Scattering theory

Solutions to Exercises #3, 5.10.2007

1. Let $\lambda \in \mathbf{R}$ (we already know that $\mathbf{C} \setminus \mathbf{R} \subset \rho(A)$). The inequality

$$\|(A-\lambda)u\| \geq M\|u\|, \quad u \in \mathscr{D}(A),$$

shows that $A-\lambda$ is injective so λ is not an eigenvalue. Since A is self-adjoint it has no residual spectrum, and we know that $\mathscr{R}(A-\lambda)$ is dense. We show that $\mathscr{R}(A-\lambda) = H$. If $f \in H$ let $f_j \in \mathscr{R}(A-\lambda)$ with $f_j \to f$, so $f_j = (A-\lambda)u_j$ for some $u_j \in \mathscr{D}(A)$ and $(A-\lambda)u_j \to f$. Then (u_j) is Cauchy in H because

$$\|u_j - u_k\| \leq \frac{1}{M}\|(A-\lambda)(u_j-u_k)\| = \frac{1}{M}\|(A-\lambda)u_j - (A-\lambda)u_k\|.$$

It follows that $u_j \to u$ for some $u \in H$, but since also $(A-\lambda)u_j \to f$ and $A-\lambda$ is closed (it is self-adjoint), we have $u \in \mathscr{D}(A)$ and $(A-\lambda)u = f$. Further, $\|u\| \leq \frac{1}{M}\|(A-\lambda)u\| = \frac{1}{M}\|f\|$.

We have shown that $A-\lambda : \mathscr{D}(A) \to H$ is bijective and that $(A-\lambda)^{-1} : f \mapsto u$ is a bounded linear operator on H with norm $\leq 1/M$. Therefore, $\lambda \in \rho(A)$. If $z \in \mathbf{C}$ and $|z-\lambda| < M$, then for $u \in \mathscr{D}(A)$ one has

$$\begin{aligned}\|(A-z)u\| &= \|(A-\lambda)u + (\lambda-z)u\| \geq \|(A-\lambda)u\| - |\lambda-z|\|u\| \\ &\geq (M-|\lambda-z|)\|u\|.\end{aligned}$$

By the first part of this problem, we obtain $z \in \rho(A)$.

2. It is enough to show that

$$\lambda \in \rho(A) \Leftrightarrow \exists\, \varepsilon > 0\ \forall\ u \in \mathscr{D}(A), \|u\| = 1 : \|(A-\lambda)u\| \geq \varepsilon.$$

If $\lambda \in \rho(A)$ and $u \in \mathscr{D}(A), \|u\| = 1$, then

$$1 = \|u\| = \|(A-\lambda)^{-1}(A-\lambda)u\| \leq C\|(A-\lambda)u\|.$$

Conversely, if $\|(A-\lambda)u\| \geq \varepsilon$ whenever $u \in \mathscr{D}(A)$ and $\|u\| = 1$, then $\|(A-\lambda)v\| \geq \varepsilon\|v\|$ for any $v \in \mathscr{D}(A)$ (take $u = v/\|v\|$), so $\lambda \in \rho(A)$ by Problem 1.

3. Let $H = L^2(\mathbf{R})$, $Au = -u''$ with domain $\mathscr{D}(A) = H^2(\mathbf{R})$. Then using distributional derivatives, and the fact that C_0^∞ is dense in H^2, we get

$$\begin{aligned} v \in \mathscr{D}(A^*) &\Leftrightarrow \exists\, v^* \in L^2 : (-u'', v) = (u, v^*) \quad \forall\, u \in H^2 \\ &\Leftrightarrow v \in L^2 \text{ and } v'' \in L^2. \end{aligned}$$

Then $\mathscr{D}(A^*) = H^2(\mathbf{R}) = \mathscr{D}(A)^1$ and $A^*u = -u''$, so A is self-adjoint and $\sigma(A) \subset \mathbf{R}$. We also have

$$\|(A-\lambda)u\|^2 = \|Au\|^2 + \lambda^2\|u\|^2 - \lambda(Au, u) - \lambda(u, Au), \quad u \in H^2.$$

Since $(Au, u) = (-u'', u) = \|u'\|^2 \geq 0$ for $u \in C_0^\infty$ and thus for $u \in H^2$, it follows for $\lambda < 0$ that

$$\|(A-\lambda)u\| \geq |\lambda|\|u\|, \quad u \in H^2.$$

By Problem 1, $\lambda \in \rho(A)$ if $\lambda < 0$, so $\sigma(A) \subset [0, \infty)$.

It remains to show that $\sigma(A) = [0, \infty)$. Let $\lambda \geq 0$, and write $\lambda = k^2$ where $k \geq 0$. The idea is that $u(x) = e^{ikx}$ satisfies $(A - \lambda)u = 0$ but is not an eigenfunction since $u \notin L^2$. However, we will approximate u by $u_j \in L^2$ with $\|u_j\| = 1$, and Problem 2 will show that $\lambda \in \sigma(A)$.

Let $\chi \in C_0^\infty(\mathbf{R})$ with $\|\chi\|_{L^2} = 1$, and let $u_j(x) = j^{-1/2}\chi(x/j)e^{ikx}$. Since $\|\chi(x/j)\|_{L^2} = j^{1/2}$ we have $u_j \in C_0^\infty(\mathbf{R})$ and $\|u_j\| = 1$. Also,

$$(A-\lambda)u_j = j^{-1/2}[\chi(x/j)((A-\lambda)e^{ikx}) - 2ikj^{-1}\chi'(x/j)e^{ikx} - j^{-2}\chi''(x/j)e^{ikx}],$$

and since $(A - \lambda)e^{ikx} = 0$ we obtain

$$\|(A-\lambda)u_j\| \leq 2j^{-3/2}k\|\chi'(x/j)\| + j^{-5/2}\|\chi''(x/j)\| \leq Cj^{-1}.$$

Thus any $\lambda \geq 0$ is an approximate eigenvalue, so $\lambda \in \sigma(A)$ by Problem 2.

4. Let $H = L^2(I), I = (0, 1)$, and for $z \in \mathbf{C}$ define $A_z : u \mapsto iu'$ with domain $\mathscr{D}(A_z) = \{u \in H^1(I)\,;\, u(1) = zu(0)\}$. The domain contains $C_0^\infty(I)$ so A_z is densely defined, and

$$v \in \mathscr{D}(A_z^*) \Leftrightarrow \exists\, v^* \in L^2 : (iu', v) = (u, v^*) \quad \forall\, u \in \mathscr{D}(A_z).$$

By looking at $u \in C_0^\infty(I)$ we obtain $\mathscr{D}(A_z^*) \subset H^1(I)$ and $A_z^*v = iv'$ for $v \in \mathscr{D}(A_z^*)$. If $u, v \in H^1(I)$ then

$$(iu', v) = i(u\bar{v})(1) - i(u\bar{v})(0) + (u, iv').$$

[1] If $v, v'' \in L^2$ then by Fourier transform $\hat{v}, \xi^2\hat{v} \in L^2$, so $\|v\|_{H^2} = \|(1+\xi^2)\hat{v}\|_{L^2} < \infty$.

Thus, we have

$$v \in \mathscr{D}(A_z^*) \Leftrightarrow v \in H^1(I),\ (z\bar{v}(1) - \bar{v}(0))u(0) = 0 \quad \forall\, u \in \mathscr{D}(A_z).$$

Since $u(0)$ can be arbitrary we get $\mathscr{D}(A_z^*) = \{v \in H^1(I)\,;\, v(1) = \frac{1}{\bar{z}}v(0)\}$. But $z = 1/\bar{z}$ iff $|z| = 1$, and it follows that A_z is self-adjoint iff $|z| = 1$.

Let $z = e^{i\theta}$ where $\theta \in [0, 2\pi)$. We first determine the eigenvalues of A_z. If $(A_z - \lambda)u = 0$ then $u' + i\lambda u = 0$ in I, which is equivalent with $(ue^{i\lambda t})' = 0$ in I. The general distributional solution is $u = Ce^{-i\lambda t}$, and this is in $\mathscr{D}(A_z)$ iff $Ce^{-i\lambda} = Cz$. We have $C = 0$ unless $\lambda \in -\theta + 2\pi\mathbf{Z}$, and thus the set of eigenvalues is exactly $-\theta + 2\pi\mathbf{Z}$.

We claim that $\sigma(A_z) = -\theta + 2\pi\mathbf{Z}$. To prove this let $\lambda \in \mathbf{R} \setminus (-\theta + 2\pi\mathbf{Z})$. We already know that $A_z - \lambda$ is injective. For surjectivity let $f \in L^2(I)$, and write the equation $(A_z - \lambda)u = f$ as $(ue^{i\lambda t})' = -ife^{i\lambda t}$. The general distributional solution in I is given by

$$u(t) = e^{-i\lambda t}(u(0) - i\int_0^t f(s)e^{i\lambda s}\,ds).$$

Since $f \in L^2(I)$ one always has $u \in H^1(I)$, and $u \in \mathscr{D}(A_z)$ provided that

$$e^{-i\lambda}(u(0) - i\int_0^1 f(s)e^{i\lambda s}\,ds) = zu(0)$$
$$\Leftrightarrow\ u(0) = i(1 - e^{i\lambda}z)^{-1}\int_0^1 f(s)e^{i\lambda s}\,ds.$$

Thus $A - \lambda$ is surjective, and $u = (A - \lambda)^{-1}f$ satisfies

$$\|u\| \le |u(0)| + \|\int_0^t f(s)e^{i\lambda s}\,ds\| \le C_{z,\lambda}\|f\|_{L^2}.$$

Therefore $\lambda \in \rho(A)$, and $\sigma(A_z) = -\theta + 2\pi\mathbf{Z}$.