Scattering theory

Solutions to Exercises #7, 16.11.2007

1. See proof of Lemma 2.2.4 in the lectures.

2. Let $u = d\sigma$ be the surface measure of S^2. Since u is a compactly supported distribution, the Fourier transform is defined pointwise for all $\xi \in \mathbf{R}^3$ and

$$\hat{u}(\xi) = \langle u, e^{-ix\cdot\xi} \rangle = \int_{S^2} e^{-i|\xi|\omega\cdot\hat{\xi}}\, d\omega.$$

Here $\hat{\xi} = \xi/|\xi|$. We choose a positive orthonormal basis $\{\eta_1, \eta_2, \hat{\xi}\}$ of $\mathbf{R}^3$, and introduce spherical coordinates

$$\omega \cdot \eta_1 = \sin\phi\, \cos\theta,$$
$$\omega \cdot \eta_2 = \sin\phi\, \sin\theta,$$
$$\omega \cdot \hat{\xi} = \cos\phi,$$

where $0 \leq \phi \leq 1$ and $0 \leq \theta < 2\pi$. Then

$$\hat{u}(\xi) = \int_0^{2\pi} \int_0^{\pi} e^{-i|\xi|\cos\phi} \sin\phi\, d\phi\, d\theta = 2\pi \int_0^{\pi} e^{-i|\xi|\cos\phi} \sin\phi\, d\phi$$
$$= 2\pi \int_{-1}^{1} e^{-i|\xi|t}\, dt = \frac{2\pi}{i|\xi|}(e^{i|\xi|} - e^{-i|\xi|}) = 4\pi \frac{\sin|\xi|}{|\xi|}.$$

3. Let $u = d\sigma$ be the surface measure of S^{n-1}. We have

$$\hat{u}(\xi) = \int_{S^{n-1}} e^{-i\omega\cdot\xi}\, d\omega = \int_{S^{n-1}} e^{-i|\xi|\omega\cdot\hat{\xi}}\, d\omega.$$

Let $\{\hat{\xi}, \eta_2, \ldots, \eta_n\}$ be a positive orthonormal basis of $\mathbf{R}^n$. We let $S^{n-1}_+ = \{\omega \in S^{n-1}\,;\, \omega\cdot\eta_n > 0\}$. If $\omega = x_1\hat{\xi} + x_2\eta_2 + \ldots + x_n\eta_n$ is identified with $x = (x_1, \ldots, x_n) \in S^{n-1}$, then

$$S^{n-1}_+ = \{(x', h(x'))\,;\, |x'| < 1\}$$

where $h(x') = \sqrt{1 - |x'|^2}$. Then we have $dS(x') = \sqrt{1 + |\nabla h(x')|^2}\, dx' = (1 - |x'|^2)^{-1/2}\, dx'$, and

$$\hat{u}(\xi) = 2\int_{S^{n-1}_+} e^{-i|\xi|x_1}\, dS(x) = 2\int_{|x'|<1} e^{-i|\xi|x_1} (1 - |x'|^2)^{-1/2}\, dx'$$
$$= 2\int_{-1}^{1} e^{-i|\xi|x_1} \int_{|x''|<\sqrt{1-x_1^2}} (1 - x_1^2 - |x''|^2)^{-1/2}\, dx''\, dx_1.$$

Here $x'' = (x_2, \dots, x_{n-1})$. One has

$$\int_{|x''|<\sqrt{1-x_1^2}} (1-x_1^2-|x''|^2)^{-1/2}\,dx'' = (1-x_1^2)^{\frac{n-3}{2}} \int_{|x''|<1} (1-|x''|^2)^{-1/2}\,dx'' = (1-x_1^2)^{\frac{n-3}{2}} \frac{1}{2}\sigma(S^{n-2})$$

by the definition of surface measure on S^{n-2}. We use that

$$\sigma(S^{n-2}) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}.$$

We have

$$\hat{u}(\xi) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_{-1}^{1} e^{-i|\xi|t}(1-t^2)^{\frac{n-3}{2}}\,dt = \frac{4\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_0^1 \cos(t|\xi|)(1-t^2)^{\frac{n-3}{2}}\,dt$$

since $(1-t^2)^{\frac{n-3}{2}}$ is even in t. From Abramowitz and Stegun, Handbook of Mathematical Functions, formula 9.1.20,

$$J_s(r) = \frac{2(\frac{1}{2}r)^s}{\pi^{\frac{1}{2}}\Gamma(s+\frac{1}{2})} \int_0^1 (1-t^2)^{s-\frac{1}{2}} \cos(rt)\,dt, \quad s > -1/2.$$

Therefore

$$\hat{u}(\xi) = (2\pi)^{n/2}|\xi|^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(|\xi|).$$

4. The claim is easy to prove if M is a hyperplane. In the case where M is a general hypersurface, we want to reduce to the hyperplane case by "flattening" M. This can be done as follows. Let $x_0 \in M$, and choose Cartesian coordinates so that $x_0 = 0$ and M is given near 0 as the graph of a C^2 function h, with $\nabla h(0) = 0$. Then for some $\delta > 0$ one has

$$M \cap B(0,\delta) = \{(x', h(x'))\,;\, |x'| < \delta\}.$$

We identify x' with $(x', h(x'))$. The normal is

$$\nu(x') = (1 + |\nabla h(x')|^2)^{-1/2}(-\nabla h(x'), 1).$$

Consider the map

$$F(y', y_n) = (y', h(y')) + y_n \nu(y').$$

Near 0, F is C^1 since h is C^2. If $\nu(y') = (\nu'(y'), \nu_n(y'))$, the Jacobian matrix $DF = (\partial_k F_j)_{j,k=1}^n$ is given by

$$DF(y', y_n) = \begin{pmatrix} I_{n-1} + y_n D\nu'(y') & \nu'(y') \\ \nabla h(y') + y_n \nabla \nu_n(y') & \nu_n(y') \end{pmatrix}.$$

One has $F(0) = 0$ and $DF(0) = I$, so the inverse function theorem shows that $F : U \to V$ is a diffeomorphism from some ball U centered at 0 onto some neighborhood V of 0.

If $\chi \in C_0^\infty(V)$ then changing coordinates $x = F(y)$ gives

$$\frac{1}{2\varepsilon} \int_{M_\varepsilon} (\chi \tilde{f})(x)\, dx = \frac{1}{2\varepsilon} \int_{F^{-1}(V \cap M_\varepsilon)} (\chi \tilde{f})(F(y)) |\det\, DF(y)|\, dy.$$

Since $F^{-1}(V \cap M_\varepsilon) = U \cap \{|y_n| < \varepsilon\}$ for ε small [1] , and since $\mathrm{supp}(\chi \circ F)$ is contained in the open ball U, the last integral may be written as

$$\frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} g(y_n)\, dy_n$$

where $g(y_n) = \int_{U_0} (\chi \tilde{f})(F(y', y_n)) |\det\, DF(y', y_n)|\, dy'$, $U_0 = U \cap \{y_n = 0\}$. Then since g is continuous,

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} g(y_n)\, dy_n = g(0) = \int_{U_0} (\chi f)(y') |\det\, DF(y', 0)|\, dy'.$$

The determinant is $(1 + |\nabla h(y')|^2)^{1/2}$, which shows that

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{M_\varepsilon} (\chi \tilde{f})(x)\, dx = \int_M \chi f\, dS.$$

The claim follows by choosing a suitable partition of unity and applying the preceding argument in each coordinate patch.

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[1] If $y \in U \cap \{|y_n| < \varepsilon\}$ then $(y', h(y')) + y_n \nu(y') \in M_\varepsilon$, so $y \in F^{-1}(V \cap M_\varepsilon)$. Conversely, let $z \in U$ and $F(z) \in M_\varepsilon$. Let $y_0 = (y', h(y'))$ be a point on $M \cap \overline{B(0, \varepsilon)}$ closest to $F(z)$. If γ is any C^2 curve on M with $\gamma(0) = y_0$, then $r(t) = |\gamma(t) - F(z)|^2$ has a local minimum at $t = 0$, hence $r'(0) = 2(y_0 - F(z)) \cdot \dot{\gamma}(0) = 0$. This implies that $y_0 - F(z)$ is orthogonal to any tangent vector of M at y_0, so $F(z) = y_0 + y_n \nu(y_0) = F(y', y_n)$ for some y_n. Then

$$|y_n| = |F(z) - y_0| = \mathrm{dist}(F(z), M \cap \overline{B(0, \varepsilon)}) < \varepsilon.$$

It follows that $z = y \in U \cap \{|y_n| < \varepsilon\}$.