1.7.10 Definition We define

$$\sigma_{pp}(A) = \{\lambda \mid \lambda \text{ is an eigenvalue of } A\}$$

$$\sigma_{cont}(A) = \sigma(A|_{H_{cont}})$$

$$\sigma_{ac}(A) = \sigma(A|_{H_{ac}})$$

$$\sigma_{sing}(A) = \sigma(A|_{H_{sing}})$$

1.7.11 Theorem

$$\sigma_{cont}(A) = \sigma_{ac}(A) \cup \sigma_{sing}(A)$$

$$\sigma(A) = \overline{\sigma_{pp}(A)} \cup \sigma_{cont}(A)$$

<u>Proof</u> Ex.

1.8. THEOREMS OF KATO, WEYL AND STONE

1.8.1 Theorem (Kato's theorem)

Assume A is self-adjoint in H and V symmetric with

$$D_A \subset D_V \qquad \text{and}$$

$$(*) \qquad \|Vu\| \leq a\|Au\| + b\|u\|$$

where $a < 1$. Then $A+V$ with domain D_A is self-adjoint.

Proof It is clear that $A+V$ is symmetric. When $u \in D_A$ we have

$$(1) \qquad \begin{aligned} \|(A+V)u\| &\geq \|Au\| - \|Vu\| \\ &\geq (1-a)\|Au\| - b\|u\|, \qquad u \in D_A \end{aligned}$$

Denote $B = A+V$. We will next show that B is closed:

Assume

$$u_n \to u \quad , \quad u_n \in D_A = D_B$$
$$B u_n \to v$$

Now

(1-a) $\| A u_n - A u_m \| \leq \| B u_n - B u_m \| + b \| u_n - u_m \|$

and thus $(A u_n)$ is Cauchy :

$$A u_n \to w$$
$$u_n \to u .$$

Since A is closed

(2) $\quad u \in D_A = D_B \subset D_V$

and $\quad A u = w .$

Finally,

$$v - Bu = (v - Au) - Vu$$
$$= \lim_{n \to \infty} Bu_n - Au_n - Vu$$
$$= \lim_{n \to \infty} Vu_n - Vu = 0$$

by (*). This with (2) proves that B is closed.

We thus need to show

$$n_\pm(B) = 0 .$$

Since all the operators

$$A_t^\pm := A + tV \pm iI, \quad 0 \le t \le 1$$

are closed and symmetric.

Now consider

$$T_t^\pm : G_A \ni (u, Au) \to A_t^\pm u$$

This is a family of bounded semi-Fredholm operators with

$$\operatorname{Ker} T_t^\pm = \{0\}$$

(Recall $A + tV$ is closed and symmetric implies $A_t^\pm$ is injective). The stability theorem of semi-Fredholm operators (Theorem 1.8.2 below) gives that

$$\operatorname{codim} \operatorname{Im} T_t^\pm = \operatorname{codim} \operatorname{Im} A_t^\pm$$

is independent of t. Thus

$$n_\pm(B) = n_\pm(A_t) = n_\pm(A) = 0$$

Thus B is self-adjoint. $\square$

1.8.2 Theorem (Stability of Fredholm operators and semi-Fredholm operators)

Assume B_1 and B_2 are Banach spaces and $T \in L(B_1, B_2)$ is Fredholm. Then if $S \in L(B_1, B_2)$ and $\|S\|$ is suff. small, then $T+S$ is Fredholm with

(1) $\quad \operatorname{ind}(T+S) = \operatorname{ind} T$ and

$$\dim \operatorname{Ker}(T+S) \leq \dim \operatorname{Ker} T$$

More generally, if $T \in L(B_1, B_2)$, $\dim \operatorname{Ker} T < \infty$ and $\operatorname{Im} T$ is closed, then $T+S$ has the same properties and (1) holds if $\|S\|$ is suff. small.

Proof c.f. Hörmander LFA.

Note that $T \in L(B_1, B_2)$ is Fredholm if

(1) $\quad \dim \operatorname{Ker} T < \infty$

(2) $\quad \dim \operatorname{Coker} T < \infty$.

If (1) holds and $\operatorname{Im} T$ is closed

we say T is semi-Fredholm.
In both cases we define

$$\operatorname{ind} T = \dim \operatorname{Ker} T - \dim \operatorname{Coker} T$$

Thus

$$\operatorname{ind} T \in \mathbb{Z} \quad \text{for } T \text{ Fredholm}$$

and

$$\operatorname{ind} T \in \mathbb{Z} \cup \{-\infty\}$$

for T semi-Fredholm.

Next we go for

1.8.3 Theorem (Weyl's Theorem)

If A is self-adjoint, we have

$$\lambda \in \sigma(A) \iff$$

$\exists \{u_n\} \subset \mathcal{D}(A)$, $\|u_n\| = 1$, $\forall n$ and

(*) $\quad \|(A-\lambda)u_n\| \to 0$ as $n \to \infty$

Proof "$\Rightarrow$" Let $\lambda \in \sigma(A)$.

Case 1°: λ is an eigenvalue.
Then $\exists\, u \in H$ s.t. $\|u\| = 1$
and $Au = \lambda u$. Take $u_n = u$
$\forall n \in \mathbb{N}$ and (*) follows

Case 2°: $\operatorname{Ker}(A-\lambda) = \{0\}$. Now $\operatorname{Im}(A-\lambda) \neq H$ but is dense. (Recall that $\sigma_{res}(A) = \emptyset$ for self-adjoint A).

Thus

$$B = (A-\lambda)^{-1}$$

is well-defined with

$$D_B = \operatorname{Im}(A-\lambda).$$

Now B cannot be bounded (why?) and thus $\exists$ seq. $\{v_n\} \subset \operatorname{Im} A-\lambda$ s.t.

$$\|v_n\| = 1 \quad \forall n$$

$$\|Bv_n\| \to \infty \quad \text{as } n \to \infty.$$

To this end let

$$u_n = \frac{w_n}{\|w_n\|} \qquad \text{where}$$

$$w_n = Bv_n = (A-\lambda)^{-1} v_n.$$

Now $\quad u_n \in D(A), \quad \|u_n\| = 1$

and

$$(A-\lambda)u_n = \frac{(A-\lambda)(A-\lambda)^{-1}v_n}{\|Bv_n\|}$$

$$= \frac{v_n}{\|Bv_n\|} \longrightarrow 0$$

as $n \to \infty$.

"$\Leftarrow$" Assume $\lambda \in \rho(A)$. Then

$$\|R_A(\lambda)u\| \leq C\|u\|$$

for some $C > 0$. Thus if $\|u_n\| = 1$

$$1 = \|u_n\| \leq C\|(A-\lambda)u_n\|$$

$\square$

Note that $\lambda \in \mathbb{C}$ is called an approximative eigenvalue if

$\forall \varepsilon > 0$, $\exists u \in D_A$ s.t.

$$\|(A-\lambda)u\| < \varepsilon\|u\|.$$

Weyl's Theorem $\Rightarrow$ For self-adjoint A

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid \lambda \text{ is an appr. eigenvalue}\}$$

1.8.4 Theorem (Stone's formula)

Let A be self adjoint in $\mathcal{H}$.

Then

$$\text{s-lim} \frac{1}{2\pi i}\int_a^b \left[(A-\lambda-i\varepsilon)^{-1} - (A-\lambda+i\varepsilon)^{-1}\right] d\lambda$$

$$= \frac{1}{2}\left(E_{[a,b]} - E_{(a,b)}\right)$$

We will need a simple lemma

1.8.5 Lemma

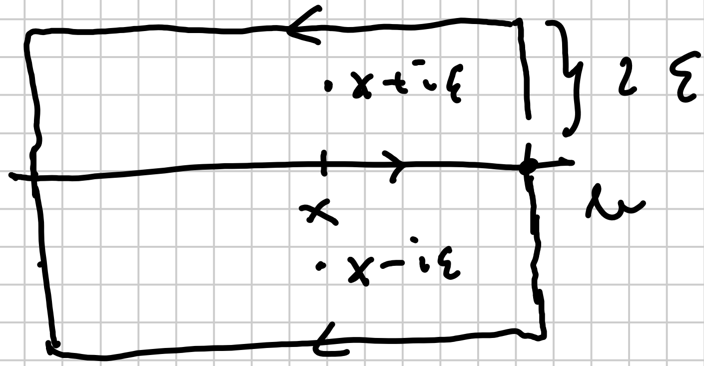
For the function

$$f_\varepsilon(x) = \frac{1}{2\pi i}\int_a^b \left(\frac{1}{x-\lambda-i\varepsilon} - \frac{1}{x-\lambda+i\varepsilon}\right) d\lambda$$

one has

$$f_\varepsilon(x) \longrightarrow \begin{cases} 0 & \text{if } x \notin [a,b] \\ 1/2 & \text{if } x = a \text{ or } x = b \\ 1 & \text{if } x \in (a,b) \end{cases}$$

Proof 1. Assume $x \in (a,b)$, then



Denote

$$\frac{1}{2\pi i}\int_a^b \frac{1}{x-i\varepsilon-t}\,dt = I_1$$

$$\frac{1}{2\pi i}\int_a^b \frac{1}{x+i\varepsilon-t}\,dt = I_2$$

By residy calculus

$$I_1 = O(\varepsilon) + I_2 + 1 \quad \text{and}$$
$$I_2 = O(\varepsilon) - I_1 - 1$$

Thus

$$I_1 - I_2 = 1 + O(\varepsilon)$$

and the claim follows.

The other claims are left as an exercise.

$\square$

Proof of Theorem 1.8.4

Let

$$f(t) = \begin{cases} 0, & t \notin [a,b] \\ 1/2, & t = 0 \text{ and } t = b \\ 1, & t \in (a,b) \end{cases}$$

Then

$$f_\varepsilon \longrightarrow f$$

pointwise and boundedly.
Theorem 1.6.1 implies that
$\| \alpha(f_\varepsilon) u - \alpha(f) u \| \to 0$
for every $u \in H$, where
$$\alpha : L^\infty_B(\mathbb{R}) \longrightarrow L(H)$$
is the spectral map.
But

$$\alpha(f_\varepsilon) = \frac{1}{2\pi i} \int_a^b (R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon))\, d\lambda$$

and
$$\alpha(f) = \int_{\mathbb{R}} f(\lambda)\, dE_\lambda = \frac{1}{2}(E_a + E_b) + E_{(a,b)}$$
$$= \frac{1}{2}\left(E_{[a,b]} + E_{(a,b)}\right)$$
Here we used E_a for $E_{\{a\}}$ and E_b for $E_{\{b\}}$.

$\blacksquare$

STONE'S THEOREM

If $A \in L(H)$, we can define

$$e^{itA} = \sum_{n=0}^{\infty} \frac{(it)^n A^n}{n!}$$

If A is unbounded we need to use spectral calculus. So assume A is self-adjoint in H and denote

$$\varphi_t(\lambda) = e^{it\lambda}$$

Now define

$$U(t) = \alpha(\varphi_t) = \varphi_t(A)$$
$$= e^{itA}$$

1.8.6 Theorem If A is self-adjoint and $U(t) = e^{itA}$, then

(i) $\forall t, s$, $U(t)$ is unitary and

$$U(t+s) = U(t)\,U(s)$$

(ii) $U(t) \xrightarrow{s} U(t_0)$ as $t \to t_0$

(iii) For $x \in D_A$

$$\frac{U(t)x - x}{t} \to iAx \quad \text{as } t \to 0$$

(iv) If

$$\exists \lim_{t \to 0} \frac{U(t)x - x}{t}$$

then $x \in D_A$.

Proof (i) Since α is algebra homomorphism we have

$$U(t)U(t)^* = \alpha(\varphi_t)\alpha(\overline{\varphi}_t)$$

$$= \alpha(\varphi_t \overline{\varphi}_t) = \alpha(1) = E_{\mathbb{R}} = I$$

Thus U is unitary. Similarly

$$U(t)U(s) = \alpha(\varphi_t \varphi_s) = \alpha(\varphi_{t+s})$$

$$= U(t+s)$$

(ii) Since

$$\varphi_t - \varphi_{t_0} = \varphi_{t_0}(\varphi_{t-t_0} - 1)$$

$\to 0$ pointwise and boundedly the claim follows

(iii) We have

$$(1) \quad \psi_t := \frac{\varphi(t) - 1}{t} = \frac{e^{it\lambda} - 1}{t} \to it\lambda$$

Now since $x \in D(A)$

$$\|\psi_t(A) - itAx\|^2$$

$$= \int |\psi_t(\lambda) - it\lambda|^2 (x, dE_\lambda x)$$

$$\le c \int |\lambda|^2 (x, dE_\lambda x) < \infty$$

the claim follows from (1) and dominated convergence theorem

(iv) Define the operator B by

$$D_B = \left\{ x \,\middle|\, \lim_{t\to 0} \frac{U(t)x - x}{t} \text{ exists} \right\}$$

and $\quad iBx = \lim_{t \to 0} \frac{U(t)x - x}{t}$

B is symmetric:

$$-i(y, Bx) = \lim_{t\to 0} \left(y, \frac{U(t)x - x}{t}\right)$$

$$= \lim_{t\to 0} \left(\frac{U(-t)y - y}{t}, x\right)$$

$$= -\lim_{t\to 0} \left(\frac{U(t)y - y}{t}, x\right)$$

$$= -i(By, x)$$

By (iii) $A \subset B$ and thus

$$B^* \subset A^* = A \implies B^* \subset A \subset B$$

Since always for symmetric operators $B \subset B^*$ we have $A = B$ $\square$

Definition If $U: \mathbb{R} \to L(H)$ satisfies (i) and (ii) it is called a *strongly continuous unitary group*

1.8.7 Theorem (Stone's theorem)
Let $U(t)$ be a strongly cont. unit. group in H. Then $\exists$ self-adjoint A on H s.t.

$$U(t) = e^{itA}$$

Proof RS I p. 266. $\square$

II SCATTERING THEORY 1

2.1. introduction

Let $P_0(D)$ be a PDO (partial differential operator) with constant real coefficients i.e

$$P_0(D)f = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$$

where $a_\alpha \in \mathbb{R}$ and

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}, \quad D_i = i\frac{\partial}{\partial x_i}$$

1.1. Theorem $P_0(D)$ with domain

$$\{u \in L^2(\mathbb{R}^n) \mid P_0(D)u \in L^2\}$$

is self-adjoint with abs. cont spectrum.

Proof

(1) $$P_0(D) = \mathcal{F}^{-1} P_0(\xi) \mathcal{F}$$
i.e.
$$P_0(D) f(x) = \mathcal{F}^{-1}\left(P_0(\xi) \hat{f}(\xi)\right)(x)$$
where $\mathcal{F}$ is the Fourier - transform

By Example 1.4.1
$$A : \varphi(\xi) \longmapsto p_0(\xi)\varphi(\xi)$$
is self - adjoint with domain
$$D_A = \{ \varphi \in L^2 \mid p_0 \varphi \in L^2 \}.$$

Since $\mathcal{F}$ is unitary we see that $P_0(D)$ is self-adjoint. Moreover by Ex. 1.4.1 the spectral projections (OVM) is given by
$$E_\Omega u = \chi(P_0(\xi) \in \Omega) u$$
We need to show that $\forall u \in L^2$
$$\mu_u = (u, E_\Omega u)$$
is abs. cont. that is

(2) $$\mu_u(\Omega) = 0 \quad \text{if} \quad m(\Omega) = 0$$

Here m denotes the Lebesgue measure. So assume $m(\Omega) = 0$. Then

$$E_\Omega u = \chi(P_0(\xi) \in \Omega] u$$

and hence the support of $E_\Omega u$ is

$$P_0^{-1}(\Omega)$$

But for polynomials

$$(3) \quad m(P_0^{-1}(\Omega)) = 0 \quad \text{if } m(\Omega) = 0.$$

Thus $E_\Omega u = 0$ and (2) holds.

We have shown μ_u is a.c.

$\square$

<u>Remark.</u> Note that (3) is necessary and sufficient condition for a multiplication operator to have only a.c. spectrum.

Denote $P_0(D) = H_0$. In scattering theory we study

$$H = H_0 + V$$

where V is a perturbation of H_0.

By Kato's theorem H is self-adjoint if V is symmetric, $D_{H_0} \subset D_V$ and

$$\|Vu\| \le a\|H_0 u\| + b\|u\|, \qquad a<1.$$

One purpose of scattering theory is to study the spectral decomposition of H.

The resolvent

$$R(z) = (H-z)^{-1} = \int (\lambda - z)^{-1}\, dE_\lambda$$

determines dE_λ since formally

(1) $$dE_\lambda = \frac{1}{2\pi i}\left(R(\lambda + i0) - R(\lambda - i0)\right).$$

Compare with Stone's formula.

Define distribution

$$\frac{1}{x \pm i0}$$

by $$\left\langle \frac{1}{\cdot \pm i0}, \varphi \right\rangle = \lim_{\varepsilon \to 0^+} \int \frac{1}{x \pm i\varepsilon}\, \varphi(x)\, dx$$

2.2 Lemma

$$\frac{1}{x+i0} - \frac{1}{x-i0} = -2\pi i\,\delta_0$$

Proof Ex.

If $V=0$, we have

$$\frac{1}{2\pi i}\big(R(\lambda+i0) - R(\lambda-i0)\big)u$$

$$= F^{-1}\big(\delta(P_0-\lambda)\,\hat{u}\big) =: v$$

when λ is not critical value i.e. $\nabla P_0 \neq 0$ when $P_0 = \lambda$.

Note: For $P_0(\xi) = \xi^2$ (H_0 ~ the Free Hamiltonian) the only critical value is 0.

If $\hat{u} \in C_0^\infty$ then

$$(2) \qquad \sup_{R>1} \int_{|x|<R} |v(x)|^2\,dx < \infty$$

In next section we study function space defined by (2) and it's dual space.

2.2 Fourier transforms of surface measures

We start with

2.2.1 Theorem Let K be a compact subset of a C'-manifold $M \subset \mathbb{R}^n$ of codimension k. If

$$u = u_0\, ds \quad \text{on } M$$

with $\operatorname{supp} u_0 \subset K$. Then

$$\int_{|\xi|<R} \hat{u}(\xi)\, d\xi \le C R^k \int_M |u_0|^2\, ds$$

where C is indep. of u and R.

Proof Partition of unity $\Rightarrow$ we may assume M is of the form

$$x'' = h(x')$$
$$x' = (x_1, \dots, x_{n-k})$$
$$x'' = (x_{n-k+1}, \dots, x_n)$$

Here $h \in C' \Rightarrow ds = a(x')\, dx'$ where a is cont. and positive.

Now u_0 is a function of x' and

$$\hat{u}(\xi) = \int e^{i(x'\cdot\xi' + h(x')\cdot\xi'')} u_0(x') a(x')\, dx'.$$

For fixed ξ'' we get from Parseval

$$\int |\hat{u}(\xi)|^2 d\xi' = \int |u_0|^2 a^2 dx' =$$

$$= \int |u_0|^2 a\, ds .$$

Integrating this w.r.t. ξ'' for $|\xi''| < R$ yields the claim. $\square$

<u>Remark 1°</u> The most important case is $k = 1$:

$$\sup_{R>0} \frac{1}{R} \int_{|\xi|<R} |\hat{u}(\xi)|^2 d\xi \leq c \int_K |u_0|^2 ds$$

<u>Remark 2°</u> Theorem 2.2.1 is optimal

<u>2.2.2 Theorem</u> Let $u \in S'$, $\hat{u} \in L^2_{loc}$ and assume

$$\limsup_{R\to\infty} \int_{|\xi|<R} |\hat{u}(\xi)|^2 d\xi / R^k < \infty .$$

Then if u is supported on M, a submanifold of codim $M = k$. Then

$u = u_0 \, ds$ and

$$\int_M |u_0|^2 ds \le \limsup_{R\to\infty} \int_{|\xi|<R} |\hat{u}(\xi)|^2 d\xi / R^k$$

Proof We assume $k=1$.

Assume

$$\begin{cases} \chi \in C_0^\infty(\mathbb{R}^n) \\ \operatorname{supp} \chi \subset \{|x|<1\} \\ \int \chi \, dx = 1 \\ \chi \text{ even} \end{cases}$$

Let $\chi_\varepsilon(x) = \chi(x/\varepsilon)\,\varepsilon^{-n}$

Now for $u_\varepsilon = \chi_\varepsilon * u$ we have

$$\hat{u}_\varepsilon(\xi) = \hat{u}(\xi)\,\hat{\chi}(\varepsilon\xi)$$

Now

$$\|u_\varepsilon\|^2 = \int |\hat{u}(\xi)|^2 |\hat{\chi}(\varepsilon\xi)|^2 d\xi$$

$$= \int_{\varepsilon|\xi|\le 1} |\hat{u}(\xi)|^2 |\hat{\chi}(\varepsilon\xi)|^2 d\xi$$

$$+\sum_{j=1}^{\infty} \int_{2^{j-1}\le|\xi|\varepsilon\le 2^j} |\hat{u}(\xi)|\,|(\chi(\varepsilon\xi))|^2\, d\xi$$

Denote

$$C_0 = \sup_{|\xi|<1} \chi(\xi)$$

$$C_j = \sup_{|\xi|\ge 2^{j-1}} |\hat{\chi}(\xi)|^2\, 2^j$$

and
$$C' = \sum_{j=0}^{\infty} C_j .$$

Now clearly

$$\|u_\varepsilon\|^2 \le C_0 \int_{\varepsilon|\xi|\le 1} |\hat{u}(\xi)|^2\, d\xi$$

$$\sum_{j=1}^{\infty} C_j\, 2^{-j} \int_{2^{j-1}\le\varepsilon|\xi|\le 2^j} |\hat{u}(\xi)|^2\, d\xi$$

$$\le C' \frac{1}{\varepsilon} K(\varepsilon)$$

where

$$K(\varepsilon) = \sup_{\varepsilon R>1} \frac{1}{R} \int_{|\xi|<R} |\hat{u}(\xi)|^2\, d\xi$$

Note that since

$$R = 2^j/\varepsilon$$

then

$$\frac{1}{R} = \varepsilon/2^j .$$

Now

$$\text{supp}\, U_\varepsilon \subset M + B(0,\varepsilon)$$

Now for $\varphi \in C_0^\infty$ we have

$$\frac{1}{\varepsilon}\int_{M_\varepsilon} |\varphi(x)|^2 dx \longrightarrow 2\int_{\text{supp}\, u} |\varphi(x)|^2 ds$$

$M_\varepsilon = \text{supp}\, U + B(0,\varepsilon)$.

Thus

$$|\langle u,\varphi\rangle|^2 = \lim_{\varepsilon\to 0} |\langle U_\varepsilon, \varphi\rangle|^2$$

$$\le C'' \int_{\text{supp}\, U} |\varphi|^2 ds \lim_{\varepsilon\to 0} K(\varepsilon)$$

This shows: $\exists\ u_0 \in L^2(M, ds)$ s.t.

$$\langle U, \varphi\rangle = \int_M u_0 \varphi\, ds$$

and

$$\int_M |u_0|^2 ds \le C'' \lim_{\varepsilon\to 0} K(\varepsilon)$$

$$= C'' \lim_{\varepsilon \to 0} \sup_{\varepsilon R > 1} \frac{1}{R} \int_{|\xi| < R} |\hat{u}(\xi)|^2 \, d\xi$$

$$\leq C'' \limsup_{R \to \infty} \frac{1}{R} \int_{|\xi| < R} |\hat{u}(\xi)|^2 \, d\xi$$

$\square$

We still need a substitute of Parseval's theorem in two different guise.

2.2.3 Theorem Let $\varphi \in C_0(\mathbb{R}^n)$ and $u = u_0 \, dS$ a L^2-density with compact support on $M \subset \mathbb{R}^n$, where M is a C^1-manifold of codim k. Then

(1) $$\lim_{R \to \infty} \int |\hat{u}(\xi)|^2 \varphi(\xi/R) \, d\xi / R^k$$

$$= (2\pi)^{n-k} \int_M |u_0(x)|^2 \Big(\int_{N_x} \varphi(\xi) \, d\sigma(\xi) \Big) dS(x)$$

where N_x is the normal plane at $x \in M$.

Proof By Theorem 2.2.1 we may assume that u is cont. and $\varphi \in C_0^\infty$.
Let $\varphi = \hat{\psi}$, with $\psi \in S$.
and
$$\psi_R(x) = R^{n-k}\psi(Rx)$$
$$\Rightarrow \quad \hat{\psi}_R(\xi) = R^{-k}\varphi(\xi/R)$$
$$\Rightarrow \quad \widehat{u * \psi_R}(\xi) = \frac{\hat{u}(\xi)\,\varphi(\xi/R)}{R^k}$$
Thus by Parseval
$$\int |\hat{u}(\xi)|^2 \varphi(\xi/R)\frac{d\xi}{R^k} = (2\pi)^n (u * \psi_R, u).$$
We need to compute
$$\lim_{R\to\infty} u * \psi_R \quad \text{on } M \,!$$
Now
$$u * \psi_R(x) = R^{n-k}\int_M u_0(y)\,\psi(R(x-y))\,ds(y)$$
where we may assume that M is

given by $x'' = h(x')$, as in Th 2.2.1, since outside the support of u_0 the integral $\to 0$ rapidly.

Now

$$u * \Psi_R(x', h(x')) =$$

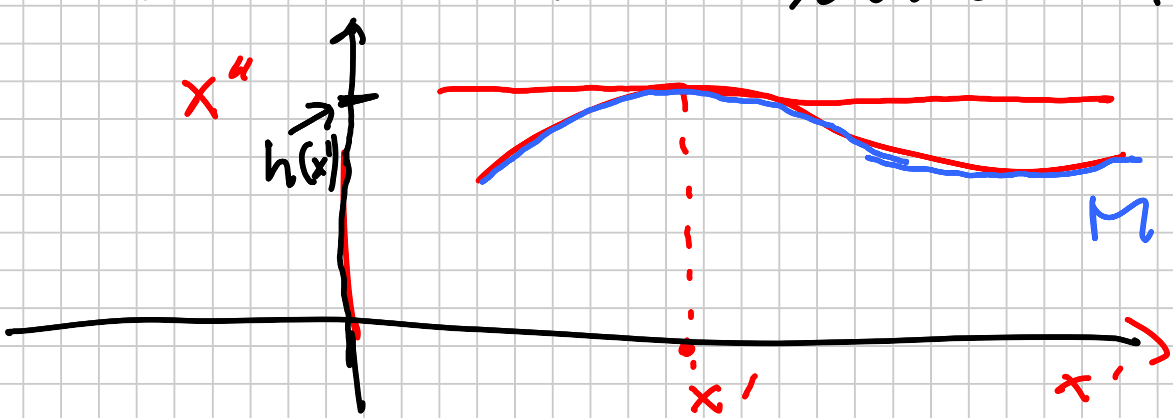
$$= \int_{\mathbb{R}^{n-1}} u_0(y')\, \Psi(R(x'-y'), R(h(x')-h(y')))\, a(y')\, dy'$$

$$= \int_{\mathbb{R}^{n-1}} u_0(x'-y')\, \Psi(Ry', R(h(x')-h(x'-y')))\, a(x-y')\, dy'$$

$$= \int u_0(x'-y'/R)\, \Psi(y', R(h(x')-h(x'-y'/R)))\, a(x'-y'/R)\, dy'$$

$$\to u_0(x') \int \Psi(y', h'(x')y')\, a(x')\, dy'$$

by dominated conv. theorem.



But the above limit is integral over the tangent plane T of M at $(x', h(x'))$ with respect to euclidean area. By Lemma 2.2.4 below

$$(*) \qquad \int_T \psi(y)\, d\sigma(y) = (2\pi)^{n-k} \int_{T^\perp} \varphi(\xi)\, d\xi$$

and the claim follows.

$\square$

2.2.4 Lemma Let $V < \mathbb{R}^n$ be lin. subspace and $d\sigma_V$ the eucl. surface measure on V. Then

$$\widehat{d\sigma_V} = (2\pi)^k\, d\sigma_{V^\perp}$$

Proof We may assume

$$V = \{ x'' = (x_{k+1}, \dots, x_n) = 0 \}$$

Thus for $\varphi \in S$ we need to show

$$(1) \qquad \int \hat{\varphi}(x', 0)\, dx' = (2\pi)^k \int \varphi(0, x'')\, dx''$$

But if $\phi(x') = \int \varphi(x', x'')\, dx''$

then

$$\phi \in S(\mathbb{R}^k)$$

$$\hat{\phi}(\xi') = \hat{\varphi}(\xi', 0)$$

Hence (1) follows from Fourier inversion formula for ϕ:

$$\int \hat{\phi}(\xi')\, d\xi' = (2\pi)^k \phi(0)$$

$\square$

Note that (*) is just the definition of $\overbrace{d\sigma_T}$!

The second guise of Parseval's theorem is

2.2.5 Theorem Let $f \in C^1(\mathbb{R}^n)$ be real valued and

$$V_t = \mathcal{F}(e^{itF} v), \quad v \in L^2$$

Then if $\varphi \in \underset{\nearrow \text{cont.}}{C^0} \cap L^\infty$ we have

(1) $(2\pi)^{-n}\int |V_t(\xi)|^2 \varphi(\xi/t)\, d\xi$

$$\xrightarrow{t\to\infty} \int |v(x)|^2 \varphi(F'(x))\, dx$$

Proof Since C_0^∞ is dense in L^2 we may assume that $v \in C_0^\infty$.

If $C_0^\infty \ni v_j \to v \in L^2$. Now

$A :=$

$$\left|\int |V_t(\xi)|^2 \varphi(\xi/t)\, d\xi - \int |v(x)|^2 \varphi(F'(x))\, dx\right|$$

$$\le \int \left(|V_t(\xi)|^2 - |V_t^j(\xi)|^2\right) \varphi(\xi/t)\, d\xi$$

$$+ \left|\int V_t^j(\xi)\, \varphi(\xi/t)\, d\xi - \int |v_j(x)|^2 \varphi(F'(x))\, dx\right|$$

$$+ \left|\int \left(|v_j(x)|^2 - |v(x)|^2\right) \varphi(F'(x))\, dx\right|$$

Note that $\|V_t^j - V_t\|_{L^2} = \|v_j - v\|_{L^2}$

Finally, the last term

$$\le \|\varphi\|_\infty \int |v_j - v| \left(|v_j| + |v|\right) dx$$

$\leq \|\varphi\|_\infty \|v_j - v\|_2 (\|v_j\|_2 + \|v\|_2)$

Now for $\varepsilon > 0$ choose first j s.t. $\|v_j - v\|_2 = \|V_t^j - V_t\|_2 \leq$
$\leq \varepsilon / 3\|\varphi\|_\infty$, then t large

then get $A \leq \varepsilon$

So we assume $v \in C_0^\infty$

Assume first that $\varphi \in S$.

If $\hat{\varphi} = \varphi$, $v_t = v e^{itF}$ and $\psi_t = t^n \psi(t\cdot)$

then

$$\mathcal{F}(v_t * \psi_t)(\xi) = \varphi(\xi/t)\, V_t(\xi)$$

Now we will show

$$(2) \quad \bar{v}_t(x) v_t * \psi_t(x) \longrightarrow |v(x)|^2 \varphi(F'(x))$$

But

$$(v_t, v_t * \psi_t) = (V_t \varphi_t, V_t)$$

$$= \int |V_t(\xi)|^2 \varphi(\xi/t)\, dt$$

Since we have the majorant $C|v(x)|$

the claim follows from (2).

To show (2)

$$\overline{v_t(x)}\, v_t * \psi_t (x) =$$

$$= \overline{v(x)}\, e^{-itF(x)} \int v(x-y)\, e^{itF(x-y)}\, t^n \psi(ty)\, dy$$

$$= \overline{v(x)} \int v(x - y/t)\, e^{it[F(x-y/t)-F(x)]}\, \psi(y)\, dy$$

$$\longrightarrow |v(x)|^2 \int e^{i\langle F'(x), y\rangle}\, \psi(y)\, dy$$

$$= |v(x)|^2\, \varphi(F'(x)).$$

To prove the extension to $\varphi \in C^0 \cap L^\infty$ is not difficult, but we skip it.

$\square$

2.2.6 Corollary Let V_t be as above and $\varphi \in C_0(\mathbb{R}^{n+1}) = \{\varphi \in C(\mathbb{R}^{n+1}) \mid \operatorname{supp}\varphi$ compact $\}$.

Then

$$(2\pi)^{-n} \int_{t \geq 0} |V_t(\xi)|^2\, \varphi(\xi/R, t/R)\, d\xi\, dt/R$$

$$\xrightarrow{R \to \infty} \int_{t \geq 0} |v(x)|^2\, \varphi(tF'(x), t)\, dt\, dx$$

Proof Let $t = s/R$. We show "≥ 0"

l.h.s.

$$= (2\pi)^{-n} \int ds \int |V_{Rs}(\xi)|^2 \varphi(\xi/R, s)\, d\xi$$

Apply Theorem 2.2.5 with replacing

$$t \longleftrightarrow R$$
$$F \longleftrightarrow sF$$
$$\varphi(\xi) \longleftrightarrow \varphi(\xi, s) \quad , s \text{ fixed}$$

to get

$$\int |V_{Rs}(\xi)|^2 \varphi(\xi/R, s)\, d\xi$$

$$\longrightarrow (2\pi)^n \int |v(x)|^2 \varphi(sF'(x), s)\, dx$$

Theorem of bounded convergence $\Rightarrow$ claim

$\square$