mathcal{B}_1 > \delta^{-m}$ so that the collection $2\mathcal{B}_1$ is still a packing. Next we pick $y_i \in (\frac{R-r}{r})B_i \cap X$ for each $B_i \in \mathcal{B}_1$ and denote $\mathcal{B}_2 = \{B(y_i, \delta R)\}_i$. It follows that \mathcal{B}_2 is a (δR) -packing of $B(x, R)$ in X with $\#\mathcal{B}_2 \geq \delta^{-m}$ and $\mu(B(y_i, \delta R)) = \nu(B(y_i, \delta R)) \geq \nu(B_i) > \varepsilon \nu(B(x, 5R)) = \varepsilon \mu(B(x, 5R))$ for all i . Thus $\text{hom}_{\delta, \varepsilon, R}(\mu, x) \geq \delta^{-m}$ and (3.16) follows. \square

Proof of Theorem 3.2. Assume to the contrary that there is a Borel set $A \subset X$ with $\mu(A) > 0$ and $0 < m < s$ such that $\dim_{\text{hom}}(\mu, x) < m < s < \overline{\dim}_{\text{loc}}(\mu, x)$ for all $x \in A$. It follows from Theorem 3.7 that there is $\delta_0 = \delta_0(m, s, N) > 0$ so that $\text{hom}_{\delta}(\mu, x) \geq \delta^{-m}$ for every $0 < \delta < \delta_0$ and for μ -almost all $x \in A$. So $\dim_{\text{hom}}(\mu, x) \geq m$ for μ -almost all $x \in A$ giving a contradiction. \square

4. LOCAL L^q -SPECTRUM

The L^q -spectrum of a measure is an essential tool in multifractal analysis and it has been investigated in many works, see e.g. [41, 22, 33, 43, 2, 19, 50] and [16, 18, 44] and references therein. We define a local version of the L^q -spectrum in a similar fashion as with the homogeneity. It turns out that the well known Hausdorff and packing dimension estimates for the measure arising from its L^q -spectrum generalise to this local setting. We relate the local L^q -dimensions to the upper and lower local dimensions as well to the upper and lower entropy dimensions.

Let μ be a measure on X , $x \in \text{spt}(\mu)$, and $q \in \mathbb{R}$. The *local L^q -spectrum of a measure μ at x* is defined by

$$\tau_q(\mu, x) = \lim_{r \downarrow 0} \liminf_{\delta \downarrow 0} \frac{\log S_{q, \delta, r}^*(\mu, x)}{\log \delta}, \quad (4.1)$$

where

$$S_{q, \delta, r}^*(\mu, x) = \sup \left\{ \sum_{B \in \mathcal{B}} \mu(B)^q : \mathcal{B} \text{ is a } \delta\text{-packing of } B(x, r) \cap \text{spt}(\mu) \right\}$$

for all $\delta, r > 0$.

Remark 4.1. (1) The limit in (4.1) exists as $S_{q,\delta,r_1}^*(\mu, x) \leq S_{q,\delta,r_2}^*(\mu, x)$ for all $\delta > 0$ and $0 < r_1 < r_2$.

(2) To show that the L^q -spectrum is concave, it is crucial to have $\liminf_{\delta \downarrow 0}$ in (4.1). See the proof of Theorem 4.4(4).

(3) If $q \geq 0$, the definition of $\tau_q(\mu, x)$ does not change if we replace $S_{q,\delta,r}^*$ by $S_{q,\delta,r}$ defined as

$$S_{q,\delta,r}(\mu, x) = \sup \left\{ \sum_{B \in \mathcal{B}} \mu(B)^q : \mathcal{B} \text{ is a } \delta\text{-packing of } B(x, r) \right\}$$

(if $q = 0$, we interpret $0^q = 0$). This simple fact will be used frequently in the proofs below.

We define the *local L^q -dimension of the measure μ at x* to be

$$\dim_q(\mu, x) = \tau_q(\mu, x)/(q - 1)$$

for all $q \in \mathbb{R}$, $q \neq 1$. To complete the definition, we set

$$\begin{aligned} \overline{\dim}_1(\mu, x) &= \limsup_{r \downarrow 0} \limsup_{\delta \downarrow 0} \frac{\int_{B(x,r)} \log \mu(B(y, \delta)) d\mu(y)}{\log \delta}, \\ \underline{\dim}_1(\mu, x) &= \liminf_{r \downarrow 0} \liminf_{\delta \downarrow 0} \frac{\int_{B(x,r)} \log \mu(B(y, \delta)) d\mu(y)}{\log \delta}, \end{aligned} \quad (4.2)$$

and call them the *upper and lower entropy dimensions of μ at x* , respectively. If these two notions coincide, we denote their common value by $\dim_1(\mu, x)$. Here and hereafter, for $A \subset X$ and a μ -measurable $f: X \rightarrow \overline{\mathbb{R}}$, we use notation $\int_A f(y) d\mu(y) = \mu(A)^{-1} \int_A f(y) d\mu(y)$ whenever the integral is well defined.

Our main results concerning the local L^q -dimensions are stated in the following theorem.

Theorem 4.2. *If μ is a Radon measure on a doubling metric space X , then*

$$\lim_{q \downarrow 1} \dim_q(\mu, x) \leq \underline{\dim}_{\text{loc}}(\mu, x) \leq \overline{\dim}_{\text{loc}}(\mu, x) \leq \lim_{q \uparrow 1} \dim_q(\mu, x) \quad (4.3)$$

for μ -almost all $x \in X$. In addition, if μ has the density point property, then

$$\underline{\dim}_{\text{loc}}(\mu, x) \leq \underline{\dim}_1(\mu, x) \leq \overline{\dim}_1(\mu, x) \leq \overline{\dim}_{\text{loc}}(\mu, x) \quad (4.4)$$

for μ -almost all $x \in X$.

Remark 4.3. (1) If μ is an s -regular measure on X , then

$$\dim_q(\mu, x) = \dim_{\text{loc}}(\mu, x) = s$$

for μ -almost all $x \in X$. Indeed, given $q \in \mathbb{R}$, we find constants $0 < c_1 < c_2 < \infty$ so that $c_1 r^s \delta^{s(q-1)} \leq S_{q,\delta,r}(\mu, x) \leq c_2 r^s \delta^{s(q-1)}$ for all $x \in \text{spt}(\mu)$ and $0 < \delta, r < r_\mu/2$. This shows that we have $\dim_q(\mu, x) = s$ for all $q \in \mathbb{R}$ and all $x \in \text{spt}(\mu)$.

(2) For any Borel set A the restriction measure $\mu|_A$ has the same upper and lower local dimension and local homogeneity dimension (see Remark 3.3(2)) as

the original measure μ for μ -almost all points in A – at least when we assume the density point property. This is not true for the L^q -dimension due to its different nature. As an example in the case $q < 1$, take $\mu = \mathcal{L}^2 + \mathcal{H}^1|_L$ on \mathbb{R}^2 , where \mathcal{L}^2 is the Lebesgue measure and $\mathcal{H}^1|_L$ is the length measure on a line $L \subset \mathbb{R}^2$. Then for any $x \in L$ we have $\dim_q(\mu, x) = 2$ and $\dim_q(\mu|_L, x) = 1$. For $q > 1$ we can define a measure on the real line by letting $\nu = \mathcal{L}^1 + \sum_{n \in \mathbb{N}} 2^{-n} \delta_{q_n}$, where $\{q_1, q_2, \dots\}$ is an enumeration of the rationals. Then $\dim_q(\nu, x) = 0$ while $\dim_q(\nu|_{\mathbb{R} \setminus \mathbb{Q}}, x) = 1$ for all $x \in \mathbb{R}$.

(3) Theorem 4.2 generalises the results [22, Theorems 1.3 and 4.1], [41, Theorem 1.1], and [19, Theorem 1.4]. See also [42, Corollary 1.3]. Examples showing that the inequalities in (4.3) can be strict are given in §6.1. Connections between that local and global spectra and dimensions are discussed later in Proposition 4.6.

Before we prove Theorem 4.2, we discuss the basic properties of the local L^q -spectrum and L^q -dimensions. We first set up some notation. For all $n \in \mathbb{N}$ we fix a (2^{-n}) -partition \mathcal{Q}_n of X . If $x \in X$ and $r > 0$, then we set $\mathcal{Q}_n(x, r) = \{Q \cap B(x, r) : Q \in \mathcal{Q}_n\}$. We remark that in the following theorem, instead of using (2^{-n}) -partitions, we could use δ_n -partitions for any decreasing sequence $\delta_n \downarrow 0$ for which there exists constants $0 < c_1, c_2 < 1$ so that $c_1 < \delta_{n+1}/\delta_n < c_2$ for all $n \in \mathbb{N}$.

Theorem 4.4. *Let μ be a Radon measure on a doubling metric space X , $x \in \text{spt}(\mu)$, $r > 0$ and $q \geq 0$. If the doubling constant of X is N , then*

(1) *it holds that*

$$\tau_q(\mu, x) = \lim_{r \downarrow 0} \liminf_{n \rightarrow \infty} \frac{\log \sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q)^q}{\log 2^{-n}}, \quad (4.5)$$

$$\overline{\dim}_1(\mu, x) = \limsup_{r \downarrow 0} \limsup_{n \rightarrow \infty} \frac{\sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q) \log \mu(Q)}{\mu(B(x, r)) \log 2^{-n}}, \quad (4.6)$$

$$\underline{\dim}_1(\mu, x) = \liminf_{r \downarrow 0} \liminf_{n \rightarrow \infty} \frac{\sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q) \log \mu(Q)}{\mu(B(x, r)) \log 2^{-n}}, \quad (4.7)$$

- (2) $\tau_1(\mu, x) = 0$,
- (3) $\min\{0, (q-1) \log_2 N\} \leq \tau_q(\mu, x) \leq \max\{0, (q-1) \log_2 N\}$,
- (4) $q \mapsto \tau_q(\mu, x)$ is concave on $[0, \infty)$,
- (5) $q \mapsto \dim_q(\mu, x)$ is continuous and decreasing on $(0, 1) \cup (1, \infty)$,
- (6) $\lim_{q \downarrow 1} \dim_q(\mu, x) \leq \underline{\dim}_1(\mu, x) \leq \overline{\dim}_1(\mu, x) \leq \lim_{q \uparrow 1} \dim_q(\mu, x)$,
- (7) $x \mapsto \liminf_{n \rightarrow \infty} \frac{1}{\log 2^{-n}} \log \sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q)^q$ is a Borel function,
- (8) $\dim_0(\mu, x) = \lim_{r \downarrow 0} \overline{\dim}_M(B(x, r) \cap \text{spt}(\mu))$.

Proof. Let us start proving the claim (1) by first validating the equality in (4.5). Let $r > 0$, $0 < \delta < r/(4\Lambda)$, and $n \in \mathbb{N}$ so that $2^{-n-1} \leq \delta < 2^{-n}$. Our first goal is

to show that for a constant $c_1 = c_1(N, \Lambda, q) < \infty$, we have

$$S_{q,\delta,r}(\mu, x) \leq c_1 \sum_{Q \in \mathcal{Q}_n(x, 2r)} \mu(Q)^q \quad (4.8)$$

Recall that N is the doubling constant of X and Λ is a fixed constant used in defining the partitions \mathcal{Q}_n . To show (4.8), we fix a δ -packing \mathcal{B} of $B(x, r)$ and let

$$\mathcal{C}_B = \{Q \in \mathcal{Q}_n(x, 2r) : Q \cap B \neq \emptyset\}$$

for all $B \in \mathcal{B}$. Since \mathcal{C}_B is a cover for B , we have

$$\mu(B)^q \leq \left(\sum_{Q \in \mathcal{C}_B} \mu(Q) \right)^q \leq (\#\mathcal{C}_B)^q \sum_{Q \in \mathcal{C}_B} \mu(Q)^q,$$

where $\#\mathcal{C}_B$ is the cardinality of \mathcal{C}_B . Notice that all the sets of \mathcal{C}_B are contained in a ball of radius $(1 + 2\Lambda)2^{-n}$ and contain a ball of radius 2^{-n} . Hence, Lemma 2.1(2) implies that $(\#\mathcal{C}_B)^q \leq c_2 = C(N, 1 + 2\Lambda)^q$ for all $B \in \mathcal{B}$ and whence

$$\sum_{B \in \mathcal{B}} \mu(B)^q \leq c_2 \sum_{B \in \mathcal{B}} \sum_{Q \in \mathcal{C}_B} \mu(Q)^q.$$

On the other hand, the cardinality of the set $\{B \in \mathcal{B} : Q \cap B \neq \emptyset\}$ is at most $c_3 = C(N, 2\Lambda + 4)$ for all $Q \in \mathcal{Q}_n(x, 2r)$ (again by Lemma 2.1(2)). Thus, (4.8) follows with $c_1 = c_2 c_3$.

To find an estimate in the other direction, we choose $m \in \mathbb{N}$ so that $2^{-m} \leq r/(2\Lambda)$. Then we use the definition of \mathcal{Q}_m to find distinct balls B_Q of radius 2^{-m} for each $Q \in \mathcal{Q}_m(x, r/2)$ so that $B_Q \cap B(x, r/2) \subset Q \subset \Lambda B_Q$. Now we use Lemma 2.1(1) to find a disjoint subcollection \mathcal{B} of the family $\{\Lambda B_Q : Q \in \mathcal{Q}_m(x, r/2)\}$ so that $c_4 \sum_{B \in \mathcal{B}} \mu(B)^q \geq \sum_{Q \in \mathcal{Q}_m(x, r/2)} \mu(\Lambda B_Q)^q$, where $c_4 = M(N, 1/\Lambda)$. Then

$$\begin{aligned} \sum_{Q \in \mathcal{Q}_m(x, r/2)} \mu(Q)^q &\leq \sum_{Q \in \mathcal{Q}_m(x, r/2)} \mu(\Lambda B_Q)^q \\ &\leq c_4 \sum_{B \in \mathcal{B}} \mu(B)^q \leq c_4 S_{q, \Lambda 2^{-m}, r}(\mu, x) \end{aligned} \quad (4.9)$$

as the balls B_Q have centres in $B(x, r)$. The equality in (4.5) now follows by taking logarithms and limits and combining (4.8) and (4.9).

To prove the equalities in (4.6) and (4.7), let $r > 0$ and $n \in \mathbb{N}$ so that $2^{-n} < r/(3\Lambda + 1)$. For each $Q \in \mathcal{Q}_n(x, r)$, we let B_Q to be a ball with radius 2^{-n} such that $B_Q \cap B(x, r) \subset Q \subset \Lambda B_Q$. Then for all $y \in Q$, we have

$$Q \subset B(y, 2^{-n+1}\Lambda) \subset 3\Lambda B_Q \subset \bigcup_{Q' \in \mathcal{C}_Q} Q',$$

where $\mathcal{C}_Q = \{Q' \in \mathcal{Q}_n(x, 2r) : Q' \cap 3\Lambda B_Q \neq \emptyset\}$. Thus

$$\begin{aligned} \sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q) \log \sum_{Q' \in \mathcal{C}_Q} \mu(Q') &\geq \sum_{Q \in \mathcal{Q}_n(x, r)} \int_Q \log \mu(B(y, 2^{-n+1}\Lambda)) d\mu(y) \\ &= \int_{B(x, r)} \log \mu(B(y, 2^{-n+1}\Lambda)) d\mu(y) \\ &\geq \sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q) \log \mu(Q). \end{aligned}$$

Moreover,

$$\begin{aligned} \sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q) \log \sum_{Q' \in \mathcal{C}_Q} \mu(Q') - \sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q) \log \mu(Q) \\ = \sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q) \log \left(1 + \frac{\sum_{Q' \in \mathcal{C}_Q \setminus \{Q\}} \mu(Q')}{\mu(Q)} \right) \\ \leq \sum_{Q \in \mathcal{Q}_n(x, r)} \sum_{Q' \in \mathcal{C}_Q \setminus \{Q\}} \mu(Q') \leq c_5, \end{aligned}$$

where $c_5 = (5\Lambda)^{\log_2 N} \mu(B(x, 2r))$. The last estimate holds since each $Q' \in \mathcal{Q}_n(x, 2r)$ is contained in at most $(5\Lambda)^{\log_2 N}$ collections \mathcal{C}_Q by Lemma 2.1(2). Putting these estimates together, we have

$$\begin{aligned} \sum_{Q \in \mathcal{Q}_{n+1}(x, r)} \mu(Q) \log \mu(Q) &\leq \int_{B(x, r)} \log \mu(B(y, \delta)) d\mu(y) \\ &\leq \sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q) \log \mu(Q) + c_5 \end{aligned}$$

for all $2^{-n}\Lambda \leq \delta \leq 2^{-n+1}\Lambda$. From this the equalities in (4.6) and (4.7) follow easily.

We will next verify the claims (2) and (3). Let M_n be the cardinality of $\mathcal{Q}_n(x, r)$. By Lemma 2.1(2), we have $M_n \leq (r2^{n+1})^{\log_2 N}$ for all $n \in \mathbb{N}$ large enough. Using Hölder's inequality, we get

$$\mu(B(x, r))^q \leq \sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q)^q \leq \mu(B(x, r))^q M_n^{1-q}$$

for all $0 < q \leq 1$ and

$$\mu(B(x, r))^q M_n^{1-q} \leq \sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q)^q \leq \mu(B(x, r))^q$$

for all $q \geq 1$. The estimates above together with (4.5) give (3), and consequently, also (2).

For all $q, p \geq 0$, $\lambda \in (0, 1)$, and $n \in \mathbb{N}$, we get, using Hölder's inequality, that

$$\sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q)^{\lambda q + (1-\lambda)p} \leq \left(\sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q)^q \right)^\lambda \left(\sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q)^p \right)^{1-\lambda}. \quad (4.10)$$

Taking logarithms and combining this with (4.5) yields $\tau_{\lambda q + (1-\lambda)p}(\mu, x) \geq \tau_{\lambda q}(\mu, x) + \tau_{(1-\lambda)p}(\mu, x)$ giving (4).

Let us next show the claims (5) and (6). It follows easily from the claims (2) and (4) that $q \mapsto \dim_q(\mu, x)$ is continuous and decreasing on both intervals $(0, 1)$ and $(1, \infty)$. In particular, the limits $\lim_{q \downarrow 1} \dim_q(\mu, x)$ and $\lim_{q \uparrow 1} \dim_q(\mu, x)$ are well defined. The claims (5) and (6) now follow if we can show that given $0 < q < 1 < p$, we have

$$\tau_q(\mu, x)/(q-1) \geq \overline{\dim}_1(\mu, x) \geq \underline{\dim}_1(\mu, x) \geq \tau_p(\mu, x)/(p-1). \quad (4.11)$$

Let $r > 0$, $n \in \mathbb{N}$, and define $h_n(q) = \log \sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q)^q$ for all $q \geq 0$. From (4.10), we see that h_n is convex. As $\mathcal{Q}_n(x, r)$ has only a finite number of elements, it is also differentiable with $h'_n(1) = \mu(B(x, r))^{-1} \sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q) \log \mu(Q)$. Thus

$$\frac{h_n(q) - h_n(1)}{q-1} \leq h'_n(1) \leq \frac{h_n(p) - h_n(1)}{p-1}.$$

Using these estimates and the fact that $h_n(1) = \log \mu(B(x, r))$ is independent of n , we may calculate

$$\begin{aligned} \frac{1}{q-1} \liminf_{n \rightarrow \infty} \frac{\log \sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q)^q}{\log 2^{-n}} &= \limsup_{n \rightarrow \infty} \frac{h_n(q) - h_n(1)}{(q-1) \log 2^{-n}} \\ &\geq \limsup_{n \rightarrow \infty} \frac{\sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q) \log \mu(Q)}{\mu(B(x, r)) \log 2^{-n}} \geq \liminf_{n \rightarrow \infty} \frac{\sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q) \log \mu(Q)}{\mu(B(x, r)) \log 2^{-n}} \\ &\geq \liminf_{n \rightarrow \infty} \frac{h_n(p) - h_n(1)}{(p-1) \log 2^{-n}} = \frac{1}{p-1} \liminf_{n \rightarrow \infty} \frac{\log \sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q)^p}{\log 2^{-n}}. \end{aligned}$$

The desired estimate (4.11) now follows when we let $r \downarrow 0$ and combine the above estimates with (4.5), (4.6), and (4.7).

The claim (7) follows easily from the fact that given $Q \in \mathcal{Q}_n$, $q \geq 0$, and $r > 0$, the function $x \mapsto \mu(Q \cap B(x, r))^q$ is upper semicontinuous. Finally, the claim (8) is a direct consequence of the definitions. \square

Remark 4.5. If $q < 0$, it may well happen that $\tau_q(\mu, x) = -\infty$ for $x \in \text{spt}(\mu)$. However, letting $q_0(x) = \inf\{q \in \mathbb{R} : \tau_q(\mu, x) > -\infty\}$, the statements (4) and (5) of Theorem 4.4 remain true for all $q > q_0(x)$ ($q \neq 1$ in (5)).

We are now ready to prove Theorem 4.2. In the proof, we use the formulas (4.5)–(4.7) in defining $\tau_q(\mu, x)$ and $\dim_q(\mu, x)$ since they are much easier to work with than the original definitions using packings. Recall from Theorem 4.4 that

changing the partitions does not change the values of $\tau_q(\mu, x)$ or $\dim_q(\mu, x)$. In the proof, we have adapted some ideas from [22] and [19] to our setting.

Proof of Theorem 4.2. We start by verifying that

$$\lim_{q \downarrow 1} \dim_q(\mu, x) \leq \underline{\dim}_{\text{loc}}(\mu, x) \quad (4.12)$$

for μ -almost all $x \in X$. If this is not the case, then we find $q > 1$, $r > 0$, $0 < \alpha < \beta < \infty$, $k \in \mathbb{N}$, and $A \subset \text{spt}(\mu)$ with $\mu(A) > 0$ so that $\underline{\dim}_{\text{loc}}(\mu, x) < \alpha$ and $\sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q)^q < 2^{(1-q)n\beta}$ for all $n \geq k$ and $x \in A$. By Theorem 4.4(7), we may assume that A is a Borel set.

Let us fix $x \in A$ for which $\mu(A \cap B(x, r/2)) > 0$ and define $\mathcal{A}_n = \{Q \in \mathcal{Q}_n(x, r) : \mu(Q) > 2^{-n\alpha}\}$ and $A_n = \bigcup_{Q \in \mathcal{A}_n} Q$ for each $n \geq k$. Then

$$\mu(A_n) \leq \sum_{Q \in \mathcal{A}_n} \mu(Q)^q \mu(Q)^{1-q} \leq 2^{n\alpha(q-1)} \sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q)^q \leq 2^{-n(\beta-\alpha)(q-1)}$$

for $n \geq k$ and whence $\sum_{n \geq k} \mu(A_n) < \infty$ yielding $\mu(\bigcap_{l=k}^{\infty} \bigcup_{n \geq l} A_n) = 0$. Thus, if $Q_n(y)$ is the unique element of \mathcal{Q}_n that contains y , then for μ -almost all $y \in B(x, r/2)$, we have $Q_n(y) \in \mathcal{Q}_n(x, r) \setminus \mathcal{A}_n$ for sufficiently large n , and thus

$$\liminf_{n \rightarrow \infty} \frac{\log \mu(Q_n(y))}{\log 2^{-n}} \geq \alpha.$$

By Theorem B.1, this implies $\underline{\dim}_{\text{loc}}(\mu, y) \geq \alpha$ for μ -almost all $y \in B(x, r/2)$. This contradiction completes the proof of (4.12).

The proof of

$$\lim_{q \uparrow 1} \dim_q(\mu, x) \geq \overline{\dim}_{\text{loc}}(\mu, x) \quad (4.13)$$

follows by using a similar argument. If (4.13) fails in a set of positive measure, we find $r > 0$, $0 < q < 1$, $0 < \alpha < \beta < \infty$, $k \in \mathbb{N}$, and a Borel set $A \subset \text{spt}(\mu)$ so that $\overline{\dim}_{\text{loc}}(\mu, x) > \beta$ while $\sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q)^q < 2^{n\alpha(1-q)}$ for all $n \geq k$ and $x \in A$. Choosing $x \in A$ with $\mu(A \cap B(x, r/2)) > 0$ and letting $\mathcal{A}_n = \{Q \in \mathcal{Q}_n(x, r) : \mu(Q) < 2^{-n\beta}\}$ and $A_n = \bigcup_{Q \in \mathcal{A}_n} Q$, we get $\sum_{n \geq k} \mu(A_n) \leq \sum_{n \geq k} 2^{-n(\beta-\alpha)(1-q)} < \infty$. Combined with Theorem B.1, this implies that $\overline{\dim}_{\text{loc}}(\mu, y) \leq \beta$ for μ -almost all $y \in B(x, r/2)$ giving a contradiction.

It remains to show (4.4) assuming that μ has the density point property. For $x \in \text{spt}(\mu)$, we choose a sequence $0 < r_k \downarrow 0$ so that

$$\underline{\dim}_1(\mu, x) = \liminf_{k \rightarrow \infty} \liminf_{n \rightarrow \infty} \frac{\sum_{Q \in \mathcal{Q}_n(x, r_k)} \mu(Q) \log \mu(Q)}{\mu(B(x, r_k)) \log 2^{-n}}.$$

Observe that if $Q_{n,k}(y)$ is the unique element of $\mathcal{Q}_n(x, r_k)$ that contains y , then we have the identity

$$\sum_{Q \in \mathcal{Q}_n(x, r_k)} \mu(Q) \log \mu(Q) = \int_{B(x, r_k)} \log(\mu(Q_{n,k}(y))) d\mu(y). \quad (4.14)$$

By Fatou's lemma and Theorem B.1, we can now estimate

$$\begin{aligned} \underline{\dim}_1(\mu, x) &= \liminf_{k \rightarrow \infty} \liminf_{n \rightarrow \infty} \frac{\int_{B(x, r_k)} \log \mu(Q_{n,k}(y)) d\mu(y)}{\mu(B(x, r_k)) \log 2^{-n}} \\ &\geq \liminf_{k \rightarrow \infty} \int_{B(x, r_k)} \liminf_{n \rightarrow \infty} \frac{\log \mu(Q_{n,k}(y))}{\log 2^{-n}} d\mu(y) \\ &\geq \liminf_{k \rightarrow \infty} \int_{B(x, r_k)} \underline{\dim}_{\text{loc}}(\mu, y) d\mu(y). \end{aligned}$$

Since the mapping $y \mapsto \underline{\dim}_{\text{loc}}(\mu, y)$ is in $L^\infty(X, \mu)$ by Lemma 2.2, the density point property implies $\lim_{r \downarrow 0} \int_{B(x, r)} \underline{\dim}_{\text{loc}}(\mu, y) d\mu(y) = \underline{\dim}_{\text{loc}}(\mu, x)$ for almost all $x \in X$ and thus we get $\underline{\dim}_1(\mu, x) \geq \underline{\dim}_{\text{loc}}(\mu, x)$ for μ -almost every $x \in X$.

Finally, let us show that $\underline{\dim}_1(\mu, x) \leq \underline{\dim}_{\text{loc}}(\mu, x)$ for μ -almost all $x \in X$. For the rest of the proof we assume that for a given $Q \in \mathcal{Q}_n(x, r)$ we always have $\mu(Q) < 1/e$, where e is the base of the logarithm that we use. This does not affect the generality as if $\mu(\{x\}) = 0$, this is always the case for small $r > 0$ and if $\mu(\{x\}) > 0$ the claim is obviously true.

For a point $x \in \text{spt}(\mu)$, we first choose $0 < r_k \downarrow 0$ so that

$$\overline{\dim}_1(\mu, x) = \limsup_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\sum_{Q \in \mathcal{Q}_n(x, r_k)} \mu(Q) \log \mu(Q)}{\mu(B(x, r_k)) \log 2^{-n}}.$$

By changing the radii r_k slightly, we can also assume that

$$\mu(S(x, r_k)) = 0 \quad (4.15)$$

for all $k \in \mathbb{N}$. Here $S(x, r) = \{y \in X : d(x, y) = r\}$ for all $x \in X$ and $r > 0$. Indeed, there are clearly at most countably many values $r > 0$ so that $\mu(S(x, r)) > 0$ and if

$$h(r) = \limsup_{n \rightarrow \infty} \frac{\sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q) \log \mu(Q)}{\mu(B(x, r)) \log 2^{-n}},$$

then

$$\lim_{t \downarrow r} h(t) = h(r). \quad (4.16)$$

The proof of (4.16) is a bit technical and we postpone it to the end of this proof.

For $y \in B(x, r_k)$ and $n \in \mathbb{N}$ we define $f_{n,k}(y) = \log \mu(Q_{n,k}(y)) / \log 2^{-n}$, where $Q_{n,k}(y)$ is as above. Moreover, we let $g_{n,k}(y) = \sup_{m \geq n} f_{m,k}(y)$. Using Theorem B.1, (4.15), and Lemma 2.2, it now follows that $\lim_{n \rightarrow \infty} g_{n,k}(y) = \overline{\dim}_{\text{loc}}(\mu, y) \leq$

$\log_2 N$ for almost all $y \in B(x, r_k)$. Thus, defining $A_{n,k} = \{y \in B(x, r_k) : f_{m,k}(y) < 2 \log_2 N \text{ for all } m \geq n\}$, we have

$$\lim_{n \rightarrow \infty} \mu(B(x, r_k) \setminus A_{n,k}) = 0 \quad (4.17)$$

for all $k \in \mathbb{N}$. Recalling (4.14), we write

$$\begin{aligned} \overline{\dim}_1(\mu, x) &\leq \limsup_{k \rightarrow \infty} \frac{1}{\mu(B(x, r_k))} \limsup_{n \rightarrow \infty} \int_{A_{n,k}} f_{n,k}(y) d\mu(y) \\ &\quad + \limsup_{k \rightarrow \infty} \frac{1}{\mu(B(x, r_k))} \limsup_{n \rightarrow \infty} \int_{B(x, r_k) \setminus A_{n,k}} f_{n,k}(y) d\mu(y). \end{aligned} \quad (4.18)$$

Here

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_{A_{n,k}} f_{n,k}(y) d\mu(y) &\leq \limsup_{n \rightarrow \infty} \int_{A_{n,k}} g_{n,k}(y) d\mu(y) \\ &\leq \int_{A_{n,k}} \overline{\dim}_{\text{loc}}(\mu, y) d\mu(y) \leq \int_{B(x, r_k)} \overline{\dim}_{\text{loc}}(\mu, y) d\mu(y) \end{aligned} \quad (4.19)$$

since $g_{n,k}$ is bounded on $A_{n,k}$. To estimate the second term in the right-hand side of (4.18), we make the following observations: If $f_{n,k} \geq c$ on $Q \in \mathcal{Q}_n(x, r_k)$, then $\mu(Q) \leq 2^{-cn}$. Also, if k is large, then the collection $\mathcal{Q}_n(x, r_k)$ has at most $2^{n \log_2 N}$ elements by Lemma 2.1(2). Hence, for an integer $M > \log_2 N$, we get

$$\begin{aligned} \int_{B(x, r_k) \setminus A_{n,k}} f_{n,k}(y) d\mu(y) &\leq M \mu(B(x, r_k) \setminus A_{n,k}) + \sum_{i=M}^{\infty} \int_{\{i < f_{n,k} \leq i+1\}} f_{n,k}(y) d\mu(y) \\ &\leq M \mu(B(x, r_k) \setminus A_{n,k}) + 2^{n \log_2 N} \sum_{i=M}^{\infty} (i+1) 2^{-ni}, \end{aligned}$$

and, consequently,

$$\lim_{n \rightarrow \infty} \int_{B(x, r_k) \setminus A_{n,k}} f_{n,k}(y) d\mu(y) = 0 \quad (4.20)$$

by recalling (4.17). Combining this with (4.18) and (4.19) yields

$$\overline{\dim}_1(\mu, x) \leq \limsup_{k \rightarrow \infty} \int_{B(x, r_k)} \overline{\dim}_{\text{loc}}(\mu, y) d\mu(y)$$

whence $\overline{\dim}_1(\mu, x) \leq \overline{\dim}_{\text{loc}}(\mu, x)$ for μ -almost every x by using the density point property and the fact that $y \mapsto \overline{\dim}_{\text{loc}}(\mu, y)$ is in $L^\infty(X, \mu)$.

It remains to prove (4.16). Since $\mu(B(x, t)) \downarrow \mu(B(x, r))$ when $t \downarrow r$, it suffices to show that

$$\limsup_{n \rightarrow \infty} \left| \left(\sum_{Q \in \mathcal{Q}_n(x, t)} \mu(Q) \log \mu(Q) - \sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q) \log \mu(Q) \right) / \log 2^{-n} \right| \longrightarrow 0$$

as $t \downarrow r$. Take $t > r$ and let $\mathcal{A}_n = \{Q \in \mathcal{Q}_n(x, t) : Q \setminus B(x, r) \neq \emptyset = Q \cap B(x, r)\}$, and $\mathcal{C}_n = \{Q \in \mathcal{Q}_n(x, t) : Q \cap S(x, r) \neq \emptyset\}$ for all $n \in \mathbb{N}$. Now

$$\begin{aligned} & \sum_{Q \in \mathcal{Q}_n(x, t)} \mu(Q) \log \mu(Q) - \sum_{Q \in \mathcal{Q}_n(x, r)} \mu(Q) \log \mu(Q) = \sum_{Q \in \mathcal{A}_n} \mu(Q) \log \mu(Q) \\ & + \sum_{Q \in \mathcal{C}_n} \left(\mu(Q) \log \mu(Q) - \mu(Q \cap B(x, r)) \log \mu(Q \cap B(x, r)) \right). \end{aligned} \quad (4.21)$$

Following the proof of (4.20), we get $\limsup_{n \rightarrow \infty} \sum_{Q \in \mathcal{A}_n} \mu(Q) \log \mu(Q) / \log 2^{-n} \leq 2(\log_2 N) \mu(B(x, t) \setminus B(x, r)) \rightarrow 0$ as $t \downarrow r$. To handle (4.21), we take $n, m \in \mathbb{N}$ and define $\mathcal{E}_m = \{Q \in \mathcal{C}_n : \mu(Q) > (1 + 2^{-m}) \mu(Q \cap B(x, r))\}$ and $E_m = \bigcup_{Q \in \mathcal{E}_m} Q$. For $Q \in \mathcal{E}_m$, we have $\mu(Q) < (2^m + 1) \mu(Q \setminus B(x, r))$. As in the proof of (4.20), we get

$$\begin{aligned} 0 & \leq \frac{1}{\log 2^{-n}} \sum_{Q \in \mathcal{E}_m} \left(\mu(Q) \log \mu(Q) - \mu(Q \cap B(x, r)) \log \mu(Q \cap B(x, r)) \right) \\ & \leq \frac{1}{\log 2^{-n}} \sum_{Q \in \mathcal{E}_m} \mu(Q) \log \mu(Q) \\ & \leq \frac{1}{\log 2^{-n}} \sum_{Q \in \mathcal{E}_m} (2^m + 1) \mu(Q \setminus B(x, r)) \log \mu(Q \setminus B(x, r)) \\ & \leq 2(2^m + 1) (\log_2 N) \mu(E_m \setminus B(x, r)) + \varepsilon(n) \\ & \leq 2(2^m + 1) (\log_2 N) \mu(B(x, t) \setminus B(x, r)) + \varepsilon(n), \end{aligned}$$

where $\varepsilon(n) \downarrow 0$ as $n \rightarrow \infty$. Again, as in the proof of (4.20), we get

$$\begin{aligned} 0 & \leq \frac{1}{\log 2^{-n}} \sum_{Q \in \mathcal{C}_n \setminus \mathcal{E}_m} \left(\mu(Q) \log \mu(Q) - \mu(Q \cap B(x, r)) \log \mu(Q \cap B(x, r)) \right) \\ & \leq \frac{1}{\log 2^{-n}} \sum_{Q \in \mathcal{C}_n \setminus \mathcal{E}_m} \left(\mu(Q) - \mu(Q \cap B(x, r)) \right) \log \mu(Q \cap B(x, r)) \\ & \leq \frac{2^{-m}}{\log 2^{-n}} \sum_{Q \in \mathcal{C}_n \setminus \mathcal{E}_m} \mu(Q \cap B(x, r)) \log \mu(Q \cap B(x, r)) \\ & \leq 2^{-m+1} (\log_2 N) \mu(B(x, r)) + \varepsilon(n) \end{aligned}$$

using the definition of \mathcal{E}_m . The desired estimate follows now by letting first $n \rightarrow \infty$, then $t \downarrow r$, and finally $m \rightarrow \infty$. \square

To finish this section, we define the global L^q -spectrum and discuss the relation between the local and global concepts. Let μ be a measure on X with a bounded

support and $q \in \mathbb{R}$. The (*global*) L^q -spectrum of μ is defined by

$$\tau_q(\mu) = \liminf_{\delta \downarrow 0} \frac{\log S_{q,\delta}^*(\mu)}{\log \delta},$$

where

$$S_{q,\delta}^*(\mu) = \sup \left\{ \sum_{B \in \mathcal{B}} \mu(B)^q : \mathcal{B} \text{ is a } \delta\text{-packing of } \text{spt}(\mu) \cap X \right\}$$

for all $\delta > 0$. The (*global*) L^q -dimension is defined analogously to the local one as

$$\dim_q(\mu) = \tau_q(\mu)/(q - 1)$$

for all $q \in \mathbb{R}$, $q \neq 1$. Moreover, the (*global*) upper and lower entropy dimensions are defined as

$$\overline{\dim}_1(\mu) = \limsup_{\delta \downarrow 0} \frac{\int_X \log \mu(B(y, \delta)) d\mu(y)}{\log \delta},$$

$$\underline{\dim}_1(\mu) = \liminf_{\delta \downarrow 0} \frac{\int_X \log \mu(B(y, \delta)) d\mu(y)}{\log \delta},$$

respectively. If they agree, then their common value is denoted by $\dim_1(\mu)$. These definitions agree with the common definitions of the L^q -spectrum and dimensions for probability measures on Euclidean spaces. See e.g. [18, 19, 23].

We also note that the global spectrum and the global dimensions enjoy the same basic properties as the local ones. In particular, all the statements of Theorems 4.4 and 4.2 have their global counterparts with trivial modifications in the proofs. (In the statements of the results, remove all limits with respect to r and replace the balls $B(x, r)$ by the whole space X .)

Proposition 4.6. *Let μ be a Radon measure on a doubling metric space X with a compact support. Then*

$$\tau_q(\mu) = \min_{x \in \text{spt}(\mu)} \tau_q(\mu, x)$$

for every $q \in \mathbb{R}$. In particular,

$$\dim_q(\mu) = \begin{cases} \max_{x \in \text{spt}(\mu)} \dim_q(\mu, x), & \text{if } q < 1, \\ \min_{x \in \text{spt}(\mu)} \dim_q(\mu, x), & \text{if } q > 1. \end{cases}$$

Proof. We first notice that $S_{q,\delta,r}^*(\mu, x)$ decreases as $r \downarrow 0$. Therefore $\tau_q(\mu) \leq \tau_q(\mu, x)$ for every $x \in \text{spt}(\mu)$.

Let us now show that there exists a point $x \in \text{spt}(\mu)$ in which $\tau_q(\mu) \geq \tau_q(\mu, x)$. First we cover $\text{spt}(\mu)$ with finitely many balls $\{B(y_i, \frac{1}{2})\}_{i=1}^{k_1}$, $y_i \in \text{spt}(\mu)$. Then, for every j and $\delta > 0$, we have

$$S_{q,\delta,\frac{1}{2}}^*(\mu, y_j) \leq S_{q,\delta}^*(\mu) \leq \sum_{i=1}^{k_1} S_{q,\delta,\frac{1}{2}}^*(\mu, y_i) \leq k_1 \max_i S_{q,\delta,\frac{1}{2}}^*(\mu, y_i). \quad (4.22)$$

Let $(\delta_j)_{j=1}^\infty$ be a decreasing sequence tending to zero so that

$$\lim_{j \rightarrow \infty} \frac{\log S_{q, \delta_j}^*(\mu)}{\log \delta_j} = \liminf_{\delta \downarrow 0} \frac{\log S_{q, \delta}^*(\mu)}{\log \delta} = \tau_q(\mu).$$

For every $j \in \mathbb{N}$, let $i_j \in \{1, \dots, k_1\}$ so that $S_{q, \delta_j, \frac{1}{2}}^*(\mu, y_{i_j}) = \max_{1 \leq i \leq k_1} S_{q, \delta_j, \frac{1}{2}}^*(\mu, y_i)$. Now for some $i \in \{1, \dots, k_1\}$ the set $\{j \in \mathbb{N} : i_j = i\}$ is infinite. Considering a suitable subsequence of $(\delta_j)_{j=1}^\infty$ and using (4.22), we get

$$\liminf_{\delta \downarrow 0} \frac{\log S_{q, \delta, \frac{1}{2}}^*(\mu, x_1)}{\log \delta} = \tau_q(\mu),$$

where $x_1 = y_i$. Next we repeat the above argument by replacing $\frac{1}{2}$ with $\frac{1}{4}$ and $\text{spt}(\mu)$ with $\text{spt}(\mu) \cap B(x_1, \frac{1}{2})$. Then we find $x_2 \in B(x_1, \frac{1}{2})$ so that

$$\liminf_{\delta \downarrow 0} \frac{\log S_{q, \delta, \frac{1}{4}}^*(\mu, x_2)}{\log \delta} = \liminf_{\delta \downarrow 0} \frac{\log S_{q, \delta, \frac{1}{2}}^*(\mu, x_1)}{\log \delta} = \tau_q(\mu).$$

Continuing inductively, we get a sequence $x_i \in \text{spt}(\mu)$ with $d(x_{i+1}, x_i) \leq 2^{-i}$ and

$$\liminf_{\delta \downarrow 0} \frac{\log S_{q, \delta, 2^{-i}}^*(\mu, x_i)}{\log \delta} = \tau_q(\mu)$$

for every $i \in \mathbb{N}$. Since $\text{spt}(\mu)$ is compact, for $x = \lim_{i \rightarrow \infty} x_i$, we eventually get

$$\liminf_{\delta \downarrow 0} \frac{\log S_{q, \delta, 2^{-i+2}}^*(\mu, x)}{\log \delta} \leq \liminf_{\delta \downarrow 0} \frac{\log S_{q, \delta, 2^{-i}}^*(\mu, x_i)}{\log \delta}$$

for all i and thus $\tau_q(\mu, x) \leq \tau_q(\mu)$. \square

5. APPLICATIONS

In this section, we use the local homogeneity estimate of Theorem 3.7 as the final step in proving various new results. In fact, understanding the conical density and porosity questions in §5.1–§5.3 below was our main motivation for investigating the local homogeneity. In addition to Theorem 3.7, the proofs will be based on already known geometric conclusions.

In §5.4, we will use the theory developed in §4 to derive a local multifractal formalism for a large class of measures defined via Moran constructions in doubling metric spaces.

5.1. Upper conical densities in Euclidean spaces. Let $G(d, n)$ be the Grassmannian manifold of all n -dimensional linear subspaces of \mathbb{R}^d and $S^{d-1} = \{y \in \mathbb{R}^d : |y| = 1\}$ the unit sphere in \mathbb{R}^d . Then for $0 < \alpha \leq 1$, $V \in G(d, d-k)$, $\theta \in S^{d-1}$, $x \in \mathbb{R}^d$ and $r > 0$ we define cones

$$X(x, r, V, \alpha) = \{y \in B(x, r) : \text{dist}(y - x, V) < \alpha|y - x|\}$$

and

$$H(x, \theta, \alpha) = \{y \in \mathbb{R}^d : (y - x) \cdot \theta > \alpha|y - x|\}.$$

With small α the cones $X(x, r, V, \alpha)$ are small cones around the translate of the subspace V by x , whereas the cone $H(x, \theta, \alpha)$ is almost a half-space from the point x to the direction θ .

The distribution of Hausdorff and packing type measures inside cones is well studied and understood, see for example [36, 48, 38, 31]. For general measures the following theorem was proved in [12, Theorem 4.1] under the assumption that the Hausdorff dimension of the measure is greater than s . We improve this result by showing that the theorem is true even if we assume a lower bound only for the packing dimension of the measure.

Theorem 5.1. *If $d \in \mathbb{N}$, $k \in \{0, \dots, d-1\}$, $s > k$, and $0 < \alpha \leq 1$, then there exists a constant $c = c(d, k, s, \alpha) > 0$ so that for every Radon measure μ on \mathbb{R}^d we have*

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{d-1} \\ V \in G(d, d-k)}} \frac{\mu(X(x, r, V, \alpha) \setminus H(x, \theta, \alpha))}{\mu(B(x, r))} > c$$

for μ -almost all $x \in \{y \in \mathbb{R}^d : \overline{\dim}_{\text{loc}}(\mu, y) > s\}$.

Proof. We can reduce the proof to verifying the following condition, see [12, Proposition 4.5]: For a given $q, K \in \mathbb{N}$ and $1 < t < \infty$ there exists a constant $\varepsilon = \varepsilon(d, k, s, q, K, t) > 0$ so that for μ -almost all $x \in \{y \in \mathbb{R}^d : \overline{\dim}_{\text{loc}}(\mu, y) > s\}$ we may find arbitrarily small radii $r > 0$ and ball families \mathcal{B} with the following properties:

- (1) $B \subset B(x, r)$ for all $B \in \mathcal{B}$.
- (2) The collection $t\mathcal{B} = \{tB : B \in \mathcal{B}\}$ is a packing.
- (3) $\mu(B) > \varepsilon\mu(B(x, 3r))$ for all $B \in \mathcal{B}$.
- (4) If $\mathcal{B}' \subset \mathcal{B}$ with $\#\mathcal{B}' \geq \#\mathcal{B}/K$ and $V \in G(d, d-k)$, then there is a translate of V intersecting at least q balls from the collection \mathcal{B}' .

We will construct the families \mathcal{B} with the help of Theorem 3.7. Let $M = M(N_d, t^{-1})$ be the constant from Lemma 2.1(1), where N_d is the doubling constant of \mathbb{R}^d . Let $m = (s + k)/2$ and choose $0 < \delta < \min\{\delta_0, \frac{1}{4}\}$ so that $4^{-k}\delta^{k-m} \geq 2KMq$, where δ_0 is as in Theorem 3.7. By Theorem 3.7 there is $\varepsilon = \varepsilon(m, s, N_d, \delta) > 0$ so that $\limsup_{r \downarrow 0} \text{hom}_{\delta, \varepsilon, r}(\mu, x) \geq \delta^{-m}$ for μ -almost all $x \in \{y \in \mathbb{R}^d : \overline{\dim}_{\text{loc}}(\mu, y) > s\}$. Fix such a point x and let $r > 0$ so that $\text{hom}_{\delta, \varepsilon, \frac{3}{4}r}(\mu, x) > \delta^{-m}/2$. Now there is a $(\frac{3}{4}\delta r)$ -packing of $B(x, \frac{3}{4}r)$, say \mathcal{B}_0 , with $\#\mathcal{B}_0 > \delta^{-m}/2$ so that $\mu(B) > \varepsilon\mu(B(x, \frac{15}{4}r)) \geq \varepsilon\mu(B(x, 3r))$ for all $B \in \mathcal{B}_0$.

Lemma 2.1(1) gives a subcollection $\mathcal{B} \subset \mathcal{B}_0$ for which $t\mathcal{B}$ is also a packing and $\#\mathcal{B} \geq \#\mathcal{B}_0/M \geq \delta^{-m}/(2M)$. Now, because $\delta \leq \frac{1}{4}$, $B \subset B(x, r)$ for each $B \in \mathcal{B}$. Thus conditions (1)–(3) hold. The only property we need to verify is the condition (4). Suppose that $\mathcal{B}' \subset \mathcal{B}$ with $\#\mathcal{B}' \geq \#\mathcal{B}/K \geq \delta^{-m}/(2KM)$,

and let $V \in G(d, d - k)$. The orthogonal projection of $B(x, r)$ into the orthogonal complement of V can be covered by $4^k \delta^{-k}$ balls of radius $\frac{3}{4} \delta r$ and so some translate of V must intersect at least

$$4^{-k} \delta^k \# \mathcal{B}' \geq \frac{4^{-k} \delta^{k-m}}{2KM} \geq q$$

balls from the collection \mathcal{B}' . Thus also (4) holds and the proof is finished. \square

5.2. Porous measures on Euclidean spaces. We first define porosity for sets. Let $A \subset \mathbb{R}^d$, $k \in \{1, \dots, d\}$, $x \in A$, and $r > 0$. We define

$$\begin{aligned} \text{por}_k(A, x, r) = \sup \{ \varrho \geq 0 : \text{there are } y_1, \dots, y_k \in \mathbb{R}^d \text{ such that for every } i \\ A \cap B(y_i, \varrho r) = \emptyset \text{ and } \varrho r + |x - y_i| \leq r, \\ \text{and } (y_i - x) \cdot (y_j - x) = 0 \text{ if } j \neq i \} \end{aligned}$$

and from this the *k-porosity of A at x* as

$$\text{por}_k(A, x) = \liminf_{r \downarrow 0} \text{por}_k(A, x, r).$$

We refer to the balls $B(y_i, \varrho r)$ in the definition as holes. The notion of *k-porosity* was introduced in [30].

When we combine this definition with the porosity for measures, defined for the first time in [15], we obtain *k-porosity for measures*: Let μ be a measure on \mathbb{R}^d , $k \in \{1, \dots, d\}$, $x \in \mathbb{R}^d$, $r > 0$, and $\varepsilon > 0$. We set

$$\begin{aligned} \text{por}_k(\mu, x, r, \varepsilon) = \sup \{ \varrho \geq 0 : \text{there are } y_1, \dots, y_k \in \mathbb{R}^d \text{ such that for every } i \\ \mu(B(y_i, \varrho r)) \leq \varepsilon \mu(B(x, r)) \text{ and } \varrho r + |x - y_i| \leq r, \\ \text{and } (y_i - x) \cdot (y_j - x) = 0 \text{ if } j \neq i \} \end{aligned}$$

and the *k-porosity of the measure μ at x* is defined to be

$$\text{por}_k(\mu, x) = \lim_{\varepsilon \downarrow 0} \liminf_{r \downarrow 0} \text{por}_k(\mu, x, r, \varepsilon).$$

It follows from [15, §2] that $\text{por}_k(\mu, x) \leq \frac{1}{2}$ for μ -almost all $x \in \mathbb{R}^d$. Observe also that all the measurability issues concerning por_k can be handled in a similar way as with por_1 , see [15, §2].

We remark that a more precise name for the porosity just defined would be lower porosity, to distinguish this notion from the upper porosity of sets and measures, see e.g. [40, 53].

We provide an upper bound for the upper local dimension of measures with *k-porosity* close to the maximum value $\frac{1}{2}$. In [5], this results was proved for $k = 1$. The first estimates for the dimension of sets with 1-porosity close to $\frac{1}{2}$ are from [38] and [49]. For more recent results on the dimension of porous sets and measures, see [30, 29, 45, 10] and [15, 26, 6, 31, 5]. It is important to notice both here and in Theorem 5.6 that even if $\text{por}_1(\mu, x) > 0$ in a set of positive μ -measure, it is

possible that $\mu(A) = 0$ for all $A \subset X$ with $\inf_{x \in A} \text{por}_1(A, x) > 0$, see [5, Theorem 4.1].

Theorem 5.2. *If $d \in \mathbb{N}$, then there exists a constant $c = c(d) > 0$ so that for every Radon measure μ on \mathbb{R}^d we have*

$$\overline{\dim}_{\text{loc}}(\mu, x) \leq d - k + \frac{c}{-\log(1 - 2 \text{por}_k(\mu, x))}$$

for μ -almost all $x \in \mathbb{R}^d$.

Remark 5.3. (1) It is rather easy to see that the upper bound in Theorem 5.2 is asymptotically sharp as $\text{por}_k(\mu, x) \uparrow \frac{1}{2}$: For each $\varrho < \frac{1}{2}$ there exists a measure μ on \mathbb{R}^d with $\text{por}_k(\mu, x) \geq \varrho$ while $\underline{\dim}_{\text{loc}}(\mu, x) \geq d - k - c/\log(1 - 2\varrho)$ for μ -almost all $x \in \mathbb{R}^d$. The easiest way to see this is to consider a regular Cantor set $C \subset \mathbb{R}$ with 1-porosity ϱ and to let μ be the natural measure on $C^k \times [0, 1]^{d-k}$.

(2) The proof of Theorem 5.2 in the case $k = 1$ given in [5] is based on an extensive use of dyadic cubes. The interplay between cubes and balls caused many technical problems, which were finally solved by considering the boundary regions of cubes separately. The method used there does not work for k -porosity when $k \geq 2$ although the statement itself has nothing to do with co-dimension being one.

(3) Using the local L^q -dimension, we could prove a slightly weaker version of Theorem 5.2. Namely,

$$\overline{\dim}_{\text{loc}}(\mu, x) \leq d - k + \frac{c(d)}{-\log(1 - 2 \text{por}_k^*(\mu, x))}$$

for μ -almost all $x \in \mathbb{R}^d$, where the locally uniform version of porosity is defined as

$$\text{por}_k^*(\mu, x) = \lim_{\varepsilon \downarrow 0} \lim_{r \downarrow 0} \liminf_{r' \downarrow 0} \text{ess inf} \{ \text{por}_k(\mu, y, r', \varepsilon) : y \in B(x, r) \}.$$

The main idea is to iterate the following estimate which is obtained by applying Lemma 5.4 below:

$$S_{q, (1 - 2 \text{por}_k^*(\mu, x))\delta, r}(\mu, x) \leq c^{1-q} (1 - 2 \text{por}_k^*(\mu, x))^{(1-q)(k-d)} S_{q, \delta, r}(\mu, x),$$

for $0 < \delta < r$. In order for this approach to work, we first need to increase the resolution by letting δ tend to zero and then let r go to zero. This means that we have to iterate the estimate inside the ball $B(x, r)$ without changing the radius r and hence we must assume some uniformity over the porosity.

Before proving Theorem 5.2, we will exhibit a couple of geometric lemmas concerning k -porous sets.

Lemma 5.4. *Let $A \subset B(x_0, r) \subset \mathbb{R}^d$ be so that $\text{por}_k(A, z, r) \geq \varrho$ for every $z \in A$. Then the set A can be covered with $c(1 - 2\varrho)^{k-d}$ balls of radius $(1 - 2\varrho)r$, where $c = c(d)$.*

Proof. The proof is based on similar geometric arguments as used in [29, Theorem 2.5], [5, Lemmas 3.4 and 3.5], and [45, Lemma 5.1]. In the proof, we will omit some of the elementary, if tedious, details.

Let $c_1, c_2, c_3 > 0$ be small constants. We may assume that $\varrho > \frac{1}{2} - c_1$. A simple compactness argument implies that \mathbb{R}^d can be covered by $m = m(d, c_2)$ cones $\{H(0, \theta_i, 1 - c_2)\}_{i=1}^m$. Observe that $H(0, \theta_i, 1 - c_2)$ is a cone to the direction $\theta_i \in S^{d-1}$ with a small opening angle.

For each point $y \in A$ denote the centres of the holes obtained from the k -porosity on the scale r by y_1, \dots, y_k . Thus, $A \cap B(y_i, \varrho r) = \emptyset$ and $|y_i - y| + \varrho r \leq r$ for every i , and $(y_i - y) \cdot (y_j - y) = 0$ whenever $i \neq j$. We observe that A may be divided into m^k sets of the form

$$A_{\mathbf{i}} = \left\{ y \in A : y_j - y \in H(0, \theta_{i_j}, 1 - c_2) \text{ for every } j \in \{1, \dots, k\} \right\}.$$

where $\mathbf{i} = (i_1, \dots, i_k) \in \{1, \dots, m\}^k$. Since $(y_i - y) \cdot (y_j - y) = 0$ for all $y \in A$ and all $i \neq j$, it follows that actually most of the sets $A_{\mathbf{i}}$ are empty. Fix \mathbf{i} so that $A_{\mathbf{i}} \neq \emptyset$ and choose x so that $A_{\mathbf{i}} \cap B(x, c_3 r) \neq \emptyset$. Define

$$M_j = B(x, 2c_3 r) \cap \partial \left(\bigcup_{y \in A_{\mathbf{i}} \cap B(x, c_3 r)} B(y_j, \varrho r) \right)$$

for all $j \in \{1, \dots, k\}$ and let

$$M = \bigcap_{j=1}^k M_j.$$

Here ∂C is the topological boundary of a given set C .

By simple (but rather technical) geometric inspections, we observe that if c_1, c_2 , and c_3 are chosen small enough (depending only on d), then the following assertions are true: If f is the orthogonal projection from M to the k -dimensional linear subspace $\bigcap_{j=1}^k \theta_{i_j}^\perp$, then

$$|f(y) - f(z)| \leq |y - z| \leq 2|f(y) - f(z)|$$

for all $y, z \in M$, so f is bi-Lipschitz with constant 2. Moreover, $\text{dist}(y, M) \leq 2\sqrt{d}(1 - 2\varrho)r$ for all $y \in A_{\mathbf{i}} \cap B(x, c_3 r)$. These estimates easily imply that $B(x, c_3 r) \cap A_{\mathbf{i}}$ may be covered by $c_4(1 - 2\varrho)^{k-d}$ balls of radius $(1 - 2\varrho)r$, where c_4 depends only on d and the choice of c_3 . On the other hand, the set $A_{\mathbf{i}} \cap B(x, r)$ is clearly covered by $2^{2d}c_3^{-d}$ balls of radius $c_3 r$ and finally A is covered by less than $m^k 2^{2d} c_3^{-d} c_4 (1 - 2\varrho)^{k-d}$ balls of radius $(1 - 2\varrho)r$. \square

Next we turn the previous lemma into a homogeneity estimate.

Lemma 5.5. *Let $0 < \varrho < \frac{1}{2}$ and let μ be a Radon measure on \mathbb{R}^d such that $\mu(A) > 0$, where $A \subset \{x \in \mathbb{R}^d : \text{por}_k(\mu, x) > \varrho\}$. Then for each $\varepsilon > 0$ there is a*

Borel set $A_\varepsilon \subset A$ with $\mu(A_\varepsilon) > 0$ such that

$$\limsup_{r \downarrow 0} \text{hom}_{1-2\varrho, \varepsilon, r}(\mu, x) < c(1-2\varrho)^{k-d}$$

for every $x \in A_\varepsilon$, where $c = c(d)$.

Proof. Let $\varepsilon > 0$ and take $r_0 > 0$ so that the set

$$A_\varepsilon = \{x \in A : \text{por}_k(\mu, x, r, \varepsilon/2) \geq \varrho \text{ for all } 0 < r < r_0\}$$

has positive μ -measure. Now take a density point $x \in A_\varepsilon$ and a radius $0 < r \leq r_0/5$ for which

$$\frac{\mu(A_\varepsilon \cap B(x, 5r))}{\mu(B(x, 5r))} > 1 - \varepsilon. \quad (5.1)$$

Let \mathcal{B} be a $((1-2\varrho)r)$ -packing of $B(x, r)$ so that $\mu(B) > \varepsilon\mu(B(x, 5r))$ for all $B \in \mathcal{B}$. Write $A_{\mathcal{B}}$ for the centres of the balls in \mathcal{B} . For each $B \in \mathcal{B}$ choose $y \in A_\varepsilon \cap B$. Because of (5.1), such a point y exists. A direct calculation using the k -porosity at y on the scale r implies that

$$\text{por}_k(A_{\mathcal{B}}, x, r) \geq \varrho - 2(1-2\varrho),$$

where x is the centre of B . Since this holds for all $x \in A_{\mathcal{B}}$, Lemma 5.4 implies that $A_{\mathcal{B}}$ may be covered by $c(1-2(\varrho-2(1-2\varrho)))^{k-d} = 5^{k-d}c(1-2\varrho)^{k-d}$ balls of radius $5(1-2\varrho)r$. Here $c = c(d)$ is the constant of Lemma 5.4. It now follows that $\#\mathcal{B} = \#A_{\mathcal{B}} \leq 10^d 5^{k-d} c(1-2\varrho)^{k-d}$ yielding the claim.

It is important to note here that we are not covering the set A_ε as it generally is not even porous. \square

Proof of Theorem 5.2. Let $c = c(d) \geq 1$ be the constant of Lemma 5.5 and let $0 < \varrho < \frac{1}{2}$. Our aim is to apply Theorem 3.7 with

$$m = d - k + \frac{\log c}{-\log(1-2\varrho)}, \quad s = m + \frac{4d \log 240}{-\log(1-2\varrho)},$$

and $\delta = 1 - 2\varrho$. Let $t = (m + s)/2$ and take $M = M(N_d, \frac{1}{10})$ from Lemma 2.1(1). Here $N_d \leq 4^d$ is the doubling constant of \mathbb{R}^d . To be able to use Theorem 3.7, we need to check that $\delta < \delta_0$, where $\delta_0 = \delta_0(m, s, N_d)$ is defined in (3.12). Inspecting the proof of Lemma 2.1(1), we estimate $M \leq 4^{d(\log_2 60+1)} = 4^d \cdot 60^{2d}$. Hence

$$\begin{aligned} \delta_0 &= (5^t/4M)^{1/(t(1-m/s))} \geq (1/(4^{d+1} \cdot 60^{2d}))^{1/(t(1-m/s))} \\ &> (240^{2d})^{-2/(s-m)} = 1 - 2\varrho = \delta \end{aligned}$$

and Theorem 3.7 is applicable. Let ε to be the constant $\varepsilon_0 = \varepsilon_0(m, s, N_d, \delta)$ of Theorem 3.7.

Proving the theorem now easily reduces to showing that $\overline{\dim}_{\text{loc}}(\mu, x) \leq s$ almost everywhere on the set $A = \{y \in \mathbb{R}^d : \text{por}_k(\mu, y) > \varrho\}$. We may assume that $\mu(A) > 0$ since otherwise there is nothing to prove. Suppose to the contrary that

there exists a set $A' \subset A$ with positive measure such that $\overline{\dim}_{\text{loc}}(\mu, x) > s$ for all $x \in A'$. Using Lemma 5.5, we find a set $A_\varepsilon \subset A'$ with $\mu(A_\varepsilon) > 0$ so that

$$\limsup_{r \downarrow 0} \text{hom}_{1-2\varrho, \varepsilon, r}(\mu, x) < c(1-2\varrho)^{-d+k} = (1-2\varrho)^{-m}$$

for all $x \in A_\varepsilon$. Now Theorem 3.7 implies that $\overline{\dim}_{\text{loc}}(\mu, x) \leq s$ for μ -almost all $x \in A_\varepsilon$. This contradiction finishes the proof. \square

5.3. Porous measures on regular metric spaces. If we consider k -porosity with $k = 1$ there is no orthogonality condition on the direction of holes. By replacing the Euclidean distance $|x - y_1|$ by $d(x, y_1)$ in the definition, it makes perfect sense to investigate 1-porosity, which we simply call porosity, in a general metric space (X, d) .

If X is an s -regular metric space, then for any $A \subset X$ with $\inf_{x \in A} \text{por}_1(A, x) \geq \varrho$, we have

$$\dim_{\text{p}}(A) \leq s - c\varrho^s,$$

see [28, Theorem 4.7]. Our result for measures in this direction is the following.

Theorem 5.6. *Assume that X is an s -regular metric space and μ is a Radon measure on X satisfying the density point property. Then*

$$\overline{\dim}_{\text{loc}}(\mu, x) \leq s - c \text{por}_1(\mu, x)^s.$$

for μ -almost all $x \in X$. Here $c > 0$ is a constant depending on the data used in defining the s -regularity.

In the proof of Theorem 5.2, we used a known estimate for k -porous sets via a density point argument. The density point property in Theorem 5.6 is needed in order to use a similar approach. To prove Theorem 5.6, we recall the following estimate from [28, Corollary 4.6]. Recall that for an s -regular measure ν , the constants a_ν , b_ν , and r_ν are such that $a_\nu r^s \leq \nu(B(x, r)) \leq b_\nu r^s$ whenever $0 < r < r_\nu$ and $x \in \text{spt}(\nu)$.

Lemma 5.7. *If ν is s -regular on X with $\text{spt}(\nu) = X$, then there exist constants $c_1, c_2, c_3 > 0$ depending only on s, a_ν, b_ν that satisfy the following: If $x \in X$, $r_p > 0$, $0 < r < c_3 \min\{r_p, r_\nu\}$, $A \subset B(x, r)$, and $\text{por}_1(A, y, r') \geq \varrho > 0$ for all $y \in A$ and $0 < r' < r_p$, then*

$$\nu(A(r'')) \leq c_1 \nu(B(x, r)) \left(\frac{r''}{r}\right)^{c_2 \varrho^s}$$

for all $0 < r'' < r$.

Proof of Theorem 5.6. Let ν be an s -regular measure on X with $\text{spt}(\nu) = X$ and let the constants $c_1, c_2, c_3 > 0$ be as in Lemma 5.7. Let $0 < \varrho' < \varrho$ and choose

$\delta' > 0$ so small that $\log(2c_1b_\nu/a_\nu)/\log(1/\delta) \leq c_2\varrho'^s$ for all $0 < \delta \leq \delta'$. We want to apply Theorem 3.7 with

$$m' = s - \frac{c_2a_\nu}{2b_\nu}(\varrho'/4)^s + \frac{\log(c_1b_\nu/a_\nu)}{-\log \delta'}, \quad s' = s - \frac{c_2a_\nu}{4b_\nu}(\varrho'/4)^s,$$

and $0 < \delta < \min\{1, \varrho'r_\nu/2, \delta', \delta_0\}$, where $\delta_0 = \delta_0(m', s', N) > 0$ is as in Theorem 3.7. Let $\varepsilon > 0$ be the constant $\varepsilon_0 = \varepsilon_0(m', s', N, \delta) > 0$ from Theorem 3.7.

It is clearly sufficient to prove that given $\varrho > 0$, we have $\overline{\dim}_{\text{loc}}(\mu, x) \leq s - c\varrho^s$ for almost all $x \in A_\varepsilon$, where

$$A_\varepsilon = \{x \in X : \text{por}(\mu, x, r, \varepsilon/2) \geq \varrho' \text{ for all } 0 < r < r_0\}.$$

We note that A_ε is a Borel set. (A careful inspection of the definitions shows that it is in fact closed.) Let $x \in A_\varepsilon$ be a density point of A_ε and take $0 < r < \min\{1, r_0/2\}$ so small that

$$\frac{\mu(A_\varepsilon \cap B(x, 5r))}{\mu(B(x, 5r))} > 1 - \varepsilon. \quad (5.2)$$

Our goal is to show that for any (δr) -packing \mathcal{B} of

$$A = \{y \in B(x, r) : \mu(B(y, \delta r)) > \varepsilon\mu(B(x, 5r))\}$$

the set $A_{\mathcal{B}} = \{y \in A : y \text{ is the centre point of some } B \in \mathcal{B}\}$ satisfy the assumptions of Lemma 5.7. Using Lemma 5.7, we are able to estimate the cardinality of \mathcal{B} and hence also $\text{hom}_{\delta, \varepsilon, r}(\mu, x)$. The desired upper bound for $\overline{\dim}_{\text{loc}}(\mu, x)$ then follows from Theorem 3.7.

Fix a (δr) -packing \mathcal{B} of A and $y \in A_{\mathcal{B}}$. Assume first that $0 < r' < 2\delta r/\varrho'$. If $B(y, \varrho'r'/4) \setminus B(y, (\frac{a_\nu}{2b_\nu})^{1/s}\varrho'r'/4) = \emptyset$, then it follows from the s -regularity of ν that

$$a_\nu(\varrho'r'/4)^s \leq \nu(B(y, \varrho'r'/4)) = \nu(B(y, (\frac{a_\nu}{2b_\nu})^{1/s}\varrho'r'/4)) \leq \frac{a_\nu}{2}(\varrho'r'/4)^s$$

which is impossible. Hence there exists a point $z \in B(y, \varrho'r'/4) \setminus B(y, (\frac{a_\nu}{2b_\nu})^{1/s}\varrho'r'/4)$. Since $\varrho'r'/4 < \delta r$, we have $A_{\mathcal{B}} \cap B(z, (\frac{a_\nu}{2b_\nu})^{1/s}\varrho'r'/4) = \emptyset$ and as $(\frac{a_\nu}{2b_\nu})^{1/s}\varrho'r'/4 + d(y, z) \leq \varrho'r'/2 < r'$, it follows that $\text{por}_1(A_{\mathcal{B}}, y, r') \geq (\frac{a_\nu}{2b_\nu})^{1/s}\varrho'r'/4$ for all $0 < r' < 2\delta r/\varrho'$.

Let us next assume that $2\delta r/\varrho' \leq r' \leq 4(\frac{b_\nu}{a_\nu})^{1/s}r$. If $A_\varepsilon \cap B(y, \delta r) = \emptyset$, then (5.2) and the definition of A imply that

$$\mu(B(y, \delta r)) \leq \mu(B(x, 5r) \setminus A_\varepsilon) < \varepsilon\mu(B(x, 5r)) \leq \mu(B(y, \delta r)).$$

Hence there exists a point $z \in A_\varepsilon \cap B(y, \delta r)$. The definition of A_ε in turn guarantees the existence of a point $w \in X$ such that $\mu(B(w, \varrho'r')) \leq \frac{\varepsilon}{2}\mu(B(z, r'))$ and $\varrho'r' + d(z, w) \leq r'$. Now

$$\varrho'r'/2 + d(y, w) \leq \varrho'r'/2 + d(y, z) + d(z, w) \leq \varrho'r' + d(z, w) \leq r'$$

and $A_{\mathcal{B}} \cap B(w, \varrho' r'/2) = \emptyset$ because for any $w' \in B(w, \varrho' r'/2)$ we have $\mu(B(w', \delta r)) \leq \mu(B(w, \varrho' r')) \leq \frac{\varepsilon}{2} \mu(B(z, r')) < \varepsilon \mu(B(x, 5r))$. Therefore $\text{por}_1(A_{\mathcal{B}}, y, r') \geq \varrho'/2$. Consequently, for $2\delta r/\varrho' \leq r' \leq 4(\frac{b_\nu}{a_\nu})^{1/s} r$ we have $\text{por}_1(A_{\mathcal{B}}, y, r') \geq (\frac{a_\nu}{b_\nu})^{1/s} \varrho'/4$.

Now let $4(\frac{b_\nu}{a_\nu})^{1/s} r < r' < r_\nu$ and put $t = \frac{1}{4}(\frac{b_\nu}{a_\nu})^{1/s} r' + 2r$. Then $t < \frac{3}{4}(\frac{a_\nu}{b_\nu})^{1/s} r'$ and thus

$$\nu(B(y, t)) \leq b_\nu t^s < a_\nu \left(\frac{3r'}{4}\right)^s \leq \nu(B(y, \frac{3}{4}r')).$$

So there exists $w \in B(y, \frac{3}{4}r') \setminus B(y, t)$. Now $A_{\mathcal{B}} \cap B(w, \frac{1}{4}(\frac{a_\nu}{b_\nu})^{1/s} r') \subset B(x, r) \cap B(w, \frac{1}{4}(\frac{a_\nu}{b_\nu})^{1/s} r') = \emptyset$ and thus $\text{por}_1(A_{\mathcal{B}}, y, r') \geq \frac{1}{4}(\frac{a_\nu}{b_\nu})^{1/s}$.

Putting the three estimates together, we have

$$\text{por}_1(A_{\mathcal{B}}, y, r') \geq (\frac{a_\nu}{2b_\nu})^{1/s} \varrho'/4$$

for all $y \in A_{\mathcal{B}}$ and $0 < r' < r_\nu$. We can now use Lemma 5.7 to obtain

$$\#\mathcal{B}a_\nu(\delta r)^s \leq \sum_{B \in \mathcal{B}} \nu(B) = \nu(A_{\mathcal{B}}(\delta r)) \leq c_1 \nu(B(x, r)) \delta^{\frac{c_2 a_\nu}{2b_\nu} (\varrho'/4)^s} \leq c_1 b_\nu r^s \delta^{\frac{c_2 a_\nu}{2b_\nu} (\varrho'/4)^s}$$

for all $0 < r < c_3 r_\nu$. Since this is true for all (δr) -packings \mathcal{B} of A , and (5.2) is true for all small $r > 0$, we get

$$\limsup_{r \downarrow 0} \text{hom}_{\delta, \varepsilon, r}(\mu, x) \leq \frac{c_1 b_\nu}{a_\nu} \delta^{\frac{c_2 a_\nu}{2b_\nu} (\varrho'/4)^s - s} < \delta^{-m'}$$

for μ -almost every $x \in A_\varepsilon$. Therefore, by Theorem 3.7, we have $\overline{\dim}_{\text{loc}}(\mu, x) \leq s' = s - \frac{c_2 a_\nu}{4b_\nu} (\varrho'/4)^s$ for μ -almost every $x \in A_\varepsilon$. This completes the proof. \square

5.4. Local multifractal analysis in metric spaces. We next introduce a class of Moran constructions in a complete doubling metric space X and show how Theorem 4.2 can be applied to calculate the local dimensions for a large class of measures defined on these Moran fractals. Then we turn to study the multifractal spectrum of these measures. Our main goal is to show that using the technique introduced in §4, we can push the standard methods used to calculate the local dimensions for self-similar measures on Euclidean spaces (see [9, 47, 18]) to obtain analogous results in doubling metric spaces with very mild regularity assumptions, see Remark 5.10.

Let $m \in \mathbb{N}$, $\Sigma = \{1, \dots, m\}^{\mathbb{N}}$, $\Sigma_n = \{1, \dots, m\}^n$ for all $n \in \mathbb{N}$, and $\Sigma_* = \{\emptyset\} \cup \bigcup_{n \in \mathbb{N}} \Sigma_n$. If $n \in \mathbb{N}$ and $\mathbf{i} \in \Sigma \cup \bigcup_{j=n}^{\infty} \Sigma_j$, then we let $\mathbf{i}|_n = (i_1, \dots, i_n)$ (and $\mathbf{i}|_0 = \emptyset$). The concatenation of two words $\mathbf{i} \in \Sigma_*$ and $\mathbf{j} \in \Sigma \cup \Sigma_*$ is denoted by \mathbf{ij} . We also set $\mathbf{i}^- = \mathbf{i}|_{n-1}$ for $\mathbf{i} \in \Sigma_n$ and $n \in \mathbb{N}$. By $|\mathbf{i}|$, we denote the length of a word $\mathbf{i} \in \Sigma_*$. We assume that $\{E_{\mathbf{i}} : \mathbf{i} \in \Sigma_*\}$ is a collection of compact subsets of X that satisfy the following conditions for some constants $0 < C_0, C_1 < \infty$:

- (M1) $E_{\mathbf{i}} \subset E_{\mathbf{i}^-}$ for all $\emptyset \neq \mathbf{i} \in \Sigma_*$.
- (M2) $E_{\mathbf{i}\mathbf{i}} \cap E_{\mathbf{i}j} = \emptyset$ if $\mathbf{i} \in \Sigma_*$ and $i \neq j$.
- (M3) For each $\mathbf{i} \in \Sigma_*$, there is $x \in E_{\mathbf{i}}$ such that $B(x, C_0 \text{diam}(E_{\mathbf{i}})) \subset E_{\mathbf{i}}$.
- (M4) $\text{diam}(E_{\mathbf{i}|_n}) \rightarrow 0$ as $n \rightarrow \infty$, for each $\mathbf{i} \in \Sigma$.

(M5) $\text{diam}(E_{i-})/\text{diam}(E_i) \leq C_1 < \infty$ for all $\emptyset \neq \mathbf{i} \in \Sigma_*$.

We define the limit set of the construction as $E = \bigcap_{n \in \mathbb{N}} \bigcup_{\mathbf{i} \in \Sigma_n} E_{\mathbf{i}}$ and given $\mathbf{i} \in \Sigma$, denote by $x_{\mathbf{i}}$ the point obtained as $\{x_{\mathbf{i}}\} = \bigcap_{n \in \mathbb{N}} E_{\mathbf{i}|_n}$. We assume that for each $i \in \{1, \dots, m\}$ there is a continuous function $r_i: E \rightarrow (0, 1)$. Given $x \in E$, and $\mathbf{i} \in \Sigma_n$, we let $r_{\mathbf{i}}(x) = \prod_{k=1}^n r_{i_k}(x)$. Moreover, we assume that

(M6) $\lim_{n \rightarrow \infty} \log \text{diam}(E_{\mathbf{i}|_n}) / \log r_{\mathbf{i}|_n}(x_{\mathbf{i}}) = 1$ uniformly for all $\mathbf{i} \in \Sigma$.

Throughout this subsection, we will assume that $\{E_{\mathbf{i}}\}$ is a collection of compact sets satisfying the assumptions (M1)–(M6). Our next lemma shows how we can obtain a δ -partition of X from the elements of $\{E_{\mathbf{i}}\}$ which are roughly of size δ .

Lemma 5.8. *For $n \in \mathbb{N}$, denote $\mathcal{E}_n = \{E_{\mathbf{i}} : \text{diam}(E_{\mathbf{i}}) \leq C_1/(C_0 2^n) < \text{diam}(E_{i-})\}$. Then there is a (2^{-n}) -partition \mathcal{Q}_n of X such that each $E_{\mathbf{i}} \in \mathcal{E}_n$ is a subset of some $Q \in \mathcal{Q}_n$ and all elements of \mathcal{Q}_n contain at most one element of \mathcal{E}_n . (The constant Λ used in defining these partitions is independent of n .)*

Proof. Consider a maximal collection \mathcal{B}_n of disjoint balls of radius 2^{-n} contained in $X \setminus \bigcup \mathcal{E}_n$. Define $\mathcal{A}_n = \mathcal{E}_n \cup \mathcal{B}_n = \{A_1, A_2, \dots\}$. For each $x \in X$, we let $i_x = \min\{j \in \mathbb{N} : \text{dist}(x, A_j) = \min_{A \in \mathcal{A}_n} \text{dist}(x, A)\}$ and set $Q_{A_i} = \{x \in X : i_x = i\}$ for all $i \in \mathbb{N}$. It is then easy to see that $\mathcal{Q}_n = \{Q_A : A \in \mathcal{A}_n\}$ is the desired (2^{-n}) -packing. Observe that each $Q \in \mathcal{Q}_n$ is a Borel set since $\bigcup_{i=1}^k Q_{A_i}$ is closed for all k . Moreover, the constant Λ of this partition depends only on the constants C_0 and C_1 as one may choose $\Lambda = C_0 C_1 + 1$. \square

Let μ be a probability measure on X with $\text{spt}(\mu) = E$. Then for each $\mathbf{i} \in \Sigma_*$, μ induces a probability vector $p_{\mathbf{i}} = (p_{\mathbf{i}}^1, \dots, p_{\mathbf{i}}^m)$ with $p_{\mathbf{i}}^i > 0$ for $i \in \{1, \dots, m\}$ such that $\mu(E_{\mathbf{i}i}) = p_{\mathbf{i}}^i \mu(E_{\mathbf{i}})$ for $i \in \{1, \dots, m\}$. Given $\mathbf{i} \in \Sigma_n$, we denote $\mu_{\mathbf{i}} := \mu(E_{\mathbf{i}}) = \prod_{j=1}^n p_{\mathbf{i}|_{j-1}}^{i_j}$. In the next theorem, we assume that the weights $p_{\mathbf{i}}$ are controlled in terms of continuous probability functions $p(x) = (p_1(x), \dots, p_m(x))$. More precisely, we assume that for each $i \in \{1, \dots, m\}$, the function $p_i: E \rightarrow (0, 1)$ is continuous with $\sum_{i=1}^m p_i(x) = 1$ for all $x \in E$ and that for some $a > 0$, we have $p_i(x) > a$ for all x and i . As with the functions r_i , we define $\mu_{\mathbf{i}}(x) = \prod_{k=1}^n p_{i_k}(x)$ when $\mathbf{i} \in \Sigma_n$.

Theorem 5.9. *Let $\{E_{\mathbf{i}} : \mathbf{i} \in \Sigma_*\}$ be a collection of compact sets that satisfy the conditions (M1)–(M6). Suppose that μ is a probability measure on E and let $p_{\mathbf{i}}$ and p be as above. If $p_{\mathbf{i}|_n} \rightarrow p(x_{\mathbf{i}})$ as $n \rightarrow \infty$ uniformly for all $\mathbf{i} \in \Sigma$, then, for all $x \in E$ and all $q \geq 0$, $\tau_q(\mu, x)$ is the unique $\tau \in \mathbb{R}$ that satisfies*

$$\sum_{i=1}^m p_i(x)^q r_i(x)^{-\tau} = 1. \quad (5.3)$$

Moreover,

$$\dim_1(\mu, x) = \dim_{\text{loc}}(\mu, x) = \frac{\sum_{i=1}^m p_i(x) \log p_i(x)}{\sum_{i=1}^m p_i(x) \log r_i(x)} \quad (5.4)$$

for μ -almost all $x \in E$.

Proof. We prove the claim (5.3). The identities (5.4) then follow from (5.3) by implicit differentiation together with (4.3) and Theorem 4.4(6).

For each $n \in \mathbb{N}$, let \mathcal{E}_n and \mathcal{Q}_n be as in Lemma 5.8. Given $\emptyset \neq \mathbf{i} \in \Sigma_*$, we denote by $Q_{\mathbf{i}}$ the unique element of $\bigcup_{n \in \mathbb{N}} \mathcal{Q}_n$ that contains $E_{\mathbf{i}}$ and does not contain $E_{\mathbf{i}^-}$ (we assume without loss of generality that $\mathcal{E}_1 = \{E_{\emptyset}\}$ so that this makes sense for all n). Let us fix $q \geq 0$, $x \in E$ and let $\mathbf{i} \in \Sigma$ so that $x = x_{\mathbf{i}}$. Let τ be as in (5.3). We first prove that $\tau_q(\mu, x) \geq \tau$. Let $0 < c < 1$. Since $p_{\mathbf{i}|_n} \rightarrow p(x_{\mathbf{i}})$ uniformly and $y \mapsto p(y)$ is continuous, we may choose n_0 so large that $p_{\mathbf{j}}^i > cp_i(x)$ whenever $i \in \{1, \dots, m\}$, $\mathbf{j} \in \Sigma_*$, and $E_{\mathbf{j}} \subset E_{\mathbf{i}|_{n_0}}$. Making n_0 larger if necessary, we may also assume that

$$cr_i(y) \leq r_i(x) \leq r_i(y)/c \quad (5.5)$$

for all $y \in E_{\mathbf{i}|_{n_0}}$ and all $i \in \{1, \dots, m\}$.

Now, for all $r > 0$, we choose $N_0 \geq n_0$ so that $Q_{\mathbf{j}} \subset B(x, r)$ whenever $\mathbf{j} \in \Sigma_*$, $|\mathbf{j}| \geq N_0$, and $E_{\mathbf{j}} \subset E_{\mathbf{i}|_{N_0}}$. Given $n \geq N_0$, let $Z_n = \{\mathbf{j} \in \Sigma_* : Q_{\mathbf{j}} \in \mathcal{Q}_n \text{ and } E_{\mathbf{j}} \subset E_{\mathbf{i}|_{N_0}}\}$. Let $\varepsilon_n = \min_{\mathbf{j} \in Z_n} (\text{diam}(E_{\mathbf{j}})/r_{\mathbf{j}}(x))^{-\tau}$. Now, denoting $c_0 = C_1^{-|\tau|} \mu_{\mathbf{i}|_{N_0}}^q$, we get an estimate

$$2^{n\tau} \sum_{\mathbf{j} \in Z_n} \mu(Q_{\mathbf{j}})^q \geq C_1^{-|\tau|} \sum_{\mathbf{j} \in Z_n} \mu_{\mathbf{j}}^q \text{diam}(E_{\mathbf{j}})^{-\tau} \geq c_0 \varepsilon_n \sum_{\mathbf{j} \in Z_n} c^{q|\mathbf{j}|} \mu_{\mathbf{j}}(x)^q r_{\mathbf{j}}(x)^{-\tau}. \quad (5.6)$$

For each $\mathbf{j} \in Z_n$, pick $y \in E_{\mathbf{j}}$. Using (M6), we may assume that $\log r_{\mathbf{j}}(y) \geq 2 \log \text{diam}(E_{\mathbf{j}})$ by making N_0 larger if necessary. Letting $r_{\max} = \max\{r_i(y) : y \in E \text{ and } i \in \{1, \dots, m\}\}$, we have

$$\log r_{\max}^{|\mathbf{j}|} \geq \log r_{\mathbf{j}}(y) \geq 2 \log \text{diam}(E_{\mathbf{j}}) \geq -2n \log 2,$$

and consequently,

$$|\mathbf{j}| \leq \frac{2 \log \text{diam}(E_{\mathbf{j}})}{\log r_{\max}} \leq \frac{-2n \log 2}{\log r_{\max}} \leq nC_2, \quad (5.7)$$

for a constant $C_2 < \infty$ independent of n . On the other hand,

$$\sum_{\mathbf{j} \in Z_n} \mu_{\mathbf{j}}(x)^q r_{\mathbf{j}}(x)^{-\tau} = \mu_{\mathbf{i}|_{n_0}}(x)^q r_{\mathbf{i}|_{n_0}}(x)^{-\tau} =: C_3 \quad (5.8)$$

by iterative use of (5.3). Putting (5.6)–(5.8) together, we get

$$\begin{aligned} \log \sum_{\mathbf{j} \in Z_n} \mu(Q_{\mathbf{j}})^q &\geq \log(2^{-n\tau} c_0 \varepsilon_n c^{qnC_2} C_3) \\ &= \log 2^{-n\tau} + \log \varepsilon_n + qnC_2 \log c + \log(c_0 C_3). \end{aligned} \quad (5.9)$$

To estimate $\log \varepsilon_n$, we choose $\mathbf{j} \in Z_n$ such that $\varepsilon_n = (\text{diam}(E_{\mathbf{j}})/r_{\mathbf{j}}(x))^{-\tau}$. Then

$$\log \varepsilon_n = -\tau \log \text{diam}(E_{\mathbf{j}}) (1 - \log r_{\mathbf{j}}(x) / \log \text{diam}(E_{\mathbf{j}})). \quad (5.10)$$

Moreover, $\log r_j(y) + |j| \log c \leq \log r_j(x) \leq \log r_j(y) - |j| \log c$ for all $y \in E_j$ by (5.5). Using (5.7), this gives

$$\frac{\log r_j(y)}{\log \text{diam}(E_j)} + C_4 \log c \leq \frac{\log r_j(x)}{\log \text{diam}(E_j)} \leq \frac{\log r_j(y)}{\log \text{diam}(E_j)} - C_4 \log c \quad (5.11)$$

for some constant $0 < C_4 < \infty$.

Using (5.9), (5.10), (5.11), and (M6), we finally get

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{\log \sum_{Q \in \mathcal{Q}_n(x,r)} \mu(Q)^q}{\log 2^{-n}} &\leq \liminf_{n \rightarrow \infty} \frac{\log \sum_{j \in Z_n} \mu(Q_j)^q}{\log 2^{-n}} \\ &\leq \tau - (qC_2 + |\tau|C_4) \log c / \log 2. \end{aligned}$$

As $c < 1$ and $r > 0$ can be chosen arbitrarily, we get, by recalling Theorem 4.4(1), that $\tau_q(\mu, x) \leq \tau$.

To prove that $\tau_q(\mu, x) \geq \tau$, we first fix $0 < c < 1$ and $r_0 > 0$ so that $cp_j^i < p_i(x)$ and $cr_i(x) < r_i(y) < \frac{1}{c}r_i(x)$ whenever $i \in \{1, \dots, m\}$ and $y \in E_j \subset B(x, r_0)$. Then, if $0 < r < r_0$, we may find $n_0 \in \mathbb{N}$ and finitely many elements $E_{\mathbf{k}} \in \mathcal{Q}_{n_0}$, $E_{\mathbf{k}} \subset B(x, r_0)$ whose union covers $B(x, r)$. For each such $E_{\mathbf{k}}$, and $n \geq n_0$, we put $Z_{n,\mathbf{k}} = \{j \in \Sigma_* : Q_j \in \mathcal{Q}_n \text{ and } E_j \subset E_{\mathbf{k}}\}$. Putting $M_n = \max_{j \in Z_{n,\mathbf{k}}} (\text{diam}(E_j)/r_j(x))^{-\tau}$, we may estimate as in (5.6) to obtain

$$2^{n\tau} \sum_{j \in Z_{n,\mathbf{k}}} \mu(Q_j)^q \leq C_5 M_n \sum_{j \in Z_{n,\mathbf{k}}} c^{-q|j|} \mu_j(x)^q r_j(x)^{-\tau}.$$

Calculating as above, this implies

$$\liminf_{n \rightarrow \infty} \frac{\log \sum_{Q \in \mathcal{Q}_n(x,r)} \mu(Q)^q}{\log 2^{-n}} \geq \tau + (qC_2 + |\tau|C_4) \log c / \log 2.$$

Letting $r \downarrow 0$ and then $c \uparrow 1$, gives $\tau_q(\mu, x) \geq \tau$. \square

Remark 5.10. (1) One can find Moran constructions that satisfy (M1)–(M6) with very mild assumptions on the space X . For instance, Theorem 5.9 can be applied in (complete) doubling metric spaces under the assumption that there exists a constant $c > 0$ such that any ball $B(x, r) \subset X$ contains two disjoint sub-balls $B(x_1, cr), B(x_2, cr) \subset B(x, r)$. Different types of Moran constructions in metric spaces have been recently studied in [46].

(2) The result is interesting already in \mathbb{R}^n . We remark that a self-similar measure on a self-similar set satisfying the strong separation conditions is a model case for Theorem 5.9 in the special case when p_i and r_i are constant, see [18]. However, as p_i and r_i are allowed to vary depending on the point, Theorem 5.9 can be applied in more general situations.

To discuss multifractal properties of the above measures, we let

$$E_\alpha = \{x \in X : \dim_{\text{loc}}(\mu, x) = \alpha\}.$$

for $\alpha \geq 0$ and define the Hausdorff and packing multifractal spectra of μ by setting $f_H(\alpha) = \dim_H(E_\alpha)$, and $f_p(\alpha) = \dim_p(E_\alpha)$. It is common to say that “ μ satisfies the multifractal formalism” if for all $\alpha \geq 0$ the values of $f_H(\alpha)$ and $f_p(\alpha)$ are given by the Legendre transform of τ , that is, if

$$f_H(\alpha) = f_p(\alpha) = \inf_{q \in \mathbb{R}} \{q\alpha - \tau_q(\mu)\}. \quad (5.12)$$

Perhaps the most classical situation in which the multifractal formalism is known to hold is the case of self-similar measures in Euclidean spaces under the strong separation condition, see e.g. [9, 18]. Next we present a generalisation of this result into metric spaces. For this, we introduce the following additional conditions on the collection $\{E_i : i \in \Sigma_*\}$.

(M7) There is $c > 0$ so that for each $Q \in \mathcal{E}_n$, there is $x \in E$ with $B(x, c2^{-n}) \subset Q$.

(M8) $\lim_{r \downarrow 0} \log r / \log(\text{diam}(E_{i|n(i,r)})) = 1$ for all $i \in \Sigma$, where $n(i, r) = \max\{n \in \mathbb{N} : B(x_i, r) \cap E \subset E_{i|n}\}$.

To handle (5.12), we have to deal with τ_q for negative values of q and for this we use the following lemma. Observe that we cannot use Theorem 4.4(1) when $q < 0$.

Lemma 5.11. *Suppose that in the setting of Theorem 5.9 also (M7) holds. Then, for all $x \in E$, $\tau_q(\mu, x)$ is determined by (5.3) also when $q < 0$.*

Proof. Let $q < 0$, $x \in E$ and let $\tau \in \mathbb{R}$ be the unique solution of (5.3). With trivial modifications to the proof of Theorem 5.9, we see that

$$\tau = \lim_{t \downarrow 0} \liminf_{n \rightarrow \infty} \frac{\log \sum_{Q \in \mathcal{Q}_{n,t}} \mu(Q)^q}{\log 2^{-n}} \quad (5.13)$$

where $\mathcal{Q}_{n,t} = \{Q \in \mathcal{Q}_n : Q \subset B(x, t) \text{ and } Q \cap E \neq \emptyset\}$. (Observe that $\text{spt}(\mu) = E$.)

In order to prove that $\tau = \tau_q(\mu, x)$, let $t > 0$, $2^{-n} \leq \delta < 2^{-n+1} < t$ and $y \in E \cap B(x, t)$. Then there is $n_0 \in \mathbb{N}$ depending only on the numbers C_0 and C_1 so that $B(y, \delta) \supset Q_i$ for some $Q_i \in \mathcal{Q}_{n+n_0, 2t}$. Thus, for any δ -packing $\{B_i\}$ of $B(x, t) \cap \text{spt}(\mu)$, we have

$$\sum_i \mu(B_i)^q \leq \sum_{Q \in \mathcal{Q}_{n+n_0, 2t}} \mu(Q)^q. \quad (5.14)$$

To get an estimate in the other direction, we fix n and t and use the assumption (M7) to find for each $Q \in \mathcal{Q}_{n,t}$ a point $y \in \text{spt}(\mu) \cap B(x, t)$ such that for $B_Q = B(y, c2^{-n})$, we have $B_Q \subset Q$. Thus, for the $(c2^{-n})$ -packing $\{B_Q : Q \in \mathcal{Q}_{n,t}\}$, we have

$$\sum_{Q \in \mathcal{Q}_{n,t}} \mu(B_Q)^q \geq \sum_{Q \in \mathcal{Q}_{n,t}} \mu(Q)^q. \quad (5.15)$$

Combining (5.13)–(5.15), and taking logarithms, it follows that $\tau_q(\mu, x) = \tau$. \square

Remark 5.12. Inspecting the proofs of Theorems 5.9 and 5.11, we observe that in the setting of these results, $\liminf_{n \rightarrow \infty}$ in the definition of $\tau_q(\mu, x)$ can actually be replaced by $\lim_{n \rightarrow \infty}$.

Theorem 5.13. *Let $\{E_{\mathbf{i}} : \mathbf{i} \in \Sigma_*\}$ be a collection of compact sets that satisfy the conditions (M1)–(M8). Suppose that $r = (r_1, \dots, r_m)$ and $p = (p_1, \dots, p_m)$ are constant functions and μ is a measure with $\text{spt}(\mu) = E$ such that $p_{\mathbf{i}|n} \rightarrow p$ uniformly for all $\mathbf{i} \in \Sigma$. If $0 \leq \alpha_{\min} \leq \alpha_{\max}$ are the asymptotic derivatives of the concave function $q \mapsto \tau_q(\mu)$, then (5.12) holds for all $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$.*

Theorem 5.13 follows from Lemma 5.14 below, by taking $\varepsilon = 0$. Observe that the mapping $q \mapsto \tau_q(\mu)$ is indeed concave and continuous on \mathbb{R} by inspecting (5.3). Moreover, one easily derives that $\alpha_{\min} = \min\{\frac{\log p_i}{\log r_i} : i \in \{1, \dots, m\}\}$ and $\alpha_{\max} = \max\{\frac{\log p_i}{\log r_i} : i \in \{1, \dots, m\}\}$. Furthermore, $\underline{\dim}_{\text{loc}}(\mu, x), \overline{\dim}_{\text{loc}}(\mu, x) \in [\alpha_{\min}, \alpha_{\max}]$ for any $x \in E$.

Lemma 5.14. *In the setting of Theorem 5.13, let $f(\alpha) = \min_{q \in \mathbb{R}}\{q\alpha - \tau_q(\mu)\}$ for $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$. If $\varepsilon \geq 0$ and $E_{\alpha, \varepsilon} = \{x \in E : \alpha - \varepsilon \leq \underline{\dim}_{\text{loc}}(\mu, x) \leq \overline{\dim}_{\text{loc}}(\mu, x) \leq \alpha + \varepsilon\}$, then*

$$f(\alpha) - c\varepsilon \leq \dim_{\mathbb{H}}(E_{\alpha, \varepsilon}) \leq \dim_p(E_{\alpha, \varepsilon}) \leq f(\alpha) + c\varepsilon \quad (5.16)$$

for a constant $c < \infty$ independent of ε .

Proof. The proof is similar to the proof of [18, Proposition 11.4]. We give some details for the convenience of the reader. Let $r = (r_1, \dots, r_m)$ and $p = (p_1, \dots, p_m)$ be the constant values of the mappings r and p . Denote moreover, $r_{\mathbf{i}} = \prod_{j=1}^n r_{i_j}$ when $\mathbf{i} \in \Sigma_n$. We set $\tau = \tau_q(\mu)$ and choose $q \in \mathbb{R}$ so that $f(\alpha) = q\alpha - \tau$. Then we define a probability measure ν on X with $\text{spt}(\nu) = E$ by setting

$$\nu(E_{\mathbf{i}i}) = p_i^q r_i^{-\tau} \nu(E_{\mathbf{i}}) = \mu_{\mathbf{i}}^q r_{\mathbf{i}}^{-\tau} = \mu(E_{\mathbf{i}})^q r_{\mathbf{i}}^{-\tau} \quad (5.17)$$

for $\mathbf{i} \in \Sigma_*$ and $i \in \{1, \dots, m\}$. Recall that $\sum_{i=1}^m p_i^q r_i^{-\tau} = 1$ by Theorem 5.9 and Lemma 5.11.

For all $\mathbf{i} \in \Sigma$ the condition (M8) implies that

$$\underline{\dim}_{\text{loc}}(\nu, x_{\mathbf{i}}) = \liminf_{n \rightarrow \infty} \log \nu(E_{\mathbf{i}|n}) / \log \text{diam}(E_{\mathbf{i}|n}), \quad (5.18)$$

$$\overline{\dim}_{\text{loc}}(\nu, x_{\mathbf{i}}) = \limsup_{n \rightarrow \infty} \log \nu(E_{\mathbf{i}|n}) / \log \text{diam}(E_{\mathbf{i}|n}), \quad (5.19)$$

and similar formulas apply for μ .

Let $\eta > 0$. Using (5.17) and (M6), we find $\delta_0 > 0$ and $n_0 \in \mathbb{N}$ so that if $0 < \delta \leq \delta_0$ and $n \geq n_0$, then

$$\begin{aligned} & \nu(\{x_{\mathbf{i}} \in E : \mu(E_{\mathbf{i}|n}) < \text{diam}(E_{\mathbf{i}|n})^{\alpha+\varepsilon+\eta}\}) \\ &= \nu(\{x_{\mathbf{i}} \in E : \mu(E_{\mathbf{i}|n})^{-\delta} \text{diam}(E_{\mathbf{i}|n})^{\delta(\alpha+\varepsilon+\eta)} \geq 1\}) \\ &\leq \sum_{\mathbf{i} \in \Sigma_n} \mu(E_{\mathbf{i}})^{-\delta} \text{diam}(E_{\mathbf{i}})^{\delta(\alpha+\varepsilon+\eta)} \nu(E_{\mathbf{i}}) \\ &= \sum_{\mathbf{i} \in \Sigma_n} \mu_{\mathbf{i}}^{q-\delta} r_{\mathbf{i}}^{\delta(\alpha+\varepsilon+\eta/2)-\tau} \leq \gamma^n, \end{aligned}$$

where $\gamma < 1$ is independent of n . For the last estimate, see [18, Lemma 11.3]. In the second to last estimate we used (M6) to conclude that $\text{diam}(E_{\mathbf{i}})^{\delta(\alpha+\varepsilon+\eta)} < r_{\mathbf{i}}^{\delta(\alpha+\varepsilon+\eta/2)}$ for all $\mathbf{i} \in \Sigma_n$. Summing the above estimate over all $n \geq n_0$, and letting $\delta \downarrow 0$, this implies that

$$\overline{\dim}_{\text{loc}}(\mu, x) \leq \alpha + \varepsilon \quad (5.20)$$

for ν -almost all $x \in X$. A similar calculation gives

$$\underline{\dim}_{\text{loc}}(\mu, x) \geq \alpha - \varepsilon \quad (5.21)$$

for ν -almost all x . Thus, in particular, we have

$$\nu(X \setminus E_{\alpha, \varepsilon}) = 0. \quad (5.22)$$

From (5.17), it follows that

$$\frac{\log \nu(E_{\mathbf{i}})}{\log \text{diam}(E_{\mathbf{i}})} = q \frac{\log \mu(E_{\mathbf{i}})}{\log \text{diam}(E_{\mathbf{i}})} - \tau \frac{\log r_{\mathbf{i}}}{\log \text{diam}(E_{\mathbf{i}})}$$

for all $\mathbf{i} \in \Sigma_n$. Using (M3) and (M6), we observe that $\log r_{\mathbf{i}} / \log \text{diam}(E_{\mathbf{i}}) \rightarrow 1$ as $|\mathbf{i}| \rightarrow \infty$. Combined with (5.18)–(5.19) and (5.20)–(5.21), this gives $\overline{\dim}_{\text{loc}}(\nu, x) \leq q\alpha + |q|\varepsilon - \tau = f(\alpha) + |q|\varepsilon$ and similarly $\underline{\dim}_{\text{loc}}(\nu, x) \geq f(\alpha) - |q|\varepsilon$ for all $x \in E_{\alpha, \varepsilon}$. Together with (5.22), these estimates readily imply that $f(\alpha) - |q|\varepsilon \leq \dim_{\text{H}}(E_{\alpha, \varepsilon}) \leq \dim_{\text{p}}(E_{\alpha, \varepsilon}) \leq f(\alpha) + |q|\varepsilon$. \square

To finish this section, we show how the local L^q -spectrum can be used in the setting of Theorem 5.9. We introduce a coarse type local multifractal formalism for the spectrum

$$f_{\text{H}}(\alpha, x) = \lim_{\varepsilon \downarrow 0} \lim_{r \downarrow 0} \dim_{\text{H}}(\{y \in B(x, r) : \alpha - \varepsilon \leq \underline{\dim}_{\text{loc}}(\mu, y) \leq \overline{\dim}_{\text{loc}}(\mu, y) \leq \alpha + \varepsilon\})$$

for $x \in X$ and $\alpha \geq 0$. The corresponding packing spectrum, $f_{\text{p}}(\alpha, x)$ is defined by replacing \dim_{H} by \dim_{p} above. Let $\alpha_{\min}(x) = \min\{\frac{\log p_i(x)}{\log r_i(x)} : i \in \{1, \dots, m\}\}$ and $\alpha_{\max}(x) = \max\{\frac{\log p_i(x)}{\log r_i(x)} : i \in \{1, \dots, m\}\}$ be the asymptotic derivatives of $q \mapsto \tau_q(\mu, x)$.

Theorem 5.15. *Let $\{E_{\mathbf{i}} : \mathbf{i} \in \Sigma_*\}$ be a collection of compact sets that satisfy the conditions (M1)–(M8). If $p_{\mathbf{i}|n} \rightarrow p(x_{\mathbf{i}})$ uniformly for all $\mathbf{i} \in \Sigma$, then*

$$f_{\mathbb{H}}(\alpha, x) = f_{\mathbb{P}}(\alpha, x) = \inf_{q \in \mathbb{R}} \{\alpha q - \tau_q(\mu, x)\}$$

for all $x \in E$ and $\alpha_{\min}(x) \leq \alpha \leq \alpha_{\max}(x)$.

Proof. Let $x \in E$, $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$, $p(x) = (p_i(x))_{i=1}^m$ and suppose that ν is the probability measure on X defined using the weights $p(x)$, that is, $\nu(E_{\mathbf{i}i}) = p_i(x)\nu(E_{\mathbf{i}})$ for each $\mathbf{i} \in \Sigma_*$ and $i \in \{1, \dots, m\}$.

Let $\varepsilon > 0$ and $c > 1$. Then, as $p_{\mathbf{i}} \rightarrow p(x_{\mathbf{i}})$ uniformly and $y \mapsto p(y)$ is continuous, there is $r_0 > 0$ so that

$$p_{\mathbf{i}}^i/c \leq p_i(x) \leq cp_{\mathbf{i}}^i$$

whenever $E_{\mathbf{i}} \subset B(x, r_0)$. This gives

$$C_2 c^{-|\mathbf{i}|} \mu(E_{\mathbf{i}}) \leq \nu(E_{\mathbf{i}}) \leq C_3 c^{|\mathbf{i}|} \mu(E_{\mathbf{i}})$$

with some constants C_2 and C_3 and, consequently,

$$\begin{aligned} \frac{\log C_3 + |\mathbf{i}| \log c}{\log \text{diam}(E_{\mathbf{i}})} + \frac{\log \mu(E_{\mathbf{i}})}{\log \text{diam}(E_{\mathbf{i}})} &\leq \frac{\log \nu(E_{\mathbf{i}})}{\log \text{diam}(E_{\mathbf{i}})} \\ &\leq \frac{\log C_2 - |\mathbf{i}| \log c}{\log \text{diam}(E_{\mathbf{i}})} + \frac{\log \mu(E_{\mathbf{i}})}{\log \text{diam}(E_{\mathbf{i}})}. \end{aligned}$$

As c can be chosen arbitrarily close to 1, and as $\text{diam}(E_{\mathbf{i}}) \leq \gamma^{|\mathbf{i}|}$ for some $\gamma < 1$ (use (M6) and the fact that $\max\{r_i(x) : x \in E \text{ and } i \in \{1, \dots, m\}\} < 1$), this implies that for small $r > 0$

$$\underline{\dim}_{\text{loc}}(\mu, y) - \varepsilon/2 \leq \underline{\dim}_{\text{loc}}(\nu, y) \leq \overline{\dim}_{\text{loc}}(\nu, y) \leq \overline{\dim}_{\text{loc}}(\mu, y) + \varepsilon/2 \quad (5.23)$$

when $y \in B(x, r)$. Observe that (5.18) and (5.19) hold for any measure whose support is contained in E by (M8). Denote $E_{\alpha, \varepsilon, r} = \{y \in B(x, r) : \alpha - \varepsilon \leq \underline{\dim}_{\text{loc}}(\mu, y) \leq \overline{\dim}_{\text{loc}}(\mu, y) \leq \alpha + \varepsilon\}$, $E_0 = \{y \in B(x, r) : \alpha - \varepsilon/2 \leq \underline{\dim}_{\text{loc}}(\nu, y) \leq \overline{\dim}_{\text{loc}}(\nu, y) \leq \alpha + \varepsilon/2\}$, and $E_1 = \{y \in B(x, r) : \alpha - \frac{3}{2}\varepsilon \leq \underline{\dim}_{\text{loc}}(\nu, y) \leq \overline{\dim}_{\text{loc}}(\nu, y) \leq \alpha + \frac{3}{2}\varepsilon\}$. Now it follows from (5.23) that $E_0 \subset E_{\alpha, \varepsilon, r} \subset E_1$. Combined with Lemma 5.14, this yields

$$f(\alpha) - c\varepsilon \leq \dim_{\mathbb{H}}(E_0) \leq \dim_{\mathbb{H}}(E_{\alpha, \varepsilon, r}) \leq \dim_{\mathbb{P}}(E_{\alpha, \varepsilon, r}) \leq \dim_{\mathbb{P}}(E_1) \leq f(\alpha) + c\varepsilon,$$

where $f(\alpha) = \inf_{q \in \mathbb{R}} \{q\alpha - \tau_q(\nu)\} = \inf_{q \in \mathbb{R}} \{q\alpha - \tau_q(\mu, x)\}$ and the constant $c < \infty$ is independent of ε . The claim now follows by letting $r \downarrow 0$ and then $\varepsilon \downarrow 0$. \square

6. EXAMPLES, OPEN PROBLEMS AND FURTHER REMARKS

6.1. Relations between different dimensions. The theory presented in §5.4 gives a firm justification for the use of the local L^q -spectrum in addition to the global one. Below, we give few more straightforward examples of situations where the local L^q -dimensions seem to be more reasonable than the global ones.

Example 6.1. We construct a probability measure μ on \mathbb{R}^d so that for all $0 \leq q < 1$ we have $\dim_q(\mu) = d$ while $\dim_q(\mu, x) = 0 = \dim_{\text{loc}}(\mu, x)$ for μ -almost all $x \in \mathbb{R}^d$.

Our measure μ will be a countable sum of weighted Dirac measures on $[0, 1]^d$. Let us denote by \mathcal{Q}^n the dyadic subcubes of $[0, 1]^d$ of side-length 2^{-n} . At step 1, we let $n_1 = 1$ and attach a point mass of size 2^{-d} to the centre point of all but one dyadic subcubes of $[0, 1]^d$ in $Q \in \mathcal{Q}^1$. Let $Q_1 \in \mathcal{Q}^1$ be the one remaining cube of measure 2^{-d} . At step 2 we choose a large integer $n_2 \in \mathbb{N}$ and attach a point mass of magnitude $2^{-n_2 d} \mu(Q_1)$ to all but one of its dyadic subcubes in $\mathcal{Q}^{n_1+n_2}$. We continue inductively, at the k :th stage we choose the one remaining cube $Q_{k-1} \in \mathcal{Q}^{n_1+\dots+n_{k-1}}$, choose a large integer n_k and attach a point mass of size $2^{-n_k d} \mu(Q_{k-1})$ to the centre points of all but one dyadic subcubes of Q_{k-1} in the collection $\mathcal{Q}^{n_1+\dots+n_k}$.

At the k :th stage we have for all $0 < q < 1$ that

$$\begin{aligned} \frac{1}{\log 2^{-n_k}} \log \sum_{Q \in \mathcal{Q}^{n_k}} \mu(Q)^q &\leq \frac{1}{\log 2^{-n_k}} \log \left(\sum_{\substack{Q \in \mathcal{Q}^{n_k} \\ Q \subset Q_{k-1}}} \mu(Q)^q \right) \\ &= \frac{\log(2^{n_k d(1-q)} \mu(Q_{k-1})^q)}{\log 2^{-n_k}} = (q-1)d + q \frac{\log \mu(Q_{k-1})}{\log 2^{-n_k}}. \end{aligned}$$

Thus, choosing the numbers n_k large enough, we can ensure that

$$\tau_q(\mu) = \liminf_{n \rightarrow \infty} \frac{1}{\log 2^{-n_k}} \log \sum_{Q \in \mathcal{Q}^{n_k}} \mu(Q)^q \leq (q-1)d.$$

Here we have used (4.5) for the global spectrum. On the other hand, it is well known and easy to see that $\tau_q(\mu) \geq (q-1)d$ for all measures μ on \mathbb{R}^d with bounded support. Whence it follows that $\dim_q(\mu) = d$. Furthermore, it is clear from the construction that $\tau_q(\mu, x) = \dim_q(\mu, x) = \dim_{\text{loc}}(\mu, x) = 0$ for μ -almost all $x \in \mathbb{R}^d$.

Example 6.2. If μ is the sum of a Dirac point mass at the origin and the Lebesgue measure on the unit cube of \mathbb{R}^d , we see that $\dim_q(\mu) = 0$ whereas $\dim_q(\mu, x) = d = \dim_{\text{loc}}(\mu, x)$ for all $q > 1$ and all $x \in [0, 1]^d \setminus \{0\}$.

Question 6.3. For $q > 1$, is it possible that $\dim_q(\mu) = 0$ while $\dim_q(\mu, x) > 0$ almost everywhere?

The authors observed already some years ago that in \mathbb{R}^d also the L^q -spectrum estimates can be used to gain information on the dimension of porous measures, but the results were somewhat weaker than the results obtained from the local homogeneity estimates in §5.2 above, see Remark 5.3(3). One motivation for investigating the local L^q -spectrum in metric spaces was to find out, which of these two methods, if any, is stronger. Also, in view of Theorems 3.2 and 4.2 it is interesting to compare $\dim_{\text{hom}}(\mu, x)$ to $\lim_{q \uparrow 1} \dim_q(\mu, x)$. In the following two examples we show that, in general, there is no relationship between these two values. We present the examples in \mathbb{R} but similar constructions work in any dimension. The

first example also shows that a measure may have large homogeneity even if it is of packing dimension zero.

Example 6.4. We construct an example in \mathbb{R} so that $\lim_{q \uparrow 1} \dim_q(\mu, x) = 0$ while $\dim_{\text{hom}}(\mu, x) = 1$ for μ -almost all $x \in \mathbb{R}$. The idea is to apply a construction resulting to a zero dimensional measure on a Cantor set. The large homogeneity is obtained by performing infinitely many (but extremely seldom so that it does not affect the value of \dim_q) construction steps where the measure is distributed almost uniformly inside the construction intervals of that level.

We first pick a sequence $0 < \varepsilon_i \downarrow 0$ and then choose integers $m_i, n_i \rightarrow \infty$ so that

$$\frac{k + \sum_{j=1}^k m_j}{\sum_{j=1}^k (n_j + m_j)} < \varepsilon_k \quad (6.1)$$

for all $k \in \mathbb{N}$. For example, we may take $\varepsilon_i = 3/i$, $m_i = i$, and $n_i = i^2$. In the first step of the construction, we put $\mu([0, 2^{-n_1}]) = \mu([1 - 2^{-n_1}, 1]) = \frac{1}{2}$. Then we divide both intervals $[0, 2^{-n_1}]$ and $[1 - 2^{-n_1}, 1]$ into 2^{m_1} dyadic subintervals of length $2^{-n_1 - m_1}$ each getting 2^{-m_1} portion of their parents measure.

We continue the construction inductively. In the k :th step, we perform the step 1 construction inside each of the construction intervals of level k just by replacing n_1 and m_1 with n_k and m_k , respectively.

As $m_k \rightarrow \infty$ it is clear that $\text{hom}_\delta(\mu, x) \approx \frac{1}{\delta}$ for all $x \in \text{spt}(\mu)$ and all small $\delta > 0$. Thus $\dim_{\text{hom}}(\mu, x) = 1$ for all $x \in \text{spt}(\mu)$. On the other hand, it follows easily from (6.1), that $\tau_q(\mu, x) = \dim_q(\mu, x) = 0 = \dim_q(\mu)$ for all $x \in \text{spt}(\mu)$ and $0 < q < 1$. Recall that by Theorem 4.4(1) we may calculate \dim_q by using dyadic intervals.

Example 6.5. We construct an example in \mathbb{R} so that $\lim_{q \uparrow 1} \dim_q(\mu, x) = 1$ but $\dim_{\text{hom}}(\mu, x) = 0$ for μ -almost all $x \in \mathbb{R}$. The idea is to perform a Cantor type construction resulting to a zero dimensional measure, but add ‘‘one-dimensional’’ perturbation which affects only a dense set of measure zero, but nevertheless, guarantees that the $\dim_q(\mu, x)$ is large for all $x \in \text{spt}(\mu)$.

Fix numbers $0 < q_k \uparrow 1$ and integers $n_k, l_k \in \mathbb{N}$ so that $n_k \rightarrow \infty$ and $\sum_{k=1}^{\infty} 2^{-l_k} < \infty$. For example, we may choose $q_k = 1 - \frac{1}{k+1}$ and $n_k = l_k = k$. In what follows, we choose a sequence of integers $m_k \rightarrow \infty$. First of these, m_1 , is taken so that

$$\frac{m_1(1 - q_1) - l_1 q_1}{n_1 l_1 + m_1} > \frac{1}{2}(1 - q_1).$$

The numbers m_2, m_3, \dots will be defined inductively below.

We begin the step 1 of the construction by setting $\mu([0, 2^{-n_1}]) = \mu([1 - 2^{-n_1}, 1]) = \frac{1}{2}$. Iterating this in a self-similar manner for l_1 steps, we get 2^{l_1} dyadic subintervals of $[0, 1]$ of length $2^{-n_1 l_1}$ each of measure 2^{-l_1} . We choose one of these intervals, say I , and divide it into 2^{m_1} dyadic subintervals of length $2^{-m_1} |I|$ and of measure $2^{-m_1} \mu(I)$. Inside the other $2^{l_1} - 1$ construction intervals of length $2^{-n_1 l_1}$ we choose

just the outermost subintervals of length $2^{-l_1 n_1 - m_1}$ and let both of these intervals have the same measure (half of the measure of their parent).

In the beginning of the step k , $k \geq 2$, we have some amount, say I_1, \dots, I_{N_k} dyadic intervals of equal length, denoted 2^{-M_k} . We perform the step 1 construction inside each of these intervals, but replace n_1 , l_1 , and m_1 by n_k , l_k , and m_k , respectively. We choose m_k so large that for each $I = I_j$, the dyadic subintervals J_i of I of size $2^{-M_k - n_k l_k - m_k}$ chosen in the construction satisfy

$$\frac{\log(\sum_i \mu(J_i)^{q_k})}{\log(2^{M_k + n_k l_k + m_k})} \geq \frac{\log(2^{m_k(1-q_k)} (2^{-l_k} \mu(I))^{q_k})}{\log(2^{M_k + n_k l_k + m_k})} > \frac{k}{k+1} (1 - q_k).$$

The former estimate is obtained by summing over the range of intervals where the measure was distributed uniformly. As $q_k \uparrow 1$, we clearly get $\lim_{q \uparrow 1} \dim_q(\mu, x) \geq 1$ for all $x \in \text{spt}(\mu)$. On the other hand, as $n_k \rightarrow \infty$, and $\sum_k 2^{-l_k} < \infty$, it follows that for μ -almost all $x \in \mathbb{R}$, we have $\text{hom}_\delta(\mu, x) \leq C$ for all $0 < \delta < 1$ with some universal constant $C < \infty$. Thus, in particular, $\dim_{\text{hom}}(\mu, x) = 0$ for almost all x .

Remark 6.6. (1) From the previous example, it follows that a strict inequality $\overline{\dim}_{\text{loc}}(\mu, x) < \lim_{q \uparrow 1} \dim_q(\mu, x)$ is possible almost everywhere in Theorem 4.2. We note that also

$$\lim_{q \downarrow 1} \dim_q(\mu, x) < \underline{\dim}_{\text{loc}}(\mu, x) \quad (6.2)$$

is possible in a set of positive measure. A simple example is given by letting $\mu = \mathcal{L}^1|_{[0,1]} + \sum_{n \in \mathbb{N}} 2^{-n} \delta_{q_n}$ where \mathcal{L}^1 is the Lebesgue measure and $\{q_1, q_2, q_3, \dots\}$ is dense in $[0, 1]$. In order to get an example where (6.2) holds almost everywhere, one can use a similar idea as in Example 6.5 but this time one has to construct a one dimensional measure with a dense zero dimensional perturbation.

(2) We note that also the other inequalities in Theorem 4.2 can be strict. For instance, see [4, Proposition 3.1].

Question 6.7. Is there any relation between the local and global entropy dimensions? Recall Proposition 4.6.

Question 6.8. Is it possible to develop a local multifractal formalism for a class of Moran constructions including (sufficiently regular) self-conformal sets?

6.2. Questions on porosity and conical density. A measure μ is called (ϱ, p) -mean porous at x if for all $\varepsilon > 0$ and for all sufficiently large n , there are at least pn values $l \in \{1, \dots, n\}$ with $\text{por}_1(\mu, x, 2^{-l}, \varepsilon) \geq \varrho$. It follows from the results of [5] that for any measure μ on \mathbb{R}^d , one has

$$\overline{\dim}_{\text{loc}}(\mu, x) \leq d - p - c(d)/\log(1 - 2\varrho)$$

for μ -almost all $x \in \{y \in \mathbb{R}^d : \mu \text{ is } (\varrho, p)\text{-mean porous at } y\}$. On the other hand, based on probabilistic ideas introduced in [24], it was recently proved in [51] that

$$\overline{\dim}_{\text{loc}}(\mu, x) \leq d - c(d)p\varrho^d$$

for μ -almost all $x \in \{y \in \mathbb{R}^d : \mu \text{ is } (\varrho, p)\text{-mean porous at } y\}$. In light of Theorems 5.2 and 5.6 it is natural to pose the following problems.

Question 6.9. If μ is a Radon measure on \mathbb{R}^d , $k \in \{1, \dots, d\}$, $0 < \varrho < 1/2$, and $0 < p < 1$, is it true that $\overline{\dim}_{\text{loc}}(\mu, x) \leq d - pk - c/\log(1 - 2\varrho)$ for μ -almost all $x \in \{y \in X : \mu \text{ is } (\varrho, p)\text{-mean } k\text{-porous at } y\}$? Here the mean k -porosity is defined just as the mean porosity but replacing por_1 by por_k .

Question 6.10. In the setting of Theorem 5.6, is it true that $\overline{\dim}_{\text{loc}}(\mu, x) \leq s - cp\varrho^s$ for μ -almost all $x \in \{y \in X : \mu \text{ is } (\varrho, p)\text{-mean porous at } y\}$?

It is also reasonable to search for conical density and/or k -porosity results in non-Euclidean spaces that have enough geometry. For instance, it would be interesting to figure out if Theorems 5.1 and 5.2 have counterparts in the Heisenberg group.

6.3. Concerning the definition of homogeneity. If we replace $\limsup_{r \downarrow 0}$ by $\liminf_{r \downarrow 0}$ in the definition of $\text{hom}_\delta(\mu, x)$ in (3.2), we denote the pointwise homogeneity dimension obtained this way by $\underline{\dim}_{\text{hom}}(\mu, x)$. However, such lower homogeneity is not very interesting from our point of view since it does not give any bounds for the local dimensions $\underline{\dim}_{\text{loc}}$ or $\overline{\dim}_{\text{loc}}$. It is easy to find measures with $\underline{\dim}_{\text{hom}}(\mu, x) = 0$ and $\underline{\dim}_{\text{loc}}(\mu, x) > 0$ almost everywhere. (It is essentially enough to construct a Cantor set $C \subset \mathbb{R}$ with $\dim_{\text{H}}(C) > 0$ such that for each $\delta > 0$ and all $x \in C$, there are arbitrarily small radii $r > 0$ such that $E \cap B(x, r) \subset B(x, \delta r)$.) One can, however, consider $\underline{\dim}_{\text{hom}}$ as a measure of antiporosity and it could turn out to be useful in connection with upper porous measures, see [40, 53].

Our definition of homogeneity is somewhat related to the (upper) average homogeneities $\overline{\text{Hom}}_k^i$ considered in [27]. It is possible to modify Example 6.4 above to obtain a measure μ on \mathbb{R} with $\overline{\text{Hom}}_k^i(\mu) = 0$ for all $k \geq 100$ and $i \geq 10$ but nevertheless $\text{hom}_\delta(\mu, x) \approx \frac{1}{\delta}$ for all $0 < \delta < 1$ and $x \in \text{spt}(\mu)$. Thus it is not possible to bound $\overline{\text{Hom}}_k^i(\mu)$ from below using $\text{hom}_\delta(\mu, x)$. On the other hand, it is clearly possible that $\text{hom}_\delta(\mu, x)$ is small in a large set even if $\overline{\text{Hom}}_k^i(\mu)$ is large for all $i \in \{1, \dots, k\}$. This reflects the fact that hom_δ is a local concept and $\overline{\text{Hom}}_k^i$ is not. It is, however, possible to consider restriction measures in small balls and define a ‘‘local upper average homogeneity’’. It is then easy to see that this local average homogeneity may be used to bound $\text{hom}_\delta(\mu, x)$ from below. This is the essential content of [12, Lemma 4.6].

A positive answer to the following question would be very interesting, although maybe a bit unlikely.

Question 6.11. Is there any kind of set dimension related to $\dim_{\text{hom}}(\mu, x)$ in the fashion of (2.1) and (2.2).

In the definition of the homogeneity dimension, $\liminf_{\delta \downarrow 0}$ in (3.3) cannot be replaced by $\lim_{\delta \downarrow 0}$ at all points, as shown in Example 6.13 below. However, the following question looks reasonable.

Question 6.12. Is $\dim_{\text{hom}}(\mu, x) = \lim_{\delta \downarrow 0} \frac{\log^+ \text{hom}_\delta(\mu, x)}{-\log \delta}$ for μ -almost all x ?

Example 6.13. We will construct a measure μ as a countable sum of Dirac point masses on \mathbb{R} so that

$$\liminf_{\lambda \downarrow 0} \frac{\log^+ \text{hom}_\lambda(\mu, 0)}{-\log \lambda} \leq \frac{1}{4} < \frac{1}{2} \leq \limsup_{\lambda \downarrow 0} \frac{\log^+ \text{hom}_\lambda(\mu, 0)}{-\log \lambda}. \quad (6.3)$$

For each $k \in \mathbb{N}$, we choose numbers $0 < \lambda_k < 1$ such that $\lambda_1 = 1/9$, $\lambda_k^{-1/2} \in \mathbb{N}$, and $2\lambda_{k+1} \leq \lambda_k^4$. In addition, we fix numbers $m_k > 0$ such that $2^k \lambda_k^{-1/2} m_{k+1} m_k^{-1} \rightarrow 0$ as $k \rightarrow \infty$. For each $k \in \mathbb{N}$, we let $\mu_k = \sum_{i=1}^k 2^{-i} \lambda_i^{1/2} m_k \sum_{j=1}^{\lambda_i^{-1/2} + 1} \delta_{10^{-k}(1-j\lambda_i)}$ and define $\mu = \sum_{k=1}^{\infty} \mu_k$. Here δ_x is the Dirac unit mass at x .

To verify (6.3), we fix $k \in \mathbb{N}$. Then, if $\varepsilon < 2^{-k} \lambda_k^{1/2}$, and $n \geq k$ is large enough, we have $\text{hom}_{\lambda_k/3, \varepsilon, 10^{-n}}(\mu, 0) \geq \lambda_k^{-1/2}$. Letting $n \rightarrow \infty$, and $\varepsilon \downarrow 0$, this implies that $\text{hom}_{\lambda_k/3}(\mu, 0) \geq \lambda_k^{-1/2}$ and the right-hand side inequality of (6.3) follows.

Next, let $\varepsilon > 0$. Then for small $r > 0$ we have $\text{hom}_{2\sqrt{\lambda_k}, \varepsilon, r}(\mu, 0) \leq \lambda_{k-1}^{-1/2} + \dots + \lambda_1^{-1/2} \leq 2\lambda_{k-1}^{-1/2} \leq \lambda_k^{-1/8}$. Letting $r \downarrow 0$ and $\varepsilon \downarrow 0$, this implies that $\text{hom}_{2\sqrt{\lambda_k}}(\mu, 0) \leq (\sqrt{\lambda_k})^{-1/4}$ and so also the left-hand side inequality of (6.3) follows.

The following question asks whether Lemma 3.4 remains true in non-complete spaces. Example 6.15 below shows that on a fixed scale, homogeneity is not necessarily a Borel function.

Question 6.14. If μ is a Radon measure on a doubling metric space X , then is $x \mapsto \dim_{\text{hom}}(\mu, x)$ defined on X a Borel function?

Example 6.15. For a fixed $r > 0$, it is easy to construct an example in which $x \mapsto \text{hom}_{\delta, \varepsilon, r}(\mu, x)$ is not a Borel function. For example, let $E \subset [0, r/2]$ be a non-Borel set with positive Lebesgue measure and define the metric space to be $X = (\mathbb{R} \times \{0\}) \cup (E \times \{r\})$ equipped with the Euclidean metric. Let μ be the length measure on X . Then with a small $\varepsilon > 0$, we have $\text{hom}_{1/2, \varepsilon, r}(\mu, x) = 3$ for $x \in E \times \{0\}$ and $\text{hom}_{1/2, \varepsilon, r}(\mu, x) = 2$ for $x \in X \setminus (E \times \{0\})$. The set $E \times \{0\}$ is clearly not a Borel subset of X .

Given $\gamma > 1$, let us define $\text{hom}_{\delta, \varepsilon, r}^\gamma$ (resp. hom_δ^γ and \dim_{hom}^γ) in a similar way as $\text{hom}_{\delta, \varepsilon, r}$ but in (3.1), replace the condition $\mu(B) > \varepsilon \mu(B(x, 5r))$ with $\mu(B) > \varepsilon \mu(B(x, \gamma r))$ (thus, $\text{hom} = \text{hom}^5$). Below, we prove that if the density point property holds, then \dim_{hom}^γ typically does not depend on γ .

Proposition 6.16. *Suppose that a Radon measure μ on a doubling metric space X has the density point property and $\gamma_2 > \gamma_1 > 1$. Then $\dim_{\text{hom}}^{\gamma_1}(\mu, x) = \dim_{\text{hom}}^{\gamma_2}(\mu, x)$ at μ -almost every $x \in X$.*

Proof. Because $\mu(B(x, \gamma_1 r)) \leq \mu(B(x, \gamma_2 r))$, it is clear that

$$\dim_{\text{hom}}^{\gamma_1}(\mu, x) \geq \dim_{\text{hom}}^{\gamma_2}(\mu, x)$$

at every point $x \in X$. Assume now that there exist $t > 0$ and a set $A \subset X$ with $\mu(A) > 0$ so that $\dim_{\text{hom}}^{\gamma_1}(\mu, x) > t > \dim_{\text{hom}}^{\gamma_2}(\mu, x)$ for every $x \in A$. Since X is doubling, there exists $M \in \mathbb{N}$ such that any ball of any radius $r > 0$ can be covered with M balls of radius $cr = \frac{\gamma_1 - 1}{6\gamma_2}r$.

For some $r_0, \varepsilon, \delta > 0$ there exists a set $A' \subset A$ with $\mu(A') > 0$ so that

$$\text{hom}_{\delta, \varepsilon, r}^{\gamma_2}(\mu, x) < \delta^{-t}/M$$

for every $0 < r < r_0$ and $x \in A'$ and

$$\text{hom}_{3c\delta}^{\gamma_1}(\mu, x) > \delta^{-t}$$

for every $x \in A'$. Because we only want the above inequalities to hold, we may slightly fluctuate the parameters δ, ε , and r in the definition of homogeneity. Thus with a similar reasoning as in the proof of Lemma 3.4, we may assume that A' is a Borel set.

Let x be a density point of A' and take $0 < r < r_0$ such that $\mu(A' \cap B(x, \gamma_1 r)) > (1 - \varepsilon/(M\delta^t))\mu(B(x, \gamma_1 r))$ and $\text{hom}_{3c\delta, \varepsilon, r}^{\gamma_1}(\mu, x) > \delta^{-t}$. Now cover $B(x, r)$ with M balls of radius cr . By the pigeon hole principle, we then have in at least one of the covering balls, say in $B(y, cr)$, a $(3c\delta r)$ -packing $\{B_i\}$ with at least δ^{-t}/M balls for which $\mu(B_i) > \varepsilon\mu(B(x, \gamma_1 r))$. Therefore, since $B(y, 6\gamma_2 cr) \subset B(x, \gamma_1 r)$, we have an estimate $\text{hom}_{\delta, \varepsilon, 3cr}^{\gamma_2}(\mu, z) > \delta^{-t}/M$ for every $z \in B(y, 2cr)$. Thus $A' \cap B(y, 2cr) = \emptyset$. On the other hand, $\mu(B(y, 2cr)) > \frac{\varepsilon}{M\delta^t}\mu(B(x, \gamma_1 r))$ which contradicts the choice of x . \square

Example 6.17. In general, the equality of Proposition 6.16 can not hold at every point $x \in X$ even when $X = \mathbb{R}^2$. To see this take

$$\mu = \sum_{k=1}^{\infty} \frac{1}{k!} \mathcal{H}^1|_{S^1(0, 2^{-k})},$$

where $\mathcal{H}^1|_{S^1(0, 2^{-k})}$ is the length measure on $S^1(0, 2^{-k}) = \{y \in \mathbb{R}^2 : |y| = 2^{-k}\}$. Then $\dim_{\text{hom}}^{3/2}(\mu, (0, 0)) = 1$, but $\dim_{\text{hom}}^{5/2}(\mu, (0, 0)) = 0$.

6.4. Necessity of the density point property. We used the density point property in many proofs. However, we do not know whether it is really needed. Thus, we pose the following “generic” question.

Question 6.18. Is the density point property a necessity in the second claim of Theorem 4.2, Theorem 5.6, and/or Proposition 6.16?

Also, to our knowledge, the following question on the local dimensions is open if μ does not have the density point property. Recall Remark 3.3(2).

Question 6.19. Suppose that μ is a measure on X , $A \subset X$ is a Borel set with $\mu(A) > 0$. Is it true that $\overline{\dim}_{\text{loc}}(\mu, x) = \overline{\dim}_{\text{loc}}(\mu|_A, x)$, $\underline{\dim}_{\text{loc}}(\mu, x) = \underline{\dim}_{\text{loc}}(\mu|_A, x)$, and/or $\dim_{\text{hom}}(\mu, x) = \dim_{\text{hom}}(\mu|_A, x)$ for μ -almost all $x \in A$.

APPENDIX A. DENSITY POINT PROPERTY

It is well known that the density point property is valid for general Radon measures in Euclidean spaces. Moreover, it holds in all metric spaces in which Besicovitch's covering theorem holds, see [20, §2.8–§2.9], [39, §2], and [21, §1]. In a general metric space, it follows from the standard $5r$ -covering estimate that the density point property is valid for measures that are locally doubling in the sense that

$$\limsup_{r \downarrow 0} \mu(B(x, 2r)) / \mu(B(x, r)) < \infty \quad (\text{A.1})$$

for μ -almost all $x \in X$, see the above references. In this section, we show that for general Radon measures the density point property may fail even if the space is compact and doubling.

Theorem A.1. *There is a compact doubling metric space Σ , a Radon measure μ on Σ , and a compact set $A \subset \Sigma$ with $\mu(A) > 0$ so that*

$$\liminf_{r \downarrow 0} \frac{\mu(A \cap B(\mathbf{i}, r))}{\mu(B(\mathbf{i}, r))} = 0 \quad (\text{A.2})$$

for all $\mathbf{i} \in A$.

Proof. Let us first construct the metric space Σ . We first choose $N_n = n2^n$ for all $n \in \mathbb{N}$ and set $I_n = \{0, \dots, N_n\}$. We also define numbers ε_n for $n \in \mathbb{N}$ by letting $\varepsilon_1 = 1$ and $\varepsilon_{n+1} = 2^{-N_n} \varepsilon_n$ for all $n \in \mathbb{N}$. Next we define functions $d_n: I_n \times I_n \rightarrow [0, \infty)$ by setting

$$d_n(i, j) = d_n(j, i) = \begin{cases} 0, & \text{if } i = j, \\ \varepsilon_n 2^{-i}, & \text{if } i \neq 0 \text{ and } j = 0, \\ \varepsilon_n (2^{-i} + 2^{-j}), & \text{if } i, j \neq 0 \text{ and } i \neq j. \end{cases}$$

The choice of ε implies that if $i, j \in I_n$ and $k, m \in I_{n+1}$, then

$$d_{n+1}(k, m) < d_n(i, j). \quad (\text{A.3})$$

We now set $\Sigma = \prod_{n=1}^{\infty} I_n$ and denote its elements by $\mathbf{i} = (i_1, i_2, \dots)$, $\mathbf{j} = (j_1, j_2, \dots)$, and so on. We also set $\Sigma_n = \prod_{j=1}^n I_j$ for all $n \in \mathbb{N}$. If $\mathbf{i} \in \Sigma$ and $n \in \mathbb{N}$, then we let $\mathbf{i}|_n = (i_1, \dots, i_n) \in \Sigma_n$. For $n \in \mathbb{N}$ and $\mathbf{i} \in \Sigma_n$ we denote $[\mathbf{i}] = \{\mathbf{j} \in \Sigma : \mathbf{j}|_n = \mathbf{i}\}$. If $\mathbf{i}, \mathbf{j} \in \Sigma$ so that $\mathbf{i} \neq \mathbf{j}$, then we let $m(\mathbf{i}, \mathbf{j}) = \min\{n \in \mathbb{N} : i_n \neq j_n\}$.

The metric $d: \Sigma \times \Sigma \rightarrow [0, \infty)$ on Σ is now defined by setting

$$d(\mathbf{i}, \mathbf{j}) = \begin{cases} 0, & \text{if } \mathbf{i}, \mathbf{j} \in \Sigma \text{ so that } \mathbf{i} = \mathbf{j}, \\ d_{m(\mathbf{i}, \mathbf{j})}(i_{m(\mathbf{i}, \mathbf{j})}, j_{m(\mathbf{i}, \mathbf{j})}), & \text{if } \mathbf{i}, \mathbf{j} \in \Sigma \text{ so that } \mathbf{i} \neq \mathbf{j}. \end{cases}$$

This is indeed a metric: the triangle inequality follows easily from (A.3) and the definition of d_n . It also follows readily that (Σ, d) is compact. To see this, let $(\mathbf{i}_j)_{j \in \mathbb{N}}$ be a sequence in Σ . To find a converging subsequence, we may clearly

assume that $A = \{\mathbf{i}_j : j \in \mathbb{N}\}$ is an infinite set. Choosing $\mathbf{i} \in \Sigma$ so that $[\mathbf{i}|_n] \cap A$ is infinite for all $n \in \mathbb{N}$, it follows that a subsequence of (\mathbf{i}_j) converges to \mathbf{i} .

Our next goal is to show that Σ is doubling. For this, we choose $\mathbf{i} \in \Sigma$, $0 < r < 1 = \text{diam}(\Sigma)$ and fix n so that $\varepsilon_{n+1} = 2^{-N_n}\varepsilon_n \leq r < \varepsilon_n$. We also choose $k \in \mathbb{N}$ so that $2^{-k}\varepsilon_n \leq r < 2^{-k+1}\varepsilon_n$. If $k > 1$, we get $B(\mathbf{i}, 2r) \subset B(\mathbf{i}, r) \cup B(\mathbf{i}_0, r) \cup B(\mathbf{i}_1, r)$, where $\mathbf{i}_0 = (i_1, \dots, i_{n-1}, 0, i_{n+1}, \dots)$ and $\mathbf{i}_1 = (i_1, \dots, i_{n-1}, k-1, i_{n+1}, \dots)$. If $k = 1$, then $B(\mathbf{i}, 2r) \subset B(\mathbf{i}, r) \cup B(\mathbf{i}_2, r) \cup B(\mathbf{i}_3, r)$, where $\mathbf{i}_2 = (i_1, \dots, i_{n-2}, 0, i_n, \dots)$ and $\mathbf{i}_3 = (i_1, \dots, i_{n-2}, N_{n-1}, i_n, \dots)$. In any case, we observe that Σ is doubling with doubling constant 3.

We let μ be the unique measure on Σ that satisfies

$$\begin{aligned}\mu([\mathbf{i}0]) &= 2^{-n}\mu([\mathbf{i}]), \\ \mu([\mathbf{i}j]) &= N_n^{-1}(1 - 2^{-n})\mu([\mathbf{i}])\end{aligned}$$

for all $j \in \{1, \dots, N_n\}$, $\mathbf{i} \in \Sigma_{n-1}$, and $n \geq 2$. Moreover, we define $A = \{\mathbf{i} \in \Sigma : i_j \neq 0 \text{ for all } j \in \mathbb{N}\}$. Then A is compact and $\mu(A) = \prod_{n=1}^{\infty} (1 - 2^{-n}) > 0$.

To show (A.2), we fix $\mathbf{i} \in A$ and define a sequence $r_n = \varepsilon_n 2^{-in}$ for all $n \in \mathbb{N}$. Given $\mathbf{i} \in A$, it then follows that $B(\mathbf{i}, r_n) = [\mathbf{i}|_n] \cup [\mathbf{i}']$ for all $n \in \mathbb{N}$, where $\mathbf{i}' = (i_1, \dots, i_{n-1}, 0) \in \Sigma_n$. Thus we get

$$\begin{aligned}\frac{\mu(A \cap B(\mathbf{i}, r_n))}{\mu(B(\mathbf{i}, r_n))} &= \frac{\mu([\mathbf{i}|_n])}{\mu([\mathbf{i}|_n]) + \mu([\mathbf{i}'])} \\ &= \frac{N_n^{-1}(1 - 2^{-n})\mu([\mathbf{i}_{n-1}])}{(N_n^{-1}(1 - 2^{-n}) + 2^{-n})\mu([\mathbf{i}_{n-1}])} = \frac{1 - 2^{-n}}{1 - 2^{-n} + n}.\end{aligned}$$

Since this tends to 0 as $n \rightarrow \infty$, the claim follows. \square

In [37], one can find a construction of an infinite dimensional (and hence nondoubling) compact metric space where the density point property fails. Concerning the above result, we are grateful to Marianna Csörnyei for her help in constructing the example. After finding this example, David Preiss told us that he has obtained a characterisation for metric spaces where all measures satisfy the density point property. This is to appear in his forthcoming book.

Finally, we note that even if the density point property fails, the following ‘‘upper density point property’’ is true for all measures in all doubling metric spaces.

Proposition A.2. *If μ is a Radon measure on a doubling metric space X and $A \subset X$ is μ -measurable, then*

$$\limsup_{r \downarrow 0} \frac{\mu(A \cap B(x, r))}{\mu(B(x, r))} = 1$$

for μ -almost all $x \in A$.

Proof. If the statement fails, there is a measure μ on X , μ -measurable sets $C \subset A \subset X$, $0 < t < 1$, and $r_0 > 0$ so that $\mu(C) > 0$ and

$$\mu(A \cap B(x, r)) < t\mu(B(x, r)) \quad (\text{A.4})$$

for all $x \in C$ and $0 < r < r_0$. Let $\varepsilon = (1 - t)/(2M)$, where $M = M(N, 1/2)$ is the constant of Lemma 2.1(1) and N is the doubling constant of X . Using the Borel regularity of μ , we find an open set $U \supset C$ so that $(1 - \varepsilon)\mu(U) < \mu(C)$. Next we choose $0 < \delta < r_0$ so small that for the set $D := \{x \in C : \text{dist}(x, X \setminus U) > \delta\}$ we have $\mu(D) > (1 - \varepsilon)\mu(U)$. Using Lemma 2.1(1), we find a δ -packing of D , say $\{B_i\}_i$, so that $\mu(\bigcup_i B_i) = c\mu(U)$, where $c \geq (1 - \varepsilon)/M$. The choice of ε implies $1 - (1 - t)c < 1 - \varepsilon$. Recalling (A.4), we now have

$$\begin{aligned} \mu(D) &\leq \sum_i \mu(D \cap B_i) + \mu(U \setminus \bigcup_i B_i) \leq \sum_i \mu(A \cap B_i) + (1 - c)\mu(U) \\ &< t\mu\left(\bigcup_i B_i\right) + (1 - c)\mu(U) = (1 - (1 - t)c)\mu(U) < (1 - \varepsilon)\mu(U) < \mu(D). \end{aligned}$$

This contradiction finishes the proof. \square

APPENDIX B. LOCAL DIMENSIONS VIA PARTITIONS

The main result of this section, Theorem B.1, is considered as folklore, but complete proofs are hard to find in the literature even for dyadic cubes of \mathbb{R}^d . Thus we offer a detailed argument. Partial proofs for corresponding Euclidean results can be found from [13, Lemma 2.3] and [44, Theorem 15.3]

We will first fix some notation to be used in this section. Let $0 < c_1, c_2 < 1$ and $\delta_n \downarrow 0$ a decreasing sequence so that $c_1 < \delta_{n+1}/\delta_n < c_2$ for all $n \in \mathbb{N}$. For each n we fix a δ_n -partition \mathcal{Q}_n of X . If $x \in X$, then we denote the unique element of \mathcal{Q}_n containing x by $Q_n(x)$. We also denote the radius of a ball B by $r(B)$ and for $Q \in \mathcal{Q}_n$ we set $\delta(Q) = \delta_n$. To simplify the notation, we set

$$\begin{aligned} \overline{\text{D}}_{\text{loc}}(\mu, x) &= \limsup_{n \rightarrow \infty} \log \mu(Q_n(x)) / \log \delta_n \\ \underline{\text{D}}_{\text{loc}}(\mu, x) &= \liminf_{n \rightarrow \infty} \log \mu(Q_n(x)) / \log \delta_n, \end{aligned}$$

for all measures μ on X and $x \in X$. Bear in mind that, a priori, the definitions of $\overline{\text{D}}_{\text{loc}}(\mu, x)$ and $\underline{\text{D}}_{\text{loc}}(\mu, x)$ depend on the choice of partitions. In Theorem B.1, we show that in μ -almost every point these quantities equal to the local dimensions and hence, the choice of partitions does not play any role. Finally, the s -dimensional Hausdorff measure is denoted by \mathcal{H}^s and the s -dimensional (radius based) packing measure by \mathcal{P}^s , see [39, §4.3] and [14, §3].

Theorem B.1. *Let μ be a Radon measure on a doubling metric space X . Then*

$$\overline{\text{dim}}_{\text{loc}}(\mu, x) = \overline{\text{D}}_{\text{loc}}(\mu, x), \quad (\text{B.1})$$

$$\underline{\text{dim}}_{\text{loc}}(\mu, x) = \underline{\text{D}}_{\text{loc}}(\mu, x) \quad (\text{B.2})$$

for μ -almost all $x \in X$.

We will make use of the following Vitali-type covering lemma for the Hausdorff measure. For a proof, see e.g. [13, Theorem 2.2] or [17, Theorem 1.10].

Lemma B.2. *Suppose that $0 < s < \infty$, $A \subset X$ and $\mathcal{V} \subset \bigcup_{n \in \mathbb{N}} \mathcal{Q}_n$ such that each $x \in A$ is contained in infinitely many elements of \mathcal{V} . Then we may find a disjoint subcollection \mathcal{Q} of \mathcal{V} so that*

$$\sum_{Q \in \mathcal{Q}} \delta(Q)^s = \infty \quad \text{or} \quad \mathcal{H}^s \left(A \setminus \bigcup_{Q \in \mathcal{Q}} Q \right) = 0.$$

Our next lemma is another covering result suitable for our purposes. It is needed mainly because the elements of \mathcal{Q}_{n+1} are not necessarily subsets of the elements in \mathcal{Q}_n . Recall from §2 that Λ is the constant used in the definition of δ -partitions to control the thinness of the partition elements.

Lemma B.3. *Let $A \subset X$ and $\mathcal{V} \subset \bigcup_{n \in \mathbb{N}} \mathcal{Q}_n$ such that $A \subset \bigcup_{Q \in \mathcal{V}} Q$. For each $Q \in \mathcal{V}$ choose $y = y_Q$ such that $B(y, \delta(Q)) \subset Q$ and set $B'_Q = B(y, \delta(Q)/2)$. Then we may find a sub-collection \mathcal{Q} of \mathcal{V} so that the balls B'_Q , $Q \in \mathcal{Q}$, are pairwise disjoint and*

$$\mu \left(\bigcup_{Q \in \mathcal{Q}} Q \right) \geq c\mu(A),$$

where $c = c(\Lambda, c_2) > 0$.

Proof. Let us fix $n_0 = n_0(\Lambda, b) \in \mathbb{N}$ so that $2\Lambda\delta_{n+n_0} \leq \delta_n/2$ for all $n \in \mathbb{N}$. Notice that this is possible since $\delta_{n+m} \leq c_2^m \delta_n$ for all $n, m \in \mathbb{N}$. Let $\mathcal{V}_n = \{Q \in \mathcal{V} : \delta(Q) = \delta_n\}$ and $\mathcal{V}^k = \bigcup_{n=0}^{\infty} \mathcal{V}_{n_0 n + k}$ for all $k \in \{1, \dots, n_0\}$. Then, for some k , we have $\mu(A \cap \bigcup_{Q \in \mathcal{V}^k} Q) \geq \mu(A)/n_0$. We define \mathcal{Q} inductively as follows: We begin by choosing $\mathcal{Q}^0 = \mathcal{V}_k$. Given \mathcal{Q}^i and $U_i = \bigcup_{j=0}^i \bigcup_{Q \in \mathcal{Q}^j} Q$, we put

$$\mathcal{Q}^{i+1} = \{Q \in \mathcal{V}_{(i+1)n_0+k} : Q \not\subset U_i\}.$$

Finally, define $\mathcal{Q} = \bigcup_{i=0}^{\infty} \mathcal{Q}^i$. Then $\bigcup_{Q \in \mathcal{Q}} Q = \bigcup_{Q \in \mathcal{V}^k} Q$ and thus $\mu(\bigcup_{Q \in \mathcal{Q}} Q) \geq \mu(A)/n_0$. It is also straightforward to check that the collection $\{B'_Q\}_{Q \in \mathcal{Q}}$ is pairwise disjoint due to the choice of n_0 . \square

The essential part of Theorem B.1 is contained in the following lemma. We remark that the lemma is well known if $\overline{\text{D}}_{\text{loc}}(\mu, x)$ and $\underline{\text{D}}_{\text{loc}}(\mu, x)$ are replaced by $\overline{\text{dim}}_{\text{loc}}(\mu, x)$ and $\underline{\text{dim}}_{\text{loc}}(\mu, x)$, respectively. See e.g. [14].

Lemma B.4. *Let μ be a Radon measure on a doubling metric space X , $A \subset X$ a μ -measurable set, and $0 < s < \infty$. If $\mu(A) < \infty$, then*

$$\overline{\text{D}}_{\text{loc}}(\mu, x) \leq s \text{ for all } x \in A \text{ implies } \text{dim}_{\text{p}}(A) \leq s, \quad (\text{B.3})$$

$$\underline{\text{D}}_{\text{loc}}(\mu, x) \leq s \text{ for all } x \in A \text{ implies } \text{dim}_{\text{H}}(A) \leq s. \quad (\text{B.4})$$

On the other hand, if $\mu(A) > 0$, then

$$\underline{\text{D}}_{\text{loc}}(\mu, x) \geq s \text{ for all } x \in A \text{ implies } \dim_{\text{H}}(A) \geq s, \quad (\text{B.5})$$

$$\overline{\text{D}}_{\text{loc}}(\mu, x) \geq s \text{ for all } x \in A \text{ implies } \dim_{\text{p}}(A) \geq s. \quad (\text{B.6})$$

Proof. We start with the claim (B.3). We first choose an open set $U \supset A$ so that $\mu(U) < \infty$. Let $t > s$, $k \in \mathbb{N}$, and

$$A_{t,k} = \{x \in A : Q_n(x) \subset U \text{ and } \mu(Q_n(x)) > \delta_n^t \text{ for all } n \geq k\}.$$

Suppose \mathcal{B} is a packing of $A_{t,k}$ and assume its elements have radius at most δ_k . For each $B \in \mathcal{B}$, we may choose $Q_B = Q_n(x) \subset U \cap B$ so that $c(c_1, \Lambda)\delta_n \geq r(B)$. This implies $\sum_{B \in \mathcal{B}} r(B)^t \leq c^t \sum_{B \in \mathcal{B}} \delta(Q_B)^t \leq c^t \sum_B \mu(Q_B) \leq c^t \mu(U)$. Hence $\mathcal{P}^t(A_{t,k}) < \infty$ and in particular $\dim_{\text{p}}(A_{t,k}) \leq t$. As $A = \bigcup_{k \in \mathbb{N}} A_{t,k}$ and $t > s$ is arbitrary, we conclude that $\dim_{\text{p}}(A) \leq s$.

The proof of (B.4) follows the argument of [13, Lemma 2.1]. Given $t > s$, and $\varepsilon > 0$, we choose an open $U \supset A$ such that $\mu(U) < \infty$ and define

$$\mathcal{V} = \left\{ Q \in \bigcup_{n \in \mathbb{N}} \mathcal{Q}_n : \delta(Q) < \varepsilon, Q \subset U, \text{ and } \delta(Q)^t < \mu(Q) \right\}.$$

Each point of A is contained in infinitely many elements of \mathcal{V} , and thus we may apply Lemma B.2 to obtain a disjoint subcollection \mathcal{Q} of \mathcal{V} so that

$$\sum_{Q \in \mathcal{Q}} \delta(Q)^t = \infty \quad \text{or} \quad (\text{B.7})$$

$$\mathcal{H}^t \left(A \setminus \bigcup_{Q \in \mathcal{Q}} Q \right) = 0. \quad (\text{B.8})$$

Since $\sum_{Q \in \mathcal{Q}} \delta(Q)^t \leq \sum_{Q \in \mathcal{Q}} \mu(Q) \leq \mu(U) < \infty$, it follows that (B.7) cannot hold and thus we have (B.8). Now

$$\mathcal{H}_\varepsilon^t(A) \leq \mathcal{H}_\varepsilon^t \left(\bigcup_{Q \in \mathcal{Q}} Q \right) \leq \Lambda^t \sum_{Q \in \mathcal{Q}} \delta(Q)^t \leq \Lambda^t \mu(U).$$

Letting $\varepsilon \downarrow 0$, we get $\mathcal{H}^t(A) < \infty$ and finally $\dim_{\text{H}}(A) \leq s$ as $t \downarrow s$.

To prove (B.5), let $t < s$. Replacing A by a subset, if necessary, we may assume that for some $k \in \mathbb{N}$ we have $\mu(Q_n(x)) < \delta_n^t$ for all $x \in A$ and $n \geq k$. Let \mathcal{U} be a countable covering of the set C and suppose its elements have diameter at most δ_k . Denoting the doubling constant of X by N , we may, recalling Lemma 2.1(2), cover each $U \in \mathcal{U}$ by $c = c(c_1, N, \Lambda)$ elements of \mathcal{Q}_n , where $c_1 \text{diam}(U) \leq \delta_n \leq \text{diam}(U)$. Denote these sets by Q_U^j . Now

$$\sum_{U \in \mathcal{U}} \text{diam}(U)^t \geq c^{-1} \sum_{U \in \mathcal{U}} \sum_j \delta(Q_U^j)^t \geq c^{-1} \sum_{U \in \mathcal{U}} \sum_j \mu(Q_U^j) \geq c^{-1} \mu(A).$$

Thus $\mathcal{H}^t(A) > 0$ and letting $t \uparrow s$, we get $\dim_{\text{H}}(A) \geq s$.

It remains to prove (B.6). Fix $\gamma < t < s$, $F \subset A$, $\varepsilon > 0$, and consider

$$\mathcal{V} = \left\{ Q \in \bigcup_{n \in \mathbb{N}} \mathcal{Q}_n : Q \cap F \neq \emptyset, \delta(Q) < \varepsilon, \text{ and } \delta(Q)^t > \mu(Q) \right\}.$$

Let \mathcal{Q} be a sub-collection of \mathcal{V} given by Lemma B.3 so that

$$\mu\left(\bigcup_{Q \in \mathcal{Q}} Q\right) \geq c_3 \mu(F). \quad (\text{B.9})$$

Here $c_3 = c_3(\Lambda, c_2)$ is the constant of Lemma B.3. For $n \in \mathbb{N}$, consider $\mathcal{Q}_n = \{Q \in \mathcal{Q} : \delta(Q) = \delta_n\}$. Since $\sum_{n \in \mathbb{N}} \delta_n^{t-\gamma} \leq c(c_2, t, \gamma) < \infty$, we observe that there is n and $c_4 = c_4(\gamma, t, c_1, c_2) > 0$, so that $\sum_{Q \in \mathcal{Q}_n} \delta(Q)^\gamma \geq c_4 \sum_{Q \in \mathcal{Q}} \delta(Q)^t$. (A simple calculation shows that choosing $c_4 = (\sum_n \delta_n^{t-\gamma})^{-1}$ will do.) Combining this with (B.9) gives $\sum_{Q \in \mathcal{Q}_n} \delta(Q)^\gamma \geq c_4 \mu(\bigcup_{Q \in \mathcal{Q}} Q) \geq c_3 c_4 \mu(F)$. For each $Q \in \mathcal{Q}_n$, we pick $y \in Q \cap F$ and define $B_Q = B(y_Q, \delta_n)$. Using the doubling condition (Lemma 2.1(1) applied to the collection $\{(2\Lambda + 1)B_Q\}$), we further find a disjoint subcollection B_1, \dots, B_k of these balls such that $k \geq c_5(N, \Lambda) \# \mathcal{Q}_n$. Thus

$$P_\varepsilon^\gamma(F) \geq k \delta_n^\gamma \geq c_5 \sum_{Q \in \mathcal{Q}_n} \delta(Q)^\gamma \geq c_3 c_4 c_5 \mu(F).$$

As $\varepsilon \downarrow 0$, we get $P^\gamma(F) \geq c_3 c_4 c_5 \mu(F)$ and finally $\mathcal{P}^\gamma(A) \geq c_3 c_4 c_5 \mu(A)$. If $\mu(A) > 0$, this implies $\dim_p(A) \geq s$ as $\gamma \uparrow s$. \square

Proof of Theorem B.1. The claims follow easily from Lemma B.4. For instance, let us prove the inequality $\overline{\text{Dim}}_{\text{loc}}(\mu, x) \leq \overline{\text{dim}}_{\text{loc}}(\mu, x)$ for μ -almost all $x \in X$. Suppose to the contrary that there are $0 < s < t < \infty$ and a set $A \subset X$ with $0 < \mu(A) < \infty$ so that $\overline{\text{Dim}}_{\text{loc}}(\mu, x) \geq t > s \geq \overline{\text{dim}}_{\text{loc}}(\mu, x)$ for all $x \in A$. Then $\dim_p(A) \geq t$ by (B.6). On the other hand, in the view of [14, Corollary 3.20(a)] (that is, (B.3) for $\overline{\text{dim}}_{\text{loc}}(\mu, x)$), we have $\dim_p(A) \leq s$. This is clearly impossible and thus we must have $\overline{\text{Dim}}_{\text{loc}}(\mu, x) \leq \overline{\text{dim}}_{\text{loc}}(\mu, x)$ for μ -almost every $x \in X$. \square

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