SCATTERING THEORY

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1. SPECTRAL THEORY FOR UNBOUNDED SELF-ADJOINT OPERATORS

Scattering theory is analysis of the Hamiltonian operator

$$H = \Delta + k^2 + V$$

with certain boundary cond. for solution at infinity.

Here $H_0 = \Delta + k^2$ is the free Hamiltonian and V the scattering potential.

H_0 is a self-adjoint unbounded operator in L^2. Thus we need to study

1.1. UNBOUNDED OPERATORS IN A HILBERT SPACE

Let $T: D(T) \longrightarrow H$ be a linear (not cont.) operator, where $D(T) \subset H$ is a linear sub-space. The set $D(T)$ is called the domain of T.

We shall assume in the sequel that $D(T)$ is dense in H.

T is closed if

$$\left.\begin{array}{l} D(T) \ni x_n \to x \\ T x_n \to y \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x \in D(T) \\ Tx = y \end{array}\right. ,$$

This is equivalent to $(D(T), \| \ \|_T)$ is Hilbert space where $\|x\|_T = (\|x\|^2 + \|Tx\|^2)^{1/2}$.

<u>Proof</u> Ex.

Adjoint operator: Let

$$A : D(A) \longrightarrow H$$
$$\cap$$
$$H$$

(densely defined). Then

$$D(A^*) = \{ y \in H \mid \exists y^* \in H \text{ s.t. } (Ax, y) = (x, y^*), \ \forall x \in D(A) \}$$

and we set

$$A^* y = y^* \quad , \text{ for } y \in D(A^*) .$$

<u>Remark</u> A^* is well-defined since A is densely defined.

1.2 SYMMETRIC AND SELF-ADJOINT OPERATORS

<u>Def</u> (i) A is symmetric if

$$(Ax, y) = (x, Ay)$$

$$\forall \quad x, y \in D(A)$$

(ii) A is self adjoint if $A = A^*$.

Remark 1 Self adjoint operators are symmetric.

Remark 2 A is symmetric iff

$$(Ax, x) \in \mathbb{R}, \quad \forall x \in D(A).$$

1.1. Proposition Assume $A : D(A) \to H$ is symmetric. Then A is self-adjoint, iff

(1) $$R(A - \lambda I) = H$$

$$\forall \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Proof Assume (1) holds for λ and $\bar{\lambda}$ with $\operatorname{Im} \lambda \neq 0$.

Then if

$$(A^* - \lambda I) u = 0$$

we have for $\forall v \in D(A)$:

$$0 = ((A^* - \lambda I) u, v)$$

$$= (u, (A - \bar{\lambda}) v) = 0$$

$$\Longrightarrow u = 0.$$

Thus

(2) $\quad \operatorname{Ker}(A^* - \lambda I) = 0$

If now $u \in D(A^*)$, we have by (1), that $\exists\, v \in D(A)$ s.t.

$$(A^* - \lambda I)u = (A - \lambda I)v.$$

Since $A \subset A^*$ we get

$$(A^* - \lambda I)(u - v) = 0$$

and by (2)

$$u = v \in D(A).$$

Thus $D(A^*) = D(A)$ and A is self-adjoint.

The converse follows from

1.2 Prop Let A be closed and symmetric. Then for $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $A - \lambda I$ is injective with closed range and

(1) $\quad |\operatorname{Im} \lambda| \, \|u\| \leq \|(A - \lambda I)u\|$

The codimension of $\operatorname{Im} A-\lambda I$ depends only on sign $\operatorname{Im}\lambda$.

<u>Proof</u> Since $A-\lambda I$ is symmetric we have for $\lambda = x+iy$

$$\|(A-\lambda I)u\|^2$$
$$=\big((A-xI)u+iyu,\ (A-xI)u+iyu\big)$$
$$=\|(A-xI)u\|^2+y^2\|u\|^2$$
$$+\big((A-xI)u,\ iyu\big)+\big(iyu,\ (A-xI)u\big)$$
$$\geq y^2\|u\|^2$$

which proves (1).

Let's prove that the range is closed:

If $u_n \in D(A)$ and $(A-\lambda)u_n \to f$..

By (1) applied to u_n-u_m we get

$$\|u_n-u_m\| \leq \frac{1}{|\operatorname{Im} z|}\|(A-\lambda)(u_n-u_m)\|$$

we have $u_n \to u$. Since $A-\lambda$ is closed we obtain

$$u\in D(A) \text{ and } (A-\lambda)u = f.$$

$\Rightarrow$ $R(A-\lambda)$ is closed.

Moreover
$u \in \operatorname{Ker}(A-\lambda) \Leftrightarrow Au = \lambda u$
$\Rightarrow$ $(u, Au) = \bar{\lambda} \|u\|^2$
Since A is symmetric is l.h.s real and hence $u = 0$.
This shows that $A-\lambda$ is injective for $\operatorname{Im} \lambda \neq 0$.
We omit the proof of the last claim. It follows from the stability of the index of bounded Fredholm operators.

$\square$

1.3 Definition The defect indeces
$$n_{\pm} = \operatorname{codim} R(A-\lambda), \quad \pm \operatorname{Im} \lambda > 0$$

Note that
$$n_{\pm} = \dim \operatorname{Ker}(A^{*}-\lambda), \quad \mp \operatorname{Im} \lambda > 0$$

1.4 Theorem If A is self-adjoint then the resolvent $R(\lambda) = (A-\lambda)^{-1}$ is bounded operator for $\operatorname{Im}\lambda \neq 0$ and

$$\|R(\lambda)\| \leq (\operatorname{Im}\lambda)^{-1}.$$

Moreover $R(\lambda)^* = R(\bar{\lambda})$ and

$$(\lambda-\mu) R(\lambda) R(\mu) = R(\lambda) - R(\mu)$$

Proof Since range of $R(\mu)$ is in $D(A)$

$$\big((A-\mu) - (A-\lambda)\big) R(\mu)$$

$$= (\lambda-\mu) R(\mu)$$

Multiplying by $R(\lambda)$ from the left we get

$$R(\lambda) - R(\mu) = (\lambda-\mu) R(\mu)$$

$\square$

1.5 Theorem Assume A is closed and symmetric. Then

(1) A is self-adjoint $\Longleftrightarrow$ $n_\pm(A) = 0$

(2) A has an self adjoint extension $\Longleftrightarrow$

$$n_+(A) = n_-(A)$$

Proof We prove (1) and sketch the proof of (2):

Since
$$n_\pm = \dim \operatorname{Ker}(A^* - \lambda),$$
$$\pm \operatorname{Im} \lambda < 0$$

If $A^* = A$ we have $n_\pm = 0$ by Prop 1.2. The proof of the converse claims follows exactly as the proof of Prop. 1.1.

The idea behind the proof of (2) is that
$$T : (A+i)u \longmapsto (A-i)u$$
is isometric if A is symmetric. Moreover
$$R(A-i) = R(T)$$
$$R(A+i) = \mathcal{D}(T)$$

If $n_+ = n_-$, T has a unitary extension $\Longleftrightarrow$ A has a self-adjoint extension.

$\square$

Friedrich's extension:

A is semi-bounded if for some $c \in \mathbb{R}$

(1) $A \geq cI \iff$
$$(Au, u) \geq c\|u\|^2$$
$\forall u \in D(A)$. Note that (1) $\iff$
$$c := \inf_{\substack{u \in D(A) \\ \|u\| = 1}} (Au, u) > -\infty$$

1.6 Theorem If A is symmetric and semi-bounded, then $\exists$ self-adjoint extension A_D with
$$(A_D u, u) \geq c\|u\|^2, \quad u \in D(A_D)$$

Outline of proof:

By replacing A with
$$A + (1-c)I$$
we may assume $c = 1$. Then

(1) $\|u\| \leq \|u\|_D$ where the new norm is define by innerproduct

(2) $$(u, v)_D = (Au, v)$$

Define D to be the completion of D_A w.r.t. (2). By (1)

$$D \subset H.$$

Finally define A_D is the restriction of A^* to $D_{A^*} \cap D$.

$\square$

1.3 PROJECTION VALUED MEASURES AND SPECTRAL THEOREMS

A projection valued measure is mapping from Borel-sets of the real line $\mathbb{R}$ to projections of H provided that the following holds:

$$E : \mathcal{B}(\mathbb{R}) \longrightarrow L(H);$$

denote $E(\Omega) = E_\Omega$.

a) E_Ω is orthogonal projection for all $\Omega \in \mathcal{B}(\mathbb{R})$

b) $E_\emptyset = 0$; $E_\mathbb{R} = I$

c) If $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$, $\Omega_n \cap \Omega_m = \emptyset$ for $n \neq m$, then

$$E_\Omega = s\text{-}\lim_{N \to \infty} \sum_{n=1}^{N} E_{\Omega_n}$$

d) $$E_{\Omega_1} E_{\Omega_2} = E_{\Omega_1 \cap \Omega_2}$$

<u>Remarks</u> (i) c) means that

$$\| E_\Omega x - \sum_{n=1}^{N} E_{\Omega_n} x \| \to 0$$

as $N \to \infty$.

(ii) It is easy to see that for $x \in H$

$$(x, E_\Omega x)$$

is a well-defined Borell measure that we denote by

$d(x, E_\lambda x)$

(similarly one defines the complex Borell-measure $d(y, E_\lambda x)$).

(iii) If we define an operator $A: H \to H$, by

$$(y, Ax) = \int_{-\infty}^{\infty} \lambda \, d(y, E_\lambda x) \qquad (*)$$

and

$$D(A) = \{x \mid \int_{-\infty}^{\infty} |\lambda|^2 \, d(x, E_\lambda x) < \infty\}$$

then A is self-adjoint.

(iv) If there exists a number $t > 0$ such that

$$E_{(-t,t)} = I$$

then A is bounded.

(v) If $(*)$ holds for every $x, y \in H$ we write symbolically

$$A = \int \lambda \, dE_\lambda$$

The converse is also true

Spectral theorem For any self-adjoint operator $A \in L(H)$ there exists a p.v.m E_λ s.t.

$$A = \int_{-\infty}^{\infty} \lambda \, dE_\lambda \quad \text{and}$$

$$D(A) = \left\{ x \,\middle|\, \int_{-\infty}^{\infty} |\lambda|^2 \, d(x, E_\lambda x) < \infty \right\}$$

Remark (i) This is one of the most powerful tools in analysis

(ii) If g is any Borell-function on $\mathbb{R}$, we may define the operator $g(A)$ by

$$g(A) = \int g(\lambda) \, dE_\lambda$$

and

$$D_{g(A)} = \left\{ x \,\middle|\, \int |g(\lambda)|^2 (x, dE_\lambda x) < \infty \right\}$$

Note that $D_{g(A)}$ is dense in H and that $g(A)$ is self-adjoint if g is real-valued.

(iii) If g is bounded then

$$\int_{-\infty}^{\infty} |g(\lambda)|^2 (x, dE_\lambda x)$$

$$\leq M \int_{-\infty}^{\infty} (x, dE_\lambda x)$$

$$= M \, (x, \underbrace{\left(\int_{-\infty}^{\infty} dE_\lambda\right)}_{= I} x)$$

$$= M \|x\|^2 < \infty$$

Thus

$$D_{g(A)} = H$$

for all self-adjoint A and consequently $g(A)$ is in $L(H)$.

1.4 Examples of self-adjoint operators

1.4.1 Example Let M be locally compact topological space and $d\mu$ a positive (Borel) measure on M. Take
$$H = L^2(d\mu).$$
Assume $\beta : M \to \mathbb{R}$ is measurable and define
$$A : H \to H \;;\; Af = \beta f,$$
$$D(A) = \{ \varphi \in L^2 \mid \beta\varphi \in L^2 \}$$

Then
(i) A is self-adjoint
(ii) E_λ is multiplication with the char. function of the set
$$\{x \mid \beta(x) < \lambda\}$$

Proof (i) If $f, g \in D(A)$, then
$$\overline{f}\, Ag = \overline{f}\, \beta\, g \in L^1$$ and (f, Ag) is well defined.

Moreover

$$(f, Ag) = \int_M f \overline{\beta g} \, d\mu =$$

$$= \int_M \beta f \bar{g} = (Af, g)$$

and thus A is symmetric. Finally

$$D(A^*) = \{ g \in L^2 \mid \exists g^* \in L^2 \text{ s.t. }$$

$$(Af, g) = (f, g^*) \; ; \; \forall f \in D(A) \}$$

$$= \{ g \in L^2 \mid \exists g^* \in L^2 \text{ s.t. }$$

$$\int \beta f \bar{g} \, d\mu = \int f \overline{g^*} \, d\mu \}$$

$$\overset{(1)}{=} \{ g \in L^2 \mid \beta g = g^* \in L^2 \}$$

$$= D(A)$$

Note that (1) follows from the density of $D(A)$.

(ii) We need to show

$$(f, Af) = \int_{\mathbb{R}} \lambda \, (f, d\chi_{\{\beta(x) < \lambda\}} f)$$

Here

$$r.h.s = \int_{\mathbb{R}} \lambda \, (f, d\chi_{\{\beta(x)<\lambda\}} f)_M$$

$$=: \int_{\mathbb{R}} \lambda \, d\nu(\lambda)$$

Here the function $\nu(\lambda)$ is given by

$$\nu(\lambda) = (f, \chi_{\{\beta(x)<\lambda\}} f)_M$$

$$= \int_M |f(x)|^2 \chi_{\{\beta(x)<\lambda\}} \, d\mu(x)$$

Since for all $s \in \mathbb{R}$

$$\int \lambda \, d\chi_{\{s<\lambda\}} = s$$

we have

$$r.h.s = \int_M \beta(x) \, |f(x)|^2 \, d\mu(x)$$

$$= (f, Af)$$

$\square$

1.4.2 Ex. Same as above but $M = \mathbb{R}$ and $\beta(\xi) = \xi$.

Moreover if λ is an eigenvalue of A, then for $\overset{\text{almost}}{\text{all}}$ $\xi \in \mathbb{R}$

$$A f(\xi) = \xi f(\xi) = \lambda f(\xi)$$

If $\mu(\{\xi_0\}) > 0$, this is possible, by defining

$$f(\xi) = \begin{cases} 1 , & \xi = \xi_0 \\ 0 , & \xi \neq \xi_0 \end{cases}$$

Thus eigenvalues correspond to such points where $d\mu$ has positive mass.

1.5 Spectrum

If $A : H \to H$ is densely defined we denote by $\rho(A)$ the resolvent set

$$\rho(A) = \{ \lambda \in \mathbb{C} \mid (A-\lambda)^{-1} \in L(H) \}$$

The complement of $\rho(A)$ in $\mathbb{C}$ is called the spectrum of A and is denoted by $\sigma(A)$.

Recall λ is eigenvalue of A if
$$Ax = \lambda x \quad , \text{ for } x \neq 0 .$$
The set of eigenvalues is called point spectrum and is denoted by
$$\sigma_p(A) .$$
The residual spectrum is
$$\sigma_r(A) = \{\lambda \mid R(A-\lambda) \subset H \text{ is not dense and } \lambda \notin \sigma_p(A)\}$$

<u>Remark</u> If A is self adjoint, then

(i) $\sigma(A) \subset \mathbb{R}$ and

(ii) $\sigma_r(A) = \emptyset$.

<u>Proof</u> (i) follows from Theorem 1.4

(ii) If $u \in H$ and $\lambda \in \mathbb{R}$ is such that
$$((A-\lambda)f, u) = 0 \quad , \quad \forall f \in D(A)$$
Then by definition

$$u \in D(A^*) = D(A)$$

and

$$(f, (A-\lambda)u) = 0 \qquad \forall f \in D(A)$$

Since $D(A)$ is dense

$$Au = \lambda u$$

and $u \in \sigma_p(A)$

$\square$