Scattering theory

Solutions to Exercises #6, 09.11.2007

1. Let $T \in L(B_1, B_2)$ be Fredholm. Assume first that T is injective, and let $n = \dim(B_2/\mathrm{im}(T))$. Then there is $S : \mathbf{C}^n \to B_2$ such that the map

$$T_1 : B_1 \times \mathbf{C}^n \to B_2, \ (x, y) \mapsto Tx + Sy,$$

is linear, bounded, and bijective. [1] By the open mapping theorem T_1 is a homeomorphism, which shows that the range $\mathrm{im}(T) = T_1(B_1 \times \{0\})$ is closed. If T was not injective we can consider $T' : B_1/\ker(T) \to B_2, [x] \mapsto Tx$. Then T' is Fredholm with range $\mathrm{im}(T)$ and injective, so $\mathrm{im}(T)$ is closed also in this case.

Let now $T_1 \in L(B_1, B_2)$ and $T_2 \in L(B_2, B_3)$ be Fredholm. Now $T_1 : \ker(T_2T_1) \to \ker(T_2)$ with kernel $\ker(T_1)$, so there is an isomorphism of $\ker(T_2T_1)/\ker(T_1)$ and a subspace of $\ker(T_2)$. Consequently $\dim \ker(T_2T_1) \leq \dim \ker(T_1) + \dim \ker(T_2) < \infty$. Also, since $\mathrm{im}(T_2T_1) \subset \mathrm{im}(T_2) \subset B_3$, we have

$$(B_3/\mathrm{im}(T_2T_1))/(\mathrm{im}(T_2)/\mathrm{im}(T_2T_1)) = B_3/\mathrm{im}(T_2),$$

so $\dim \mathrm{coker}(T_2T_1) \leq \dim \mathrm{im}(T_2)/\mathrm{im}(T_2T_1) + \dim \mathrm{coker}(T_2)$. But $T_2 : B_2/\mathrm{im}(T_1) \to \mathrm{im}(T_2)/\mathrm{im}(T_2T_1)$ is surjective, so $\dim \mathrm{im}(T_2)/\mathrm{im}(T_2T_1) \leq \dim \mathrm{coker}(T_1)$. This shows that T_2T_1 is Fredholm.

2. Assume first that T is bijective. Then T has a bounded inverse by the open mapping theorem, and

$$T + S = T(I + T^{-1}S).$$

If $\|S\|$ is small enough then $\|T^{-1}S\| < 1/2$, so $I + T^{-1}S$ is invertible by Neumann series. Then also $T + S$ is bijective, so $\dim \ker(T + S) \leq \dim \ker(T) = 0$, $\dim \mathrm{coker}(T+S) \leq \dim \mathrm{coker}(T) = 0$, and $\mathrm{ind}(T+S) = \mathrm{ind}(T) = 0$.

If $T : B_1 \to B_2$ is Fredholm but not bijective, there exists a closed subspace V_1 of B_1 and a finite dimensional subspace V_2 of B_2 such that $B_1 =$

[1] In fact, let $S_0 : \mathbf{C}^n \to B_2/\mathrm{im}(T)$ be an isomorphism and let $q : B_2 \to B_2/\mathrm{im}(T)$ be the quotient map. If $\{e_1, \ldots, e_n\}$ is a basis of $\mathbf{C}^n$ we define Se_j to be some element of $q^{-1}(S_0e_j)$, and define S on $\mathbf{C}^n$ by linearity. Then T_1 is linear and bounded. If $T_1(x, y) = 0$ then $0 = qT_1(x, y) = qSy = S_0y$ so $y = 0$ and then also $x = 0$. For surjectivity let $z \in B_2$ and consider the equation $Tx + Sy = z$. Applying q gives $S_0y = qz$, and the choice $y = S_0^{-1}(qz)$ gives $q(z - Sy) = 0$ so there is $x \in B_1$ with $z - Sy = Tx$. Thus T_1 is bijective.

$V_1 \oplus \ker(T)$ and $B_2 = V_2 \oplus \mathrm{im}(T)$. [2] Let $q_2 : B_2 \to B_2/V_2$ be the quotient map, and define $T' = q_2 T|_{V_1}$ and $S' = q_2 S|_{V_1}$. Then $T', S' : V_1 \to B_2/V_2$ and T' is bijective and $\|S'\| \le \|S\|$. If $\|S\|$ is small enough then $T' + S'$ is bijective. We prove the statements for $T + S$ in four steps.

Step 1: $\dim \ker(T + S) \le \dim \ker(T)$.

If $x \in \ker(T + S)$ then $x = v_1 + w$ where $v_1 \in V_1$ and $w \in \ker(T)$. Thus $(T + S)v_1 = -Sw$, so $(T' + S')v_1 = -q_2 Sw$ and consequently

$$x = (I - (T' + S')^{-1} q_2 S)w. \tag{1}$$

If $\{w_1, \ldots, w_m\}$ is a basis of $\ker(T)$, the corresponding vectors $\{x_1, \ldots, x_m\}$ span $\ker(T + S)$.

Step 2. $\dim \operatorname{coker}(T + S) \le \dim \operatorname{coker}(T)$.

Since $T' + S'$ is bijective, any $y \in B_2$ has the form $y = (T + S)v_1 + v_2$ for some $v_1 \in V_1$ and $v_2 \in V_2$. Thus in $B_2/\mathrm{im}(T+S)$, $[y] = [v_2]$. Consequently

$$B_2/\mathrm{im}(T + S) = \{[v_2]\, ;\, v_2 \in V_2\}, \tag{2}$$

so $\dim \operatorname{coker}(T + S) \le \dim V_2 = \dim \operatorname{coker}(T)$.

Step 3. $\ker(T) \cong (T + S)^{-1}(V_2)$.

The computation leading to (1) shows that any $x \in (T+S)^{-1}(V_2)$ is of the form (1) for some $w \in \ker(T)$. Conversely, if $x = (I - (T' + S')^{-1} q_2 S)w$ for some $w \in \ker(T)$, then $q_2(T + S)x = 0$ so $x \in (T + S)^{-1}(V_2)$. We see that $I - (T' + S')^{-1} q_2 S$ gives for $\|S\|$ small an isomorphism $\ker(T) \cong (T + S)^{-1}(V_2)$.

Step 4. $\mathrm{ind}(T + S) = \mathrm{ind}(T)$.

Consider the map M which is a restriction of $T + S$ between finite dimensional spaces,

$$M : (T + S)^{-1}(V_2) \to V_2,\ x \mapsto (T + S)x.$$

By the rank-nullity theorem for matrices,

$$\dim \ker(M) + \dim \mathrm{im}(M) = \dim (T + S)^{-1}(V_2).$$

[2] This is clear in a Hilbert space, since any closed subspace has an orthogonal complement. In a Banach space it is easy to show that any closed subspace with finite dimension or codimension has a complement, see Rudin, Functional Analysis, Lemma 4.21.

By Step 3 we have $\dim (T+S)^{-1}(V_2) = \dim \ker(T)$, and clearly $\ker(M) = \ker(T+S)$. Finally, we have the isomorphism

$$V_2 \cong \mathrm{im}(M) \oplus \mathrm{coker}(T+S), \; v_2 \mapsto (P_{\mathrm{im}(M)} v_2, [v_2]).$$

Here we have used (2). The result follows.

3. One has
$$\frac{1}{x-i\varepsilon} - \frac{1}{x+i\varepsilon} = \frac{2i\varepsilon}{x^2+\varepsilon^2} = 2\pi i j_\varepsilon(x)$$
where $j(x) = \frac{1}{\pi}\frac{1}{1+x^2}$ and $j_\varepsilon(x) = \varepsilon^{-1} j(x/\varepsilon)$. Since $\int j(x)\,dx = 2\pi^{-1}\int_0^\infty (1+x^2)^{-1}\,dx$, the substitution $x = \tan\theta$ gives $\int j(x)\,dx = 2\pi^{-1}\int_0^{\pi/2} d\theta = 1$. Therefore, for $\varphi \in C_c(\mathbf{R})$

$$\Big\langle \frac{1}{x-i\varepsilon} - \frac{1}{x+i\varepsilon}, \varphi \Big\rangle = 2\pi i \int j_\varepsilon(x)\varphi(x)\,dx = 2\pi i \int j(x)\varphi(\varepsilon x)\,dx.$$

The last expression has the limit $2\pi i\varphi(0)$ as $\varepsilon \to 0$ by dominated convergence.

Let $\varphi \in C_c(\mathbf{R})$ and define

$$\varphi_\varepsilon(t) = \frac{1}{2\pi i}\int \Big(\frac{1}{t-\lambda-i\varepsilon} - \frac{1}{t-\lambda+i\varepsilon}\Big)\varphi(\lambda)\,d\lambda = (j_\varepsilon * \varphi)(t).$$

Since φ is bounded and uniformly continuous, one has $\varphi_\varepsilon \to \varphi$ in L^∞ as $\varepsilon \to 0$ [3] . Let A be self-adjoint and let $d\mu_v$ be the spectral measure for $v \in H$. Then

$$|([\varphi_\varepsilon(A) - \varphi(A)]v, v)| = \Big|\int [\varphi_\varepsilon(t) - \varphi(t)]\,d\mu_v(t)\Big| \le \|\varphi_\varepsilon - \varphi\|_{L^\infty}\|v\|^2$$

since $\mu_v(\mathbf{R}) = \|v\|^2$. It follows that $\varphi_\varepsilon(A) \to \varphi(A)$ in the operator norm as $\varepsilon \to 0$. This ends the proof because

$$\varphi_\varepsilon(A) = \frac{1}{2\pi i}\int (R(\lambda+i\varepsilon) - R(\lambda-i\varepsilon))\varphi(\lambda)\,d\lambda.$$

[3] Let $\delta > 0$, and choose $R > 0$ so that $\int_{|y|\ge R} j(y)\,dy \le (4\|\varphi\|_{L^\infty})^{-1}\delta$. Then choose $\varepsilon_0 > 0$ so that $|\varphi(x-\varepsilon_0 y) - \varphi(x)| \le \delta/2$ if $|y| \le R$ and $x \in \mathbf{R}$. Then

$$|\varphi_\varepsilon(x) - \varphi(x)| = \Big|\int j(y)[\varphi(x-\varepsilon y) - \varphi(x)]\,dy\Big| \le 2\|\varphi\|_{L^\infty}\int_{|y|\ge R} j(y)\,dy + \sup_{|y|\le R}|\varphi(x-\varepsilon y) - \varphi(x)|.$$

This is $\le \delta$ if $\varepsilon < \varepsilon_0$.

4. The function V is real valued, so the operator $u \mapsto Vu$ with domain $H^2(\mathbf{R}^3)$ is symmetric. Since $-\Delta$ with domain $H^2(\mathbf{R}^3)$ is self-adjoint, by Kato's theorem $-\Delta + V$ will be self-adjoint if for some $a < 1$ one has

$$\|Vu\|_{L^2} \le a\|-\Delta u\|_{L^2} + b\|u\|_{L^2}, \quad u \in H^2(\mathbf{R}^3).$$

We have $\|V_1 u\|_{L^2} \le \|V_1\|_{L^\infty}\|u\|_{L^2}$, so it is enough to consider V_2. Now if $f \in L^2$ then for $t > 0$

$$\begin{aligned} \|(-\Delta + it)^{-1} f\|_{L^\infty} &\le (2\pi)^{-n}\|((-\Delta + it)^{-1} f)\hat{}\|_{L^1} \\ &\le (2\pi)^{-n} \int_{\mathbf{R}^3} |(|\xi|^2 + it)^{-1}\hat{f}(\xi)|\, d\xi \\ &\le (2\pi)^{-n} \int_{\mathbf{R}^3} (|\xi|^4 + t^2)^{-1/2}|\hat{f}(\xi)|\, d\xi \\ &\le (2\pi)^{-n}\Big(\int_{\mathbf{R}^3} \frac{1}{t^2 + |\xi|^4}\, d\xi\Big)^{1/2}\|\hat{f}\|_{L^2}. \end{aligned}$$

Here $\int_{\mathbf{R}^3} \frac{1}{t^2+|\xi|^4}\, d\xi = t^{-1/2}\int_{\mathbf{R}^3} \frac{1}{1+|\xi|^4}\, d\xi \le C_0 t^{-1/2}$ for some absolute constant C_0. It follows that for any $\varepsilon > 0$ there is $t > 0$ such that

$$\|(-\Delta + it)^{-1} f\|_{L^\infty} \le \varepsilon\|f\|_{L^2}, \quad f \in L^2.$$

Then for $u \in H^2(\mathbf{R}^3)$, the choice $f = (-\Delta + it)u$ gives

$$\begin{aligned} \|V_2 u\|_{L^2} &\le \|u\|_{L^\infty}\|V_2\|_{L^2} \le \varepsilon\|V_2\|_{L^2}\|(-\Delta + it)u\|_{L^2} \\ &\le \varepsilon\|V_2\|_{L^2}\|-\Delta u\|_{L^2} + \varepsilon t\|V_2\|_{L^2}\|u\|_{L^2}. \end{aligned}$$

Choosing ε small enough gives the desired norm estimate with $a < 1$ (in fact one may take a arbitrarily close to 0).