Scattering theory

Solutions to Exercises #5, 19.10.2007

1. Let $\lambda \in \sigma_{\text{ess}}(A)$ and let K be compact and self-adjoint. Assuming Weyl's criterion, there is a sequence $(u_j) \subset \mathscr{D}(A)$, $\|u_j\| = 1$, with $u_j \to 0$ weakly and $\|(A - \lambda)u_j\| \to 0$. Any weakly convergent sequence is bounded, and therefore (Ku_j) converges to some $\tilde{u} \in H$ after taking a subsequence if necessary. But $(Ku_j, v) = (u_j, Kv) \to 0$ for any $v \in H$, so $\tilde{u} = 0$ and so $\|(A + K - \lambda)u_j\| \to 0$. The new sequence (u_j) is a Weyl sequence for $A + K$, so $\lambda \in \sigma_{\text{ess}}(A + K)$. The other direction, $\sigma_{\text{ess}}(A + K) \subset \sigma_{\text{ess}}(A)$, follows by writing $A = (A + K) - K$ and using the first part.

2. Let $\lambda \in \sigma_{\text{ess}}(A)$ and $\dim \ker(A - \lambda) = \infty$. Choose an orthonormal set $\{u_j\}_{j=1}^{\infty}$ in $\ker(A - \lambda)$. Then $(u_j) \subset \mathscr{D}(A)$, $\|u_j\| = 1$, and $(A - \lambda)u_j = 0$ for all j. One has

$$\sum_{j=1}^{\infty} |(u_j, v)|^2 \leq \|v\|^2, \quad v \in H,$$

and therefore $(u_j, v) \to 0$ for all $v \in H$. This shows that (u_j) is a Weyl sequence.

3. Let $\lambda \in \sigma_{\text{ess}}(A)$ and $\dim \ker(A - \lambda) < \infty$. Consider the map

$$A_\lambda := A - \lambda : \mathscr{D}(A_\lambda) \to \ker(A - \lambda)^\perp,$$

where $\mathscr{D}(A_\lambda) = \mathscr{D}(A) \cap \ker(A - \lambda)^\perp$. By Ex. 4, Problem 4, A_λ is self-adjoint and $0 \in \sigma(A_\lambda)$. Since $\ker(A_\lambda) = \{0\}$ and A_λ has no residual spectrum, the range $\mathscr{R}(A_\lambda)$ is dense.

We have that $A_\lambda^{-1} : \mathscr{R}(A_\lambda) \to \mathscr{D}(A_\lambda)$ is unbounded. If $0 < \dim \ker(A - \lambda) < \infty$ this follows from Ex. 4, Problem 4. If $\ker(A - \lambda) = \{0\}$ this is true since otherwise one would have $\|u\| = \|A_\lambda^{-1} A_\lambda u\| \leq C\|(A - \lambda)u\|$ for $u \in \mathscr{D}(A)$, which is a contradiction by Ex. 2, Problem 1.

Since A_λ^{-1} is unbounded, there is a sequence $(v_j) \subset \mathscr{R}(A_\lambda)$ with $\|v_j\| = 1$ and $\|A_\lambda^{-1} v_j\| \to \infty$. Define

$$u_j = \frac{A_\lambda^{-1} v_j}{\|A_\lambda^{-1} v_j\|}.$$

Then $u_j \in \mathscr{D}(A) \cap \ker(A - \lambda)^\perp$, $\|u_j\| = 1$, and $\|(A - \lambda)u_j\| \to \infty$.

It remains to show that $(u_j, v) \to 0$ for all $v \in H$. In fact, it is enough to show this for v in a dense set. If $v \in \mathscr{D}((A_\lambda^{-1})^*)$ then

$$(u_j, v) = \frac{1}{\|A_\lambda^{-1} v_j\|}(A_\lambda^{-1} v_j, v) = \frac{1}{\|A_\lambda^{-1} v_j\|}(v_j, (A_\lambda^{-1})^* v) \to 0.$$

Also, $\mathscr{D}((A_\lambda^{-1})^*)$ is dense in H since $\mathscr{R}(A_\lambda) \subset \mathscr{D}((A_\lambda^{-1})^*)$, which follows because $(A_\lambda^{-1} u, A_\lambda f) = (u, f)$ for $u \in \mathscr{D}(A_\lambda^{-1})$. The proof is finished.

4. Let A be self-adjoint in H and let $(u_j) \subset \mathscr{D}(A)$, $\|u_j\| = 1$, with $u_j \to 0$ weakly and $\|(A - \lambda)u_j\| \to 0$. Then λ is an approximate eigenvalue, so $\lambda \in \sigma(A)$ by Ex. 3, Problem 2. If dim $\ker(A - \lambda) = \infty$ then $\lambda \in \sigma_{\text{ess}}(A)$ by definition.

   Assume that $\ker(A - \lambda)$ is finite dimensional, so it has an orthonormal basis $\phi_1, \ldots, \phi_N$. Let $P : u \mapsto \sum_{m=1}^N (u, \phi_m)\phi_m$ be the orthogonal projection onto $\ker(A - \lambda)$, and let $P_\perp = I - P$ be the orthogonal projection onto $\ker(A - \lambda)^\perp$. Since $u_j \to 0$ weakly,

$$\|Pu_j\|^2 = \sum_{m=1}^N |(u_j, \phi_m)|^2 \to 0 \quad \text{as } j \to \infty,$$

   and so $\|P_\perp u_j\| \to 1$. Then for j sufficiently large we may define

$$v_j = \frac{1}{\|P_\perp u_j\|} P_\perp u_j.$$

   Since $u_j \in \mathscr{D}(A)$, one has $P_\perp u_j \in \mathscr{D}(A)$ and so $v_j \in \mathscr{D}(A) \cap \ker(A - \lambda)^\perp$. Also, $\|v_j\| = 1$, and

$$\|(A - \lambda)v_j\| = \frac{1}{\|P_\perp u_j\|}\|(A - \lambda)u_j\| \to 0.$$

   This shows that A_λ^{-1} is unbounded, since otherwise one would have $1 = \|v_j\| = \|A_\lambda^{-1} A_\lambda v_j\| \leq C\|(A - \lambda)v_j\| \to 0$, contradiction.

   Since we know that $\lambda \in \sigma(A)$, dim $\ker(A - \lambda) < \infty$ and A_λ^{-1} is unbounded, by Ex. 4, Problem 4 it must be true that $\lambda \in \sigma_{\text{ess}}(A)$.