PLANAR SOBOLEV HOMEOMORPHISMS AND HAUSDORFF DIMENSION DISTORTION

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Abstract. We investigate how planar Sobolev-Orlicz homeomorphisms map sets of Hausdorff dimension less than two. With the correct gauge functions the generalized Hausdorff measures of the image sets are shown to be zero.

1. Introduction

We study Sobolev mappings $f \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$, where Ω is an open subset of $\mathbb{R}^2$. This means that the components of the mapping $f \colon \Omega \to \mathbb{R}^2$ have locally integrable distributional partial derivatives. We would like to understand how much these mappings can increase the size of small sets. There are many different ways to define the size of a set. Here we will use the standard and generalized Hausdorff measures. This approach is a natural continuation to the study of Lusin condition (N), which requires that $\mathcal{L}^2(f(A)) = 0$ for every $A \subset \Omega$ with $\mathcal{L}^2(A) = 0$. By $\mathcal{L}^2(A)$ we mean the 2-dimensional Lebesgue measure of A.

Without assuming $f \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$ to be a homeomorphism the Lusin condition (N) holds if we suppose that $|Df|^p \in L^1_{\mathrm{loc}}(\Omega)$ with $p > 2$, see [8]. For a homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$ a result by Reshetnyak [9] says that f satisfies Lusin condition (N) under the weaker assumption $|Df|^2 \in L^1_{\mathrm{loc}}(\Omega)$. Later this was improved in [4] by showing that already $|Df|^2 \log^{-1}(e + |Df|) \in L^1_{\mathrm{loc}}(\Omega)$ suffices for a homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$ to satisfy the Lusin condition (N).

When we replace the Lebesgue measure with Hausdorff measures we have the following result from [1]: Every homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$, with $|Df|^p \in L^1_{\mathrm{loc}}(\Omega)$ for some $p > 2$, maps sets of Hausdorff dimension less than two to sets of Hausdorff dimension less than two. Motivated by the results on the Lusin condition (N), we study here what happens to sets of Hausdorff dimension less than two when we assume only that $|Df|^2 \log^{\lambda-1}(e + |Df|) \in L^1_{\mathrm{loc}}(\Omega)$ for some $\lambda > 0$. It is clear that sets of Hausdorff dimension less than two can be mapped to sets of Hausdorff

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dimension two with such mappings, see for example [2, 5]. The usual Hausdorff measures are therefore too rough to be used to measure the image sets. This is where the generalized Hausdorff measures come into play. For them we obtain the following result.

Theorem 1.1. *Let Ω be an open set in $\mathbb{R}^2$ and $f\colon \Omega \to f(\Omega)$ a homeomorphism of class $W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^2)$ with*

$$|Df|^2 \log^{\lambda-1}(e+|Df|) \in L^1_{\mathrm{loc}}(\Omega)$$

for some $\lambda > 0$. Then, with $h_\lambda(t) = t^2 \log^\lambda \frac{1}{t}$,

$$\mathcal{H}^{h_\lambda}(f(E)) = 0$$

for every set $E \subset \Omega$ for which $\dim_{\mathrm{H}}(E) < 2$.

Theorem 1.1 was proved in [5] for sets with Minkowski dimension less than two. There the result was conjectured to hold also with sets of Hausdorff dimension less than two. Theorem 1.1 proves this conjecture. The proof follows the approach used in [5]. The paper [5] also contains an example which shows that the result we prove here is sharp: there exists a homeomorphism f that maps a Cantor set of Hausdorff dimension less than two to a set of positive $\mathcal{H}^{h_\lambda}$-measure, with $|Df|^2\log^{t-1}(e+|Df|) \in L^1_{\mathrm{loc}}(\Omega)$ for every $t < \lambda$.

A weaker distortion estimate was proved in [6]. Namely that for $f \in W^{1,1}(\Omega;\mathbb{R}^2)$ with $|Df|^2\log^\lambda(e+|Df|) \in L^1_{\mathrm{loc}}(\Omega)$ sets with Hausdorff dimension less than two are mapped to sets with $\mathcal{H}^{h_\lambda}$-measure zero.

2. Proof of the theorem

Let us first introduce some notation. An r-neighborhood of a set A is written as $A + r = \{x : \mathrm{dist}(A, x) < r\}$. By $\mathcal{L}^k$ we mean the k-dimensional Lebesgue measure. We also use the generalized Hausdorff measure $\mathcal{H}^h$, which is defined for a set A as $\mathcal{H}^h(A) = \lim_{\delta\to 0}\mathcal{H}^h_\delta(A)$, where

$$\mathcal{H}^h_\delta(A) = \inf\left\{\sum_{i=1}^\infty h(\operatorname{diam} U_i) \ : \ A \subset \bigcup_{i=1}^\infty U_i, \operatorname{diam}(U_i) \le \delta\right\}$$

and h is a non-decreasing function with $h(0) = 0$. When the gauge function is $h(t) = t^s$, we have the usual s-dimensional Hausdorff measure which we also write as $\mathcal{H}^s$. The Hausdorff dimension of a set A is

$$\dim_{\mathrm{H}}(A) = \inf\{s : \mathcal{H}^s(A) = 0\} = \sup\{s : \mathcal{H}^s(A) = \infty\}.$$

We denote the weak differential of the function f by $Df\colon \mathbb{R}^2 \to \mathbb{R}^2$ and its operator-norm as $|Df(x)| = \sup\{|Df(x)\xi| : |\xi| = 1\}$. The Jacobian determinant is written as $J_f(x) = \det Df(x)$.

We are now ready to start with the proof. The main auxiliary ingredient in the proof is the following lemma from [5].

Lemma 2.1. *Let Ω be an open set in $\mathbb{R}^2$ and $f\colon \Omega \to f(\Omega)$ a homeomorphism in $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$. Then there exists a set $F \subset f(\Omega)$ with $\mathcal{H}^{3/2}(F) = 0$ such that for all $y \in f(\Omega) \setminus F$ there exist constants $C_y > 0$ and $r_y > 0$ such that*

$$\operatorname{diam}(f^{-1}(B(y,r))) \leq C_y r^{1/2}$$

for all $0 < r < r_y$.

Using the previous lemma we are able to cover almost all of the target $f(\Omega)$ with a countable collection of sets where the homeomorphism f maps uniformly nicely.

Lemma 2.2. *Let Ω be an open set in $\mathbb{R}^2$ and $f\colon \Omega \to f(\Omega)$ a homeomorphism in $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$. Then there exists a decomposition $f(\Omega) = \bigcup_{i=0}^{\infty} F_i$, where $\mathcal{H}^{3/2}(F_0) = 0$ and for each F_i, $i = 1, 2, \ldots$, there exist constants $1 \leq C_i < \infty$ and $R_i > 0$ such that*

$$f^{-1}\left((f(A) \cap F_i) + \left(\frac{r}{C_i}\right)^2\right) \subset A + r$$

for every $A \subset \Omega$ and for every $r \in (0, R_i)$.

Proof. As in [5, Lemma 3.2] we may represent $f(\Omega)$ as

$$f(\Omega) = F_0 \cup \bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty}\left\{ y \in f(\Omega) \;\middle|\; \operatorname{diam}(f^{-1}(B(y,r))) \leq k r^{\frac{1}{2}} \text{ for all } r \in (0, \frac{1}{j}) \right\},$$

where $F_0 = F$ is the set in Lemma 2.1. □

Proof of Theorem 1.1. Take s so that $\dim_{\mathrm{H}}(E) < s < 2$. Let F_i, C_i and R_i be as in Lemma 2.2. We will show that $\mathcal{H}^{h_\lambda}(F_i \cap f(E)) = 0$ for all $i \in \mathbb{N}$. This clearly holds for F_0. Let $i \geq 1$.

Because we are considering a homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$, its Jacobian is either non-negative almost everywhere in Ω or non-positive almost everywhere in Ω, see for example [7, Chapter 1]. We may assume that $J_f \geq 0$ almost everywhere in Ω. Because

$$|Df|^2 \log^{\lambda-1}(e + |Df|) \in L^1_{\mathrm{loc}}(\Omega)$$

we have by [3, Corollary 9.1] that $J_f \log^{\lambda}(e + J_f) \in L^1_{\mathrm{loc}}(\Omega)$. Take now $\epsilon > 0$. By the absolute continuity of the integral there is $\delta > 0$ so that

$$\int_A J_f(x) \log^{\lambda}(e + J_f(x))dx < \epsilon$$

for every $A \subset \Omega$ for which $\mathcal{L}^2(A) < \delta$. Write $\sigma = \frac{2-s}{2}$. Let $0 < r_0 < 1$ be so small that $2\log^\lambda(\frac{C_i}{r_0}) \le r_0^{-\sigma}$. Because $\mathcal{H}^s(E) = 0$, there exists a countable collection of balls $\{B(x_j, r_j)\}_{j=1}^\infty$ with radii less than $\min\{r_0, R_i, \frac{1}{C_i}\}$ so that

$$\sum_{j=1}^{\infty} 4\pi r_j^s < \min\{\epsilon, \delta\}$$

and $E \subset \bigcup_{j=1}^\infty B(x_j, r_j)$.

Now write $F_{i,j} = F_i \cap f(B(x_j, r_j))$ for every $j \in \mathbb{N}$. Then by Lemma 2.2 we have $f^{-1}(F_{i,j} + R_{i,j}) \subset B(x_j, 2r_j)$, where $R_{i,j} = (\frac{r_j}{C_i})^2 \le r_0$.

Next we use the $5r$-covering theorem on the target side to find a pairwise disjoint collection of balls $\{B(y_k, R_{i,j(k)})\}_{k\in K}$ from

$$\{B(x, R_{i,j}) : x \in F_{i,j}, j \in \mathbb{N}\}$$

so that

$$F_i \cap f(E) \subset \bigcup_{k\in K} B(y_k, 5R_{i,j(k)}).$$

By estimating that any ball of radius $10r$ in $\mathbb{R}^2$ can be covered by 1000 balls of radius r we see that

$$\begin{aligned}
\frac{1}{1000}\mathcal{H}^{h_\lambda}_{r_0}(F_i \cap f(E)) &\le \sum_{k\in K} R_{i,j(k)}^2 \log^\lambda(\frac{1}{R_{i,j(k)}}) \\
&\le \sum_{k\in K} \mathcal{L}^2(B(y_k, R_{i,j(k)})) \log^\lambda(\frac{1}{R_{i,j(k)}}) \\
&\le \sum_{k\in K} \int_{f^{-1}(B(y_k,R_{i,j(k)}))} \log^\lambda(\frac{1}{R_{i,j(k)}}) J_f(x)dx \\
&\le \sum_{k\in K} \Bigg(\int_{\left\{z\in f^{-1}(B(y_k,R_{i,j(k)})):J_f(z)<r_{j(k)}^{-\sigma}\right\}} \log^\lambda(\frac{1}{R_{i,j(k)}}) J_f(x)dx \\
&\quad + \int_{\left\{z\in f^{-1}(B(y_k,R_{i,j(k)})):J_f(z)\ge r_{j(k)}^{-\sigma}\right\}} \log^\lambda(\frac{1}{R_{i,j(k)}}) J_f(x)dx \Bigg) \\
&\le \sum_{k\in K} r_{j(k)}^{-2\sigma} \mathcal{L}^2(f^{-1}(B(y_k, R_{i,j(k)}))) \\
&\quad + \sum_{k\in K} \frac{\log^\lambda(1/R_{i,j(k)})}{\log^\lambda(e + 1/r_{j(k)}^\sigma)} \int_{f^{-1}(B(y_k,R_{i,j(k)}))} J_f(x) \log^\lambda(e + J_f(x))dx,
\end{aligned}$$

where in the third inequality we have used [4, Lemma 3.2] and in the last one the fact that $\log^\lambda(\frac{1}{R_{i,j(k)}}) = 2\log^\lambda(\frac{C_i}{r_{j(k)}}) \le r_{j(k)}^{-\sigma}$.

Our task is to estimate the two remaining sums. By grouping the balls according to $j(k)$, we get for the first sum an estimate

$$\sum_{k\in K} r_{j(k)}^{-2\sigma}\mathcal{L}^2(f^{-1}(B(y_k,R_{i,j(k)}))) = \sum_{j=1}^{\infty} r_j^{s-2} \sum_{\substack{k\in K\\ j(k)=j}} \mathcal{L}^2(f^{-1}(B(y_k,R_{i,j})))$$
$$\leq \sum_{j=1}^{\infty} r_j^{s-2}\mathcal{L}^2(B(x_j,2r_j)) = \sum_{j=1}^{\infty} 4\pi r_j^s < \epsilon.$$

To estimate the second sum, first note that for every $k\in K$ we have

$$\frac{\log^\lambda(1/R_{i,j(k)})}{\log^\lambda(e+1/r_{j(k)}^\sigma)} = \frac{2^\lambda\log^\lambda(C_i/r_{j(k)})}{\log^\lambda(e+1/r_{j(k)}^\sigma)} \leq \frac{2^\lambda\log^\lambda(1/r_{j(k)}^2)}{\log^\lambda(1/r_{j(k)}^\sigma)} = \frac{4^\lambda}{\sigma^\lambda}.$$

From this we get

$$\sum_{k\in K}\frac{\log^\lambda(1/R_{i,j(k)})}{\log^\lambda(e+1/r_{j(k)}^\sigma)}\int_{f^{-1}(B(y_k,R_{i,j(k)}))} J_f(x)\log^\lambda(e+J_f(x))dx$$
$$\leq \frac{4^\lambda}{\sigma^\lambda}\sum_{k\in K}\int_{f^{-1}(B(y_k,R_{i,j(k)}))} J_f(x)\log^\lambda(e+J_f(x))dx$$
$$\leq \frac{4^\lambda}{\sigma^\lambda}\int_{\bigcup_{k\in K} f^{-1}(B(y_k,R_{i,j(k)}))} J_f(x)\log^\lambda(e+J_f(x))dx$$
$$\leq \frac{4^\lambda}{\sigma^\lambda}\int_{\bigcup_{j=1}^\infty B(x_j,2r_j)} J_f(x)\log^\lambda(e+J_f(x))dx \leq \frac{4^\lambda}{\sigma^\lambda}\epsilon,$$

since the sets $\{f^{-1}(B(y_k,R_{i,j(k)}))\}_{k\in K}$ are pairwise disjoint and

$$\mathcal{L}^2\left(\bigcup_{j=1}^\infty B(x_j,2r_j)\right) \leq \sum_{j=1}^\infty 4\pi r_j^2 \leq \sum_{j=1}^\infty 4\pi r_j^s < \delta.$$

The proof is now finished as we have shown that

$$\mathcal{H}_{r_0}^{h_\lambda}(F_i\cap f(E)) \leq 1000\left(1+\frac{4^\lambda}{\sigma^\lambda}\right)\epsilon$$

for every $r_0,\epsilon>0$ and $i\in\mathbb{N}$. $\square$

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