2.3.7 Theorem Assume $u \in B$.

Then
$$\int_{-\infty}^{\infty} \|u(x_1)\|_{L^2} \, dx_1 \le \sqrt{2} \, \|u\|_B$$

Proof We write
$$u = \sum u_j \quad , \quad \operatorname{supp} u_j \subset X_j$$

Now
$$\int_{-\infty}^{\infty} \|u_j(x_1)\|_{L^2} \, dx_1 \le$$
$$\le (2R_j)^{1/2} \left(\int_{-\infty}^{\infty} \|u_j(x_1)\|_{L^2}^2 \, dx_1 \right)^{1/2}$$
$$= (2R_j)^{1/2} \left(\int_{X_j} |u(x)|^2 \, dx \right)^{1/2}$$
$$= (2R_j)^{1/2} \|u_j\|_{L^2} \quad , \quad R_j = 2^j$$

Note that in $\operatorname{supp} u_j(x_1)$
$$-R_j \le x_1 \le R_j \, .$$

Thus
$$\|u\|_B^2 = \sum R_j^{1/2} \|u_j\|_{L^2} \ge$$
$$\frac{1}{\sqrt{2}} \sum \int_{-\infty}^{\infty} \|u_j(x_1)\|_{L^2} \, dx_1 \ge$$

$$\geq \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \| \sum u_j(x_1) \|_{L^2} \, dx_1$$

$$= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \| u(x_1) \|_{L^2} \, dx_1 .$$

$\square$

Note that Theorem 2.3.7 implies

$$B \subset L^1(\mathbb{R}, L^2(\mathbb{R}^{n-1}))$$

Thus by duality

$$L^\infty(\mathbb{R}, L^2(\mathbb{R}^{n-1})) \subset B^*$$

and

$$\|u\|_{B^*} \leq \sqrt{2} \sup_{x_1} \|u(x_1)\|_{L^2}$$

GENERALIZATIONS

Assume

$$c = (c_j)_{j=1}^{\infty} , \quad c_j > 0$$

s.t.

$$c_j/M \leq c_{j+1} \leq c_j M ,$$

for some $M > 0$ and for all $j \in \mathbb{N}$.

We define

$$u \in B_c \iff$$

$$\sum c_j \|u_j\|_{L^2} < \infty$$

Thus $B = B_c$ with $c_j = R_j^{1/2} = 2^{j/2}$

Thus, as before,

$v \in B^* \iff$

$$\sup_j c_j^{-1} \|v_j\|_{L^2} < \infty$$

The following reveals the connection between B_c and the weighted L^2-space

$$L^2_s = \{ v \mid \int (1+|x|^2)^s |u(x)|^2 dx < \infty \}$$

2.3.8 Lemma Let $N \in \mathbb{N}$ be the smallest number with

$$2^N > M$$

Then

$$\|v\|_{B^c} \le C_M \, c_k \left(2^{kN} \|v\|_{L^2_{-N}} + 2^{-kN} \|v\|_{L^2_N} \right)$$

holds $\forall k \in \mathbb{N}$.

Proof

$$R_j^2/4 = 2^{2j}/4 \le 2^{2(j-1)+2}/4$$

$$= 1 + R_{j-1}^2 \le 1 + |x|^2 \le 1 + R_j^2 \le 2R_j^2$$

$\Longrightarrow$

$$\|v\|_{L^2_{-N}} \geq \left(\int_{X_j} (1+|x|^2)^{-N} |v(x)|^2 dx \right)^{1/2}$$

$$\geq (2R_j)^{-N} \|v_j\|_{L^2}$$

and

$$\|v\|_{L^2_N} \geq (R_j/2)^N \|v_j\|_{L^2}$$

Thus

$$\|v\|_{B_c} \leq \left(\sum_{j=1}^{k} c_j \|v_j\|_{L^2} + \sum_{j=k}^{\infty} c_j \|v_j\|_{L^2} \right)$$

$$\leq \sum_{j=1}^{k} c_j (2R_j)^N \|v\|_{L^2_{-N}}$$

$$+ \sum_{j=k}^{\infty} c_j (R_j/2)^{-N} \|v\|_{L^2_N}$$

$$= 2^N \left(\sum_{j=1}^{k} c_j R_j^N \|v\|_{L^2_{-N}} + \sum_{j=k}^{\infty} c_j R_j^{-N} \|v\|_{L^2_N} \right)$$

If we denote

$$\alpha_j = c_j R_j^N$$

$$\beta_j = c_j R_j^{-N}$$

we have

$$\alpha_{j+1}/\alpha_j = 2^N \frac{c_{j+1}}{c_j} > 2^N/M =: \frac{1}{q}$$

and

$$\beta_{j+1}/\beta_j = 2^{-N}\frac{c_{j+1}}{c_j} < 2^{-N} M = q$$

$$0 < q < 1$$

Thus

$$2^{+N} \|v\|_{B_c} \leq$$

$$\leq \|v\|_{L^2_{-N}} \sum_{j=1}^{k} \alpha_j + \|v\|_{L^2_N} \sum_{j=k}^{\infty} \beta_j$$

Here da le the geometric progression

$$\alpha_j = \alpha_k \frac{\alpha_j}{\alpha_k} = \alpha_k \left[\frac{\alpha_j}{\alpha_{j-1}} \frac{\alpha_{j-1}}{\alpha_{j-2}} \cdots \frac{\alpha_{k-1}}{\alpha_k}\right]$$

$$\leq \alpha_k\, q^{k-j}$$

and

$$\beta_j = \beta_k \frac{\beta_j}{\beta_{j-1}} \cdots \frac{\beta_{k-1}}{\beta_k} \leq {}_k\, q^{j}$$

Thus

$$\sum_{j=1}^{k} \alpha_j \leq \alpha_k \sum_{j\leq k} q^{k-j} \leq \frac{\alpha_k}{1-q}$$

$$= \frac{c_k R_k^N}{1 - 2^{-N} M}$$

$$\sum_{j=k}^{\infty} \beta_j \leq \frac{\beta_k}{1-q} = \frac{c_k R_k^{-N}}{1-2^{-N} M}$$

and the claim follows with

$$C_M = \frac{2^{-N}}{1-2^{-N}} M$$

$\square$

2.3.9 Theorem (Interpolation theorem)

Let $M < N \leq M+1$ be an integer.

Then $\exists\ C'_M$ s.t.

$$\| T : L^2_{\pm N} \to L_{\pm N} \| \leq A$$

$$\Rightarrow \quad \| T : B_c \to B_c \| \leq C'_M A$$

Proof Write

$$u = \sum u_j$$

Now

$$\|u\|_{B_c} = \sum c_k \|u_k\|_{L^2} \geq$$

$$\geq c_N \sum c_k \left(R_k^N \|u_k\|_{L^2_{-N}} + R_k^{-N} \|u_k\|_{L^2_N} \right)$$

On the other hand by Lemma 2.3.8

$$\|Tu\|_{B_c} \le \sum c_k \|Tu_k\|_{B_c} \le$$

$$\le C_M \sum c_k \left(R_k^N \|Tu_k\|_{L^2_{-N}} + R_k^{-N} \|Tu_k\|_{L^2_N} \right)$$

$$\le C_M \sum A c_k \left(R_k^N \|u_k\|_{L^2_{-N}} + R_k^{-N} \|u_k\|_{L^2_N} \right)$$

$\blacksquare$

2.3.10 Corollary Assume

$$\sum_{|\alpha| \le N} \|D^\alpha \lambda\| \le C_0$$

$$\lambda \in C^N$$

Then

$$\|\lambda(D) u\|_{B_c} \le C''_M C_0 \|u\|_{B_c}$$

Proof Note

$$\|u\|_{L^2_N} \sim \sum_{|\alpha| \le N} \|D^\alpha \hat{u}\|_{L^2}$$

Now

$$\|\lambda(D) u\|_{L^2_N} \le C_1 \sum_{|\alpha| \le N} \|D^\alpha \lambda(\xi) \hat{u}(\xi)\|_{L^2}$$

$$\le C_0 C_N \sum_{|\alpha| \le N} \|D^\alpha \hat{u}\| \le C'_N \|u\|_{L^2_N}$$

Thus
$$\| \Lambda(D) : L^2_N \to L^2_N \| \le C_N C_0$$
Since L^2_{-N} is the dual of L^2_N and $\Lambda(D)$ is symmetric we have
$$\| \Lambda(D) : L^2_{-N} \to L^2_{-N} \|$$
$$= \| \Lambda(D) : L^2_N \to L^2_N \|$$
and the claim follows from Theorem 2.3.9.

$\blacksquare$

2.3.11 Corollary Let $X_j \subset \mathbb{R}$, $j = 1, 2$. $\Psi : X_1 \to X_2$ diffeom. If $\chi \in C_0^\infty(X_1)$, define
$$Tu = \mathcal{F}^{-1}(\chi(\hat{u} \circ \Psi)), \quad u \in B_c$$
Then
$$T : B_c \to B_c \quad \text{boundedly}$$
and the norm depends only on
$$\|\chi\|_{C^N}, \ \|\Psi\|_{C^{N+1}} \text{ and } \|\Psi^{-1}\|_{C^{N+1}}$$

Proof What is T^*?

$$(u, T^* v) = (Tu, v) = (\mathcal{F}^{-1}(\chi\, \hat{u}\circ\psi), v)$$

$$= (\chi\, \hat{u}\circ\psi, \hat{v}) = \int \chi\, \hat{u}\circ\psi(\xi)\, \hat{v}(\xi)\, d\xi$$

$$= \int \underbrace{\chi\circ\psi^{-1}}_{\chi_1}(\eta)\, \hat{u}(\eta)\, \hat{v}\circ\psi^{-1} J_{\psi^{-1}}(\eta)\, d\eta$$

$$= \langle \mathcal{F}u, \chi_1\, \hat{v}\circ\psi^{-1} J_{\psi^{-1}} \rangle$$

$$= \langle u, \mathcal{F}^{-1}(\chi_1\, \hat{v}\circ\psi^{-1} J_{\psi^{-1}}) \rangle$$

Thus

$$T^* v = \mathcal{F}^{-1}(\chi_1\, \hat{v}\circ\psi^{-1} J_{\psi^{-1}})$$

and consequently T^* is the same form as T.

We need to show

$$T : L^2_N \to L^2_N \quad \text{boundedly}$$

$$T^* : L^2_N \to L^2_N \quad \text{boundedly}$$

But

$$\|Tu\|_{L^2_N} \sim \sum_{|\alpha|\le N} \|D^\alpha \widehat{Tu}\|_{L^2}$$

$$= \sum_{|\alpha|\le N} \| D^\alpha (\chi\, \hat{u} \circ \psi) \|_{L^2}$$

$$= \sum_{|\alpha|\le N} \sum_{\beta\le\alpha} \| \tbinom{\beta}{\alpha} D^\beta \chi \; D^{\alpha-\beta} \hat{u} \circ \psi \|_{L^2}$$

$$\le \sum_{|\gamma|\le N} c \; \| (D^\gamma \hat{u}) \circ \psi \|_{L^2(X_1)}$$

$$\le c \sum_{|\gamma|\le N} \| D^\gamma \hat{u} \| = c \| u \|_{L^2_N}$$

Similarly for T^* and the claim follows from Theorem 2.3.9

$\boxed{\text{ü}}$

2.3.12 Theorem (Localizing the Fourier transform of B_c functions)

Let $\chi \in C_0^\infty(\mathbb{R}^n)$. Then

$$\int \| \chi(D-\eta) u \|^2_{B_c} \, d\eta \le C_{M,\chi} \| u \|^2_{B_c}$$

Here $\mathcal{F}(\chi(D-\eta)u)(\xi) = \chi(\xi-\eta)\,\hat{u}$.

<u>Proof</u> essentially similar to the proof of Theorem 2.3.9.

$\boxed{\text{ü}}$

2.3.13 Corollary If $\chi \in C_0^\infty(\mathbb{R}^n)$, $\|\chi\|_{L^2} = 1$, then

$$\|u\|_{B_c^*} \le C_{M,\chi} \int \|\chi(D-\eta)u\|^2_{B_c^*}\, d\eta$$

for $u \in L^2_{loc} \cap S'$.

<u>Proof</u> By Theorem 2.3.12 we have

$$\begin{aligned}
|(u,v)| &= |(\hat{u},\hat{v})| \\
&= \left| \int \chi(\xi-\eta)^2 \hat{u}(\xi)\, \hat{v}(\xi)\, d\eta \right| \\
&= \left| \int (\chi(D-\eta)u, \chi(D-\eta)v)\, d\eta \right| \\
&\le \int \|\chi(D-\eta)u\|_{B_c^*} \|\chi(D-\eta)v\|_{B_c}\, d\eta \\
&\le C_{M,\chi}^{1/2} \left(\int \|\chi(D-\eta)u\|^2_{B_c^*}\, d\eta \right)^{1/2} \|v\|_{B_c}
\end{aligned}$$

for $u \in L^2_{loc} \cap S'$ and $v \in S$.

Thus the claim follows. $\square$

We need one more generalization in

2.3.14 Corollary Assume $\tilde{\mu} \in C^1(\bar{\mathbb{R}}_+, \mathbb{R}_+)$ with

(i) $\quad (1+t)|\tilde{\mu}'(t)| \leq N\tilde{\mu}(t) \quad , t \geq 0 .$

Then if $\chi \in C_0^\infty$ and $\|\chi\|_{L^2} = 1$ it follows for $\mu(x) = \tilde{\mu}(|x|)$.

(i) $\int \|\mu \chi(D-\eta) u\|_B \, d\eta \leq C_{N,\chi} \|\mu u\|_B$

and

(ii) $\quad \|\mu u\|_{B^*} \leq C_{N,\chi} \int \|\mu \chi(D-\eta) u\|_{B^*} d\eta$

__Proof__ From (1) it follows that (Ex)

$$\frac{(1+s)^N}{(1+t)^N} \leq \frac{\tilde{\mu}(s)}{\tilde{\mu}(t)} \leq \frac{(1+t)^N}{(1+s)^N}$$

Now define

$$C_j = R_j^{1/2} \tilde{\mu}(R_j) \qquad , R_j = 2^j$$

and $\quad M = 2^{N+1/2}$

Now

$$C_j / M = \frac{R_j^{1/2} \tilde{\mu}(R_j)}{2^N}$$

$$\leq R_j^{1/2} \left(\frac{1+R_{j+1}}{1+R_j} \right)^N \frac{\mu(R_{j+1})}{2^N}$$

$$\leq \sqrt{2}\, R_{j+1}^{1/2}\, 2^N\, \tilde{\nu}\left(\frac{R_{j+1}}{2^{N+1/2}}\right)$$

$$= c_{j+1} \leq \dots \leq M\, c_j$$

Thus (c_j) is admissible sequence for defining B_c and B_c^*. Moreover

$$\|\nu u\|_B \sim \|u\|_{B_c}, \quad \text{since}$$

$$\|\nu u\|_B = \sum_j R_j^{1/2} \|\nu u_j\|_{L^2}$$

$$\sim \sum_j R_j^{1/2}\, \tilde{\nu}(R_j)\, \|u_j\|_{L^2}$$

$$= \sum_j c_j \|u_j\|_{L^2} = \|u\|_{B_c}$$

Thus (i) is equivalent with the claim of Theorem 2.3.12.

By taking

$$c_j = \frac{R_j^{1/2}}{\nu(R_j)}$$

we obtain (ii) from Corollary 2.3.13

2.4. DIVISION BY FUNCTIONS WITH SIMPLE ZEROS

Recall that

$u \in \mathcal{D}'^{k}(X) \Longleftrightarrow \forall K \subset X$ cop

$$|\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq k} \|\partial^\alpha \varphi\|_{L^\infty}$$

holds $\forall \varphi \in C_0^\infty(K)$.

Moreover

$$u \in \mathcal{D}'^{k} \Longrightarrow \partial^\alpha u \in \mathcal{D}'^{k+|\alpha|}$$

and $\partial^\alpha : \mathcal{D}'^{k} \longrightarrow \mathcal{D}'^{k+|\alpha|}$ is cont.

We are interested in the limit's

$$\lim_{\varepsilon \to +0} (p \pm i\varepsilon)^{-1} =: (p \pm i0)^{-1}$$

where p is real valued polynomial. This can be considered of as a pull back of the one dimensional distributions

$$(t \pm i0)^{-1} = \lim_{\varepsilon \to 0^+} (t \pm i\varepsilon)^{-1}$$

2.4.1 **Lemma** The limits

$$\lim_{\varepsilon \to 0^+} (t \pm i\varepsilon)^{-1}$$

exist in $\mathcal{D}'^{1}(\mathbb{R})$

<u>Proof</u> Note that

$$\log (t \pm i\varepsilon) \longrightarrow \log (t \pm i0)$$

when log is defined in $\mathbb{C}$ with cut in positive real axis.

Since $L^1_{loc} = \mathcal{D}'^{0}$ and

$$\frac{d}{dx} : \mathcal{D}'^{0} \to \mathcal{D}'^{1}$$

is cont. the claim follows from

$$\frac{d}{dx} \log (t \pm i\varepsilon) = \frac{1}{t \pm i\varepsilon} .$$

$\square$

To prove the corresponding claim for

$$\lim (p \pm i\varepsilon)^{-1}$$

we will need to recall the pull back theorem

2.4.2 Theorem Assume

$$X_1 \Subset \mathbb{R}^{n_1}, \quad X_2 \Subset \mathbb{R}^{n_2},$$

$$f : X_1 \to X_2 \quad C^\infty\text{-map}$$

with

$$f'(x) \text{ surjective} \quad \forall x \in X_1 .$$

Then $\exists \; f^* : D'(X_2) \to D'(X_1)$

i.e.

$$f^* u = u \circ f \quad , \text{ when } u \in C^0(X_2)$$

Moreover $\quad f^* : D'^k \to D'^k \quad , \forall k$

<u>Remark</u> If $f \in C^{k+1}$ instead of $f \in C^\infty$, then

$$f^* : D'^k \to D'^k$$

2.4.3 Theorem If $p \in C^2(X)$, $X \Subset \mathbb{R}^n$ is real valued with only simple zeros, then the limits

$$\lim_{\varepsilon \to +0} \frac{1}{p \pm i\varepsilon} =: \frac{1}{p \pm i0}$$

exist in $D'^1(X)$

Proof Let $u_\varepsilon(t) = \dfrac{1}{t+i\varepsilon}$ and

$$u_0(t) = \frac{1}{t+i0}$$

Then

$$\frac{1}{p+i\varepsilon} = p^* u_\varepsilon$$

Note that since $Dp(x) \neq 0$ when $p=0$. If $p(x_0) = 0$ we may apply Theorem 2.4.2 in the neighbourhood of Ω of x_0 and obtain

$$\frac{1}{p+i\varepsilon} \to p^*(u_0) \quad \text{in } \mathcal{D}'(\Omega).$$

If $p(x_0) \neq 0$ then clearly

$$\frac{1}{p(x)+i\varepsilon} \to \frac{1}{p(x)} \quad \text{in}$$

in the neighbourhood of x_0.
Localization yields the claim

$\square$