A NOTE ON THE LEVEL SETS OF HÖLDER CONTINUOUS FUNCTIONS ON THE REAL LINE

TAPIO RAJALA

Abstract. We construct an α-Hölder continuous functions on the real line so that all it's level sets have positive $(1-\alpha)$-dimensional Hausdorff measure.

1. Introduction

In [5] Kirchheim and Serra Cassano studied intrinsic regularity of surfaces in the Heisenberg group. They constructed an $\mathbb{H}$-regular surface in $\mathbb{H}^1$ that has Euclidean Hausdorff dimension $\frac{5}{2}$. Part of the construction was a $\frac{1}{2}$-Hölder continuous function $h\colon \mathbb{R} \to \mathbb{R}$ whose level sets $h^{-1}(t)$ have Euclidean Hausdorff dimension at least $\frac{1}{2}$. It remained open whether one could construct a $\frac{1}{2}$-Hölder continuous function for which the level sets have positive $\frac{1}{2}$-dimensional Hausdorff measure. They posed the question in the following slightly more general setting.

> Can one find α-Hölder continuous functions such that all level sets $f^{-1}(t)$ (or at least for all t from a set of positive measure) are of positive $(1-\alpha)$-dimensional Hausdorff measure?

We answer this question in the affirmative with a construction where in each step we make a selection between two fixed self-affine constructions. As will be shown, the construction can also be done with modulus of continuity slightly better than $\sqrt{|x-y|}$, so that it meets the requirements of [5, Lemma 3.5] of Kirchheim and Serra Cassano. However, for any such construction almost all of the level sets must have zero $\frac{1}{2}$-dimensional Hausdorff measure.

The method of alternating between two constructions that is used in this note was later reused in [7, Section 5] by the author together with Vilppolainen to sharpen the results of Balogh, Tyson and Warhurst [2]. It is also now known that on the level of dimension, the level set behaviour of the function in Proposition 2.4 is typical for α-Hölder continuous functions: Anttila, Bárány and Käenmäki have shown in [1, Theorem 1.6] that for a prevalent α-Hölder continuous function the Hausdorff dimension of Lebesgue positively many level sets is $1-\alpha$. Notice that the function in our Proposition 2.4 has the stronger property of having positive Hausdorff measure for all level sets.

Date: July 8, 2026.

2000 *Mathematics Subject Classification.* Primary 26A16, 28A78.

The author acknowledges the support of the Academy of Finland, project #211229 and the support of Vilho, Yrjö and Kalle Väisälä fund. (This note was first published on the author's webpage in 2008.) The author also thanks the anonymous referee for the several comments on the note.

The construction of Proposition 2.4 can be made with more zigzags than 3. By doing this, one can make a countable collection of affine constructions covering a dense set of parameters α, see [3, Section 2.4] for comparison.

2. The construction

Let us first fix some notation. For $x \in \mathbb{R}$ we write its integer part as $\lfloor x \rfloor = \sup\{k \in \mathbb{Z} : k \leq x\}$. The collection of all subsets of a set X will be written as $\mathcal{P}(X) = \{A \subset X\}$.

Definition 2.1. Let (X, d_X) and (Y, d_Y) be two metric spaces. For $\alpha \in]0,1]$ we call a function $f\colon X \to Y$ α-Hölder continuous if there exists a constant $C > 0$ so that for all $x, y \in X$ the inequality

$$d_Y(f(x), f(y)) \leq C d_X(x,y)^\alpha$$

holds.

We denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure. For the definition of Hausdorff measure and its basic properties, see for instance [6].

We will prove the positivity of Hausdorff measure of level sets by defining a measure with μ via mass distribution. The connection to Haudsorff measure follows from the next Lemma. For the proof see for example [6, Theorem 6.9].

Lemma 2.2. *Let μ be a Radon measure on $\mathbb{R}$, $0 < c < \infty$ and $A \subset \mathbb{R}$ measurable so that*

$$\limsup_{r \to 0} \frac{\mu(B(x,r))}{(2r)^s} \leq c$$

for all $x \in A$. Then $\mathcal{H}^s(A) \geq \frac{\mu(A)}{c2^s}$.

For the optimality of the results we will use the following coarea inequality [4, 2.10.25].

Lemma 2.3. *Let X, Y be two metric spaces and $f\colon X \to Y$ a Lipschitz map. Suppose $A \subset X$ is a measurable set, $0 \leq k < \infty$ and $0 \leq m < \infty$. Then*

$$\int_Y \mathcal{H}^k\left(A \cap f^{-1}(y)\right) d\mathcal{H}^m(y) \leq C(\mathrm{Lip} f)^m \mathcal{H}^{k+m}(A),$$

where C is a constant depending on k and m.

Now let us begin with the construction. We define for every $m \in \mathbb{N}$, $k \in \{0,1,2\}$ and $i \in \{0, 1, \ldots, m-1\}$ a mapping $f^m_{mk+i}\colon \mathbb{R}^2 \to \mathbb{R}^2$ with

$$f^m_{mk+i}((x,y)) = \left(\frac{x + mk + i}{3m}, (-1)^k \frac{y+i}{m} + \frac{1 + (-1)^{k+1}}{2}\right).$$

Using these, we define the set functions $F^m\colon \mathcal{P}(\mathbb{R}^2) \to \mathcal{P}(\mathbb{R}^2)$, $m \in \mathbb{N}$, by letting

$$F^m(A) = \bigcup_{i=0}^{3m-1} f^m_i(A)$$

for every $A \subset \mathbb{R}^2$. See Figure 1 to see how function F^4 maps the unit square.

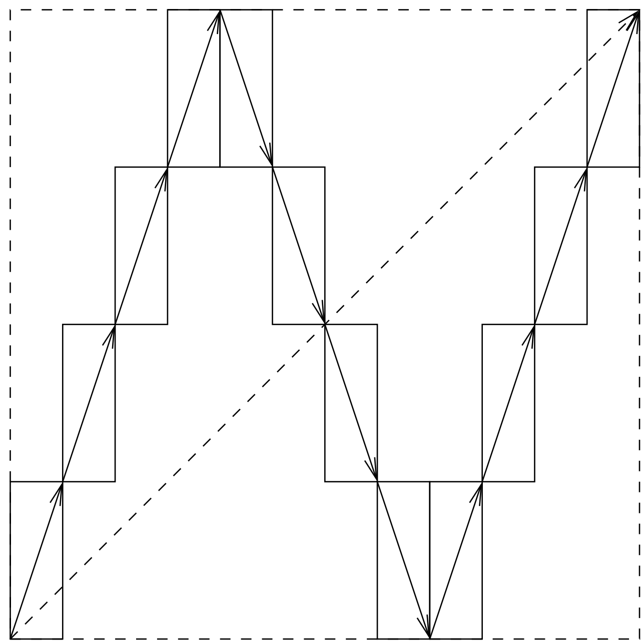

FIGURE 1. An illustration how the family of mappings $(f_i^4)_{i=0}^{11}$ maps $[0,1]^2$.

Recall that by an attractor of a family of contracting maps $f_i\colon \mathbb{R}^d \to \mathbb{R}^d$, $i \in I$, we mean the unique compact set $E \subset \mathbb{R}^d$ for which

$$\bigcup_{i\in I} f_i(E) = E.$$

The self-affine set we get from a function F^m, $m \geq 2$ as the attractor of the mapping family $(f_i^m)_{i=0}^{3m-1}$ is already a graph of a $\frac{\log m}{\log 3+\log m}$-Hölder continuous function on the interval $[0,1]$. In particular, the attractor of $(f_i^3)_{i=0}^{8}$ is a graph of a $\frac{1}{2}$-Hölder continuous function whose level sets have positive $\frac{1}{2}$-dimensional Hausdorff measure. To get the rest of the desired α-Hölder continuous functions with α between $\frac{\log m}{\log 3+\log m}$ and $\frac{\log(m+1)}{\log 3+\log(m+1)}$ we use both of the functions F^m and F^{m+1}.

Proposition 2.4. *Let $\alpha \in]0,1]$. Then there is an α-Hölder continuous function $h\colon \mathbb{R} \to \mathbb{R}$ so that all the level sets $h^{-1}(t)$ have positive $(1-\alpha)$-dimensional Hausdorff measure.*

Proof. In the case $\alpha = 1$ take $h(x) = x$. Let $\alpha \in]0,1[$. Find the largest $m \in \mathbb{N}$ so that

$$\alpha \geq \frac{\log m}{\log 3 + \log m}.$$

Next we define the portion ϱ which tells us how much of the construction should be done using the function F^{m+1}. The rest of the construction will use the function F^m. Let

$$\varrho = \frac{\alpha \log 3 - (1-\alpha)\log m}{(1-\alpha)(\log(m+1) - \log m)} \in [0,1[$$

so that the equality
$$3 = (3(m+1))^{\rho(1-\alpha)}(3m)^{(1-\rho)(1-\alpha)}$$
holds. Define a sequence of closed sets $(G_i)_{i=0}^{\infty}$ with
$$G_i = F^{j_1} \circ F^{j_2} \circ \cdots \circ F^{j_i}([0,1]^2),$$
where $j_k = m+1$ if $\lfloor \varrho k \rfloor > \lfloor \varrho(k-1) \rfloor$ and $j_k = m$ otherwise. Let $G = \bigcap_i G_i$. Because all of the G_i are closed, so is G. Clearly the projection of G to the x-axis is the whole interval $[0,1]$ and since the height of each section $G_i \cap \{x = p\}$ goes to zero when i goes to infinity, the set G is a graph of a function. Call this function g. and extend it to be a function $h\colon \mathbb{R} \to \mathbb{R}$ by defining
$$h(x+k) = ((k+1)^\alpha - k^\alpha)g(x) + k^\alpha \tag{2.1}$$
for every $k \in \mathbb{N} \cup \{0\}$ and $x \in [0,1]$, and by $h(x) = -h(-x)$ for $x < 0$. Note that while the mappings f_i^1 are not contracting, the construction still works when $m = 1$ since then $\varrho > 0$. Let $t \in [0,1]$ and consider the set $h^{-1}(t)$.

To see that $\mathcal{H}^{1-\alpha}(h^{-1}(t)) > 0$ we construct a measure μ on $[0,1]$ by distributing the measure at each step n evenly between the construction intervals of $G_n \cap \{(x,t) : x \in \mathbb{R}\}$. Then for any $z \in h^{-1}(t)$ and $0 < r < 1$ let $n \in \mathbb{N}$ be such that
$$(3m)^{-(1-\varrho)(n+1)}(3(m+1))^{-\varrho(n+1)} \le r < (3m)^{-(1-\varrho)n}(3(m+1))^{-\varrho n}.$$
For any construction interval $X_{\mathtt{i}} \subset G_n \cap \{(x,t) : x \in \mathbb{R}\}$ we have
$$r < 3^{-n}(m+1)^{-\varrho n} m^{-(1-\varrho)n} \le 3^{-n}(m+1)^{-\lfloor \varrho n \rfloor} m^{-(n - \lfloor \varrho n \rfloor)} = \operatorname{diam}(X_{\mathtt{i}})$$
Thus, $B(z,r)$ intersects at most two such intervals $X_{\mathtt{i}}$ and hence
$$\begin{aligned}\mu(B(z,r)) &\le 2\mu(X_{\mathtt{i}}) = 2 \cdot 3^{-n} = 2(3(m+1))^{-n\rho(1-\alpha)}(3m)^{-n(1-\rho)(1-\alpha)} \\ &= 2(3(m)^{(1-\rho)}(m+1)^{\rho})^{1-\alpha}\left((3m)^{-(1-\varrho)(n+1)}(3(m+1))^{-\varrho(n+1)}\right)^{1-\alpha} \\ &\le 2(3(m)^{(1-\rho)}(m+1)^{\rho})^{1-\alpha} r^{1-\alpha}.\end{aligned}$$
Now by Lemma 2.2 it follows that $\mathcal{H}^{1-\alpha}(h^{-1}(t)) > 0$.

Next we check that the function h is α-Hölder continuous. Let $x, y \in \mathbb{R}$. Assume first that $|x - y| \ge 1$ and let $k \in \mathbb{N}$ be such that $k \le |x-y| < k+1$. Then
$$|h(x) - h(y)| \le 2 \cdot (k+1)^\alpha \le 2\left(\frac{k+1}{k}\right)^\alpha |x-y|^\alpha.$$
It remains to show that h is locally α-Hölder continuous. Towards this, suppose that $|x-y| \in [0,1]$. Let $n \in \mathbb{N}$ be so that
$$(3m)^{-(1-\varrho)(n+1)}(3(m+1))^{-\varrho(n+1)} \le |x-y| < (3m)^{-(1-\varrho)n}(3(m+1))^{-\varrho n}.$$
Then, since x and y are contained in two consecutive level n construction rectangles,
$$\begin{aligned}|h(x) - h(y)| &= |g(x) - g(y)| \le 2 \cdot m^{-(1-\varrho)n-1}(m+1)^{-\varrho n+1} \\ &= 2\frac{m+1}{m}3^{-\frac{\alpha n}{1-\alpha}} = 2\frac{m+1}{m}((3m)^{-(1-\varrho)n}(3(m+1))^{-\varrho n})^\alpha \\ &\le 18(m+1)^2|x-y|^\alpha.\end{aligned}$$

If $x, y \in [k, k+1]$ for some $k \in \mathbb{N}$, then by (2.1) and the previous estimate we have

$$|h(x) - h(y)| \le |g(x-k) - g(y-k)| \le 18(m+1)^2|x-y|^\alpha.$$

By gluing the previous estimate at integer points, we have the α-Hölder continuity for all $x, y \in \mathbb{R}$ with $|x-y| < 1$. □

Remark 2.5. As was noted in [5], if $f\colon [0,1] \to [0,1]$ is α-Hölder continuous then it is Lipschitz continuous from $[0,1]$ to X, where X is $[0,1]$ with distance $d(x,y) = |x-y|^\alpha$. Then by Lemma 2.3

$$\int_{[0,1]} \mathcal{H}_d^{\frac{1}{\alpha}-1}(f^{-1}(y))d\mathcal{H}^1(y) \le C\mathcal{H}_d^{\frac{1}{\alpha}}([0,1]),$$

and thus almost all level sets of f have finite $(\frac{1}{\alpha}-1)$-dimensional Hausdorff measure with respect to the distance d meaning that they have finite $(1-\alpha)$-dimensional Hausdorff measure with respect to the Euclidean distance. This shows that our construction is in some sense optimal.

By changing the distribution of constructing functions F^m we can produce graphs of functions with other kinds of moduli of continuity. As an example we combine the functions F^m so that they give a function satisfying the lemma of Kirchheim and Serra Cassano [5, Lemma 3.5].

Proposition 2.6. *There is a function $h\colon \mathbb{R} \to \mathbb{R}$ such that*

(i) *for all $t \in \mathbb{R}$ the Euclidean Hausdorff dimension of $h^{-1}(t)$ is at least $\frac{1}{2}$.*
(ii) *for each $m \ge 1$ we have*

$$\lim_{r\to 0_+} \frac{\log((\frac{1}{r}))^m}{\sqrt{r}} \sup\{|h(x) - h(y)|, |x-y| \le r\} = 0.$$

Proof. We will mainly use the function F^3 to construct the graph, but on a small part of construction steps we use the function F^4 to improve the modulus of continuity. This time we define a sequence of closed sets $(G_i)_{i=0}^\infty$ with

$$F^{j_1} \circ F^{j_2} \circ \cdots \circ F^{j_i}([0,1]^2),$$

where $j_k = 4$ if $\sqrt{k} \in \mathbb{N}$ and $j_k = 3$ otherwise. Again, we set $G = \bigcap_i G_i$ and let h be the extension of the map g as in the proof of Proposition 2.4.

Let us first prove that the condition (i) is satisfied. Take a $t \in [0,1]$ and construct a measure μ on $[0,1]$ by using the intervals of $G_i \cap \{(x,t) : x \in \mathbb{R}\}$. At each step i distribute the measure evenly on the construction intervals. Now take an $\epsilon > 0$ and an $n_\epsilon \in \mathbb{N}$ so that $9^{-n\epsilon} \le (\frac{3}{4})^{\frac{1}{2}\lfloor\sqrt{n}\rfloor}$ for every $n \ge n_\epsilon$. Let $z \in h^{-1}(t)$ and $r > 0$ be such that

$$9^{-n-1+\lfloor\sqrt{n+1}\rfloor}12^{-\lfloor\sqrt{n+1}\rfloor} \le r < 9^{-n+\lfloor\sqrt{n}\rfloor}12^{-\lfloor\sqrt{n}\rfloor} = \mathrm{diam}(X_{\mathtt{i}}),$$

for some $n \ge n_\epsilon$, where $X_{\mathtt{i}} \subset G_n \cap \{(x,t) : x \in \mathbb{R}\}$, is any construction interval of level n. Then, the ball $B(z,r)$ intersects at most two construction intervals of level n and so we have

$$\begin{aligned}\mu(B(z,r)) &\le 2\mu(X_{\mathtt{i}}) \le 2\cdot 3^{-n} \le 2\cdot 3^{-n}9^{n\epsilon}\left(\frac{3}{4}\right)^{\frac{1}{2}\lfloor\sqrt{n}\rfloor} \le 2\cdot 3^{-n}9^{n\epsilon}\left(\frac{3}{4}\right)^{\left(\frac{1}{2}-\epsilon\right)\lfloor\sqrt{n}\rfloor} \\ &= 2(9^{-n+\lfloor\sqrt{n}\rfloor}12^{-\lfloor\sqrt{n}\rfloor})^{\frac{1}{2}-\epsilon} \le 24r^{\frac{1}{2}-\epsilon}.\end{aligned}$$

Again by Lemma 2.2 we get $\mathcal{H}^{\frac{1}{2}-\epsilon}(\{h^{-1}(t)\}) > 0$.

We still need to show that property (ii) is satisfied. Take $0 < r < 1$ and the largest $n \in \mathbb{N}$ so that $r \leq 9^{-n+\lfloor\sqrt{n}\rfloor}12^{-\lfloor\sqrt{n}\rfloor}$. Now for every $x, y \in \mathbb{R}$ with $|x-y| < r$ we have

$$|h(x) - h(y)| \leq 2 \cdot 3^{-n+\lfloor\sqrt{n}\rfloor}4^{-\lfloor\sqrt{n}\rfloor}.$$

Therefore, for any $m \in \mathbb{N}$ we get

$$\begin{aligned}
&\frac{\log((\frac{1}{r}))^m}{\sqrt{r}} \sup\{|h(x)-h(y)|, |x-y| \leq r\} \\
&\qquad \leq \frac{((n+1-\lfloor\sqrt{n+1}\rfloor)\log 9 + \lfloor\sqrt{n+1}\rfloor \log 12)^m}{3^{-n-1+\lfloor\sqrt{n+1}\rfloor}12^{-\frac{1}{2}\lfloor\sqrt{n+1}\rfloor}} 2 \cdot 3^{-n+\lfloor\sqrt{n}\rfloor}4^{-\lfloor\sqrt{n}\rfloor} \\
&\qquad \leq ((n+1)\log 12)^m 24 \left(\frac{3}{4}\right)^{\frac{1}{2}\lfloor\sqrt{n}\rfloor} \to 0
\end{aligned}$$

as $r \to 0$ (implying that $n \to \infty$). $\square$

Remark 2.7. Using the same inequality as in Remark 2.5 we see that any function satisfying the conditions of the previous proposition has $\mathcal{H}^1$-almost every level sets with zero $\frac{1}{2}$-dimensional Hausdorff measure: Let $f\colon [0,1] \to [0,1]$ be a function so that

$$|f(x) - f(y)| \leq c\frac{\sqrt{|x-y|}}{-\log(|x-y|)}$$

for every $x \neq y$. Consider on the domain the distance $d(x,y) = |x-y|^{\frac{1}{2}}$. In this distance f is Lipschitz and then similarly as in Remark 2.5 by Lemma 2.3 we get

$$\int_{f(I)} \mathcal{H}^1_d(f^{-1}(y))d\mathcal{H}^1(y) \leq C\mathrm{Lip}(f|_I)\mathcal{H}^2_d(I)$$

for any subinterval $I \subset [0,1]$. Notice that we have

$$\mathrm{Lip}(f|_I) \leq \frac{c}{-\log(\mathrm{diam}(I))}$$

with respect to the distance d. Thus,

$$\begin{aligned}
\int_{[0,1]} \mathcal{H}^1_d(f^{-1}(y))d\mathcal{H}^1(y) &= \sum_{i=0}^{k-1} \int_{f([i/k,(i+1)/k])} \mathcal{H}^1_d(f^{-1}(y))d\mathcal{H}^1(y) \\
&\leq \sum_{i=0}^{k-1} C\mathrm{Lip}(f|_{[i/k,(i+1)/k]})\mathcal{H}^2_d([i/k,(i+1)/k]) \\
&\leq C\frac{c}{-\log(1/k)}\mathcal{H}^2_d([0,1]) \to 0,
\end{aligned}$$

as $k \to \infty$. This proves the claim.

References

[1] R. Anttila, B. Bárány and A. Käenmäki, *Level sets of prevalent Hölder functions*, Proc. Amer. Math. Soc. **153** (2025), no. 5, 2023–2035.

[2] Z. Balogh, J. Tyson and B. Warhurst, *Sub-Riemannian vs. Euclidean dimension comparison and fractal geometry on Carnot groups*, Adv. Math. **220** (2009), no. 2, 560–619.

[3] Z. Buczolich and B. Maga, *Exact formula on upper box dimension of generic Hölder level sets*, preprint (2026), arXiv:2602.09749.

[4] H. Federer, *Geometric measure theory*, Springer-Verlag, New York, (1969).

[5] B. Kirchheim, F. Serra Cassano, *Rectifiability and parameterization of intrinsic regular surfaces in the Heisenberg group*. Ann. Sc. Norm. Super. Pisa Cl. Sci. **3(4)** (2004), 871–896.

[6] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces: Fractals and rectifiability*, Cambridge University Press, Cambridge, (1995).

[7] T. Rajala and M. Vilppolainen, *Weakly controlled Moran constructions and iterated functions systems in metric spaces*, Illinois J. Math. **55** (2011), no. 3, 1015–1051.

University of Jyväskylä, Department of Mathematics and Statistics, P.O. Box 35, FI-40014 University of Jyväskylä, Finland

Email address: `tapio.m.rajala@jyu.fi`