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A NOTE ON THE LEVEL SETS OF HÖLDER CONTINUOUS FUNCTIONS ON THE REAL LINE

Abstract

We construct α-Hölder continuous functions on the real line so that all their level sets have positive $(1-\alpha)$-dimensional Hausdorff measure.

1 Introduction

Bernd Kirchheim and Francesco Serra Cassano constructed an $\mathbb{H}$-regular surface in $\mathbb{H}^1$ that has Euclidean Hausdorff dimension $\frac{5}{2}$. Here $\mathbb{H}$ refers to the Heisenberg group. See [2] for the construction and definitions. Part of the construction was a $\frac{1}{2}$-Hölder continuous function $h : \mathbb{R} \to \mathbb{R}$ whose level sets $h^{-1}(t)$ had Euclidean Hausdorff dimension at least $\frac{1}{2}$. It remained open whether or not one could construct a $\frac{1}{2}$-Hölder continuous function for which the level sets have positive $\frac{1}{2}$-dimensional Hausdorff measure. They posed the question in the following slightly more general setting.

> Can one find α-Hölder continuous functions such that all level sets $f^{-1}(t)$ (or at least for all t from a set of positive measure) are of positive $(1-\alpha)$-dimensional Hausdorff measure?

Answer to this question is *yes*. We will prove this by varying a self-affine construction. As will be shown the construction can also be done with a better modulus of continuity than $\sqrt{|x-y|}$, so that it meets the requirements of the construction of Kirchheim and Serra Cassano. However, it is true that for any such construction almost all of the level sets must have zero $\frac{1}{2}$-dimensional Hausdorff measure.

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2 The construction

Let us first fix some notation. For $x \in \mathbb{R}$ write its integer part as $\lfloor x \rfloor = \sup\{k \in \mathbb{Z} : k \leq x\}$. The collection of all subsets of a set X will be written as $\mathcal{P}(X) = \{A \subset X\}$.

Definition 2.1. *For $\alpha \in]0,1]$ call a function $f : X \to Y$ α-Hölder continuous if there exists a constant $C > 0$ so that for all $x, y \in X$ the inequality*

$$d_Y(f(x), f(y)) \leq C d_X(x,y)^\alpha$$

holds.

For the definition of Hausdorff measure and its basic properties we advice the reader to consult [3].

Now let us begin with the construction. Define for every $m \in \mathbb{N}$, $k \in \{0,1,2\}$ and $i \in \{0,1,\ldots,m-1\}$ a mapping $f^m_{mk+i} : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$f^m_{mk+i}((x,y)) = (\frac{x+mk+i}{3m}, (-1)^k \frac{y+i}{m} + \frac{1+(-1)^{k+1}}{2}).$$

Using these define set functions $F^m : \mathcal{P}(\mathbb{R}^2) \to \mathcal{P}(\mathbb{R}^2)$, $m \in \mathbb{N}$, by letting

$$F^m(A) = \bigcup_{i=0}^{3m-1} f^m_i(A)$$

for every $A \subset \mathbb{R}^2$. Look at Figure 1 to see how function F^4 maps the unit square.

Recall that by an attractor of a family of contracting maps $f_i : \mathbb{R}^d \to \mathbb{R}^d$, $i \in I$, we mean the unique compact set $E \subset \mathbb{R}^d$ for which

$$\bigcup_{i \in I} f_i(E) = E.$$

The self-affine set we get from a function F^m, $m \geq 2$ as the attractor of the mapping family $(f^m_i)_{i=0}^{3m-1}$ is already a graph of a $\frac{\log m}{\log 3 + \log m}$-Hölder continuous function on the interval $[0,1]$. In particular, the attractor of $(f^3_i)_{i=0}^8$ is a graph of a $\frac{1}{2}$-Hölder continuous function whose level sets have positive $\frac{1}{2}$-dimensional Hausdorff measure. To get the rest of the desired α-Hölder continuous functions with α between $\frac{\log m}{\log 3 + \log m}$ and $\frac{\log(m+1)}{\log 3 + \log(m+1)}$ we use both functions F^m and F^{m+1}.

Proposition 2.2. *Let $\alpha \in]0,1]$. Then there is an α-Hölder continuous function $h : \mathbb{R} \to \mathbb{R}$ so that all the level sets $h^{-1}(t)$ have positive $(1-\alpha)$-dimensional Hausdorff measure.*

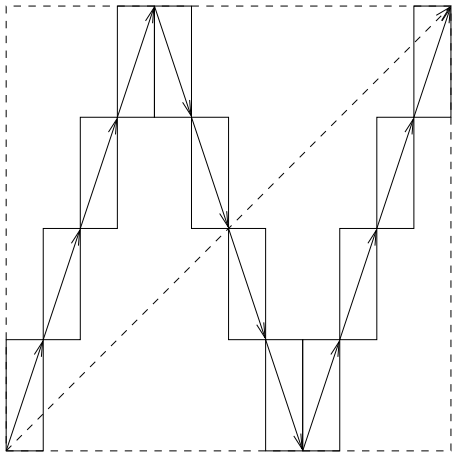

Figure 1: An illustration how the family of mappings $(f_i^4)_{i=0}^{11}$ maps $[0,1]^2$.

Proof. In the case $\alpha = 1$ take $h(x) = x$. Let $\alpha \in]0,1[$. Find the largest $m \in \mathbb{N}$ so that

$$\alpha \geq \frac{\log m}{\log 3 + \log m}.$$

Next we define the portion ϱ which tells how much of the construction should be done using the function F^{m+1}. The rest of the construction will use the function F^m. Let

$$\varrho = \frac{\alpha \log 3 - (1-\alpha)\log m}{(1-\alpha)(\log(m+1) - \log m)} \in [0,1[.$$

Define a sequence of closed sets $(G_i)_{i=0}^{\infty}$ with

$$G_i = F^{j_1} \circ F^{j_2} \circ \cdots \circ F^{j_i}([0,1]^2),$$

where $j_k = m+1$ if $\lfloor \varrho k \rfloor > \lfloor \varrho(k-1) \rfloor$ and $j_k = m$ otherwise. Let $G = \cap_i G_i$. Because all of the G_i are closed, so is G. Clearly the projection of G to the x-axis is the whole interval $[0,1]$ and since the height of each section $G_i \cap \{x = p\}$ goes to zero when i goes to infinity, the set G is a graph of a function. Call this function g and extend it to be a function $h : \mathbb{R} \to \mathbb{R}$ by defining

$$h(x+k) = g(x) + k$$

for every $k \in \mathbb{Z}$ and $x \in [0,1]$. Note that while the mappings f_i^1 are not contracting, the construction still works when $m = 1$ since then $\varrho > 0$. Let $t \in [0,1]$ and consider the set $h^{-1}(t)$.

To see that $\mathcal{H}^{1-\alpha}(h^{-1}(t)) > 0$ we construct a measure μ on $[0,1]$ by distributing the measure at each step evenly between the constructing intervals of $G_i \cap \{(x,t) : x \in \mathbb{R}\}$. Then for any interval $X_{\mathtt{i}} \subset G_n \cap \{(x,t) : x \in \mathbb{R}\}$ we have

$$\begin{aligned}\mu(X_{\mathtt{i}}) = 3^{-n} &\leq \Big(\frac{m+1}{m}\Big)^{1-\alpha}((3m)^{-(1-\varrho)n+1}(3(m+1))^{-\varrho n-1})^{1-\alpha} \\ &\leq \Big(\frac{m+1}{m}\Big)^{1-\alpha}\mathrm{diam}(X_{\mathtt{i}})^{1-\alpha}.\end{aligned}$$

Now from this it immediately follows that $\mathcal{H}^{1-\alpha}(h^{-1}(t)) > 0$. See for example [3, Theorem 6.9]

Next we check that the function h is α-Hölder continuous. Let $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$ so that

$$(3m)^{-(1-\varrho)(n+1)}(3(m+1))^{-\varrho(n+1)} \leq |x-y| < (3m)^{-(1-\varrho)n}(3(m+1))^{-\varrho n}.$$

Then

$$\begin{aligned}|h(x)-h(y)| &\leq 2 \cdot m^{-(1-\varrho)n-1}(m+1)^{-\varrho n+1} = 2\frac{m+1}{m}3^{-\frac{\alpha n}{1-\alpha}} \\ &= 2\frac{m+1}{m}((3m)^{-(1-\varrho)n}(3(m+1))^{-\varrho n})^{\alpha} \\ &\leq 18(m+1)^2|x-y|^{\alpha}.\end{aligned}$$

□

Remark 2.3. *As was noted in [2], if $f : [0,1] \to [0,1]$ is α-Hölder continuous then it is Lipschitz continuous from $[0,1]$ to X, where X is $[0,1]$ with metric $d(x,y) = |x-y|^{\frac{1}{\alpha}}$. Then by inequality [1, 2.10.25]*

$$\int_{[0,1]} \mathcal{H}^{1-\alpha}(f^{-1}(y))d\mathcal{H}^{\alpha}_d(y) \leq C\mathcal{H}^1([0,1]), \tag{1}$$

we see that almost all level sets of f have finite $(1-\alpha)$-dimensional Hausdorff measure. Notice that here $\mathcal{H}^{\alpha}_d$ is the α-dimensional Hausdorff measure on X with the metric d which is the same as 1-dimensional Hausdorff measure on $[0,1]$ with the Euclidean metric. This means that our construction is in some sense optimal.

By changing the distribution of constructing functions F^m we can produce graphs of functions with different kinds of moduli of continuity. As an example we combine the functions F^m so that they give a function satisfying the lemma of Kirchheim and Serra Cassano [2, Lemma 3.5].

Proposition 2.4. *There is a function $h : \mathbb{R} \to \mathbb{R}$ such that*

(i) for all $t \in \mathbb{R}$ the Euclidean Hausdorff dimension of $h^{-1}(t)$ is at least $\frac{1}{2}$.

(ii) for each $m \geq 1$ we have

$$\lim_{r \to 0_+} \frac{\log((\frac{1}{r}))^m}{\sqrt{r}} \sup\{|h(x) - h(y)|, |x - y| \leq r\} = 0.$$

Proof. We will mainly use the function F^3 to construct the graph, but on small part of construction steps we use the function F^4 to improve the modulus of continuity. This time define a sequence of closed sets $(G_i)_{i=0}^{\infty}$ with

$$F^{j_1} \circ F^{j_2} \circ \cdots \circ F^{j_i}([0,1]^2),$$

where $j_k = 4$ if $\sqrt{k} \in \mathbb{N}$ and $j_k = 3$ otherwise. Again set $G = \cap_i G_i$ and let h be the extension of the map g as in the proof of Proposition 2.2.

Let us first prove that the condition (i) is satisfied. Take a $t \in [0,1]$ and construct a measure μ on $[0,1]$ by using the intervals of $G_i \cap \{(x,t) : x \in \mathbb{R}\}$. At each step i distribute the measure evenly on the consrtuction intervals. Now take an $\epsilon > 0$ and an $n_\epsilon \in \mathbb{N}$ so that $9^{-n\epsilon} \leq (\frac{3}{4})^{\frac{1}{2}\lfloor\sqrt{n}\rfloor}$ for every $n \geq n_\epsilon$. Then for each construction interval $X_{\mathtt{i}} \subset G_n \cap \{(x,t) : x \in \mathbb{R}\}$, $n \geq n_\epsilon$, we have

$$\mu(X_{\mathtt{i}}) \leq 3^{-n} \leq 3^{-n} 9^{n\epsilon} \left(\frac{3}{4}\right)^{\frac{1}{2}\lfloor\sqrt{n}\rfloor} = (9^{-n+\lfloor\sqrt{n}\rfloor} 12^{-\lfloor\sqrt{n}\rfloor})^{\frac{1}{2}-\epsilon} = \operatorname{diam}(X_{\mathtt{i}})^{\frac{1}{2}-\epsilon}.$$

Again by recalling the easier direction of Frostman's lemma ([3, Theorem 6.9]) we get $\mathcal{H}^{\frac{1}{2}-\epsilon}(\{h^{-1}(t)\}) > 0$.

We still need to show that property (ii) is satisfied. Take $0 < r < 1$ and the largest $n \in \mathbb{N}$ so that $r \leq 9^{-n+\lfloor\sqrt{n}\rfloor} 12^{-\lfloor\sqrt{n}\rfloor}$. Now for every $x, y \in \mathbb{R}$ with $|x - y| < r$ we have

$$|h(x) - h(y)| \leq 2 \cdot 3^{-n+\lfloor\sqrt{n}\rfloor} 4^{-\lfloor\sqrt{n}\rfloor}.$$

Therefore for any $m \in \mathbb{N}$ we get

$$\begin{aligned}
&\frac{\log((\frac{1}{r}))^m}{\sqrt{r}} \sup\{|h(x) - h(y)|, |x - y| \leq r\} \\
&\leq \frac{((n + 1 - \lfloor\sqrt{n+1}\rfloor) \log 9 + \lfloor\sqrt{n+1}\rfloor \log 12)^m}{3^{-n-1+\lfloor\sqrt{n+1}\rfloor} 12^{-\frac{1}{2}\lfloor\sqrt{n+1}\rfloor}} 2 \cdot 3^{-n+\lfloor\sqrt{n}\rfloor} 4^{-\lfloor\sqrt{n}\rfloor} \\
&\leq ((n+1) \log 12)^m 24 \left(\frac{3}{4}\right)^{\frac{1}{2}\lfloor\sqrt{n}\rfloor} \to 0
\end{aligned}$$

as $r \to 0$ (and then $n \to \infty$). □

Remark 2.5. *Using the same inequality as in Remark 2.3 we see that for any function satisfying the conditions of the previous proposition the set of levels with positive* $\frac{1}{2}$*-dimensional Hausdorff measure must have zero* 1*-dimensional Hausdorff measure: Let* $f : [0,1] \to [0,1]$ *be a function so that*

$$|f(x) - f(y)| \leq c \frac{\sqrt{|x-y|}}{-\log(|x-y|)}$$

for every $x \neq y$*. Denote* $\delta(t) = \frac{\sqrt{t}}{-\log(t)}$ *and move the irregularity from the function to the metric by considering the target side with a metric* $d(x,y) = \delta^{-1}(|x-y|)$*. Note that the metric is well defined with small distances since* δ *is convex on the interval* $[0, e^{-2\sqrt{2}}]$*. In the new metric* f *is Lipschitz and by the inequality (1) we see that there exists some* $C > 0$ *so that*

$$\mathcal{H}_d^{\frac{1}{2}}(\{y \in [0,1] : \mathcal{H}^{\frac{1}{2}}(f^{-1}(y)) > 0\}) < C.$$

But now take any $\epsilon > 0$ *and cover* $\{y \in [0,1] : \mathcal{H}^{\frac{1}{2}}(f^{-1}(y)) > 0\}$ *by sets* A_i *with diameter less than* $\epsilon e^{-\frac{1}{2\epsilon}}$*. Then*

$$\sum_i diam(A_i) \leq \epsilon \sum_i (\delta^{-1}(diam(A_i)))^{\frac{1}{2}}$$

and hence

$$\mathcal{H}^1(\{y \in [0,1] : \mathcal{H}^{\frac{1}{2}}(f^{-1}(y)) > 0\}) = 0.$$

References

[1] H. Federer, *Geometric measure theory*, Springer-Verlag, New York, (1969).

[2] B. Kirchheim, F. Serra Cassano, *Rectifiability and parameterization of intrinsic regular surfaces in the Heisenberg group*. Ann. Sc. Norm. Super. Pisa Cl. Sci. **3(4)** (2004), 871–896.

[3] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces: Fractals and rectifiability*, Cambridge University Press, Cambridge, (1995).