2.3 Some Function Spaces

2.3.1 Määr We define the Besov space B by

$$B = \Big\{ v \in L^2_{loc}(\mathbb{R}^n) \;\Big|\; \|v\|_B^2 =$$

$$= \Big(\int_{|x| \le 1} |v|^2 \Big)^{1/2} + \sum_{j=1}^{\infty} \Big(2^j \int_{2^{j-1} \le |x| \le 2^j} |v|^2 \, dx \Big)^{1/2} < \infty \Big\}$$

2.3.2 Lemma

$$B \subset L^2$$

Proof Define

$$v_j = v \, \chi\{2^{j-1} \le |x| \le 2^j\}, \quad j = 2, \dots$$

$$v_1 = v \, \chi\{|x| \le 1\}$$

Then

$$\sum_{j=1}^{N} v_j = v \, \chi\{|x| \le 2^N\}$$

and

$$\Big(\int_{|x| \le 2^N} |v|^2 \Big)^{1/2} \le \Big\| \sum_{j=1}^{N} v_j \Big\|_{L^2}$$

$$\le \sum_{j=1}^{N} \|v_j\|_{L^2}$$

But

$$\|v\|_B = \|v_1\|_{L^2} + \Big(\sum_{j=1}^{\infty} 2^j \|v_j\|_{L^2}^2 \Big)^{1/2}$$

Thus

$$\int_{|x|\le 2^N} |v(x)|^2 \le \sum_{j=1}^{N} \|v_j\|_{L^2}$$

$$\le \sum_{j=1}^{N} 2^{-j} \|2^j v_j\|_{L^2}$$

$$\le \left(\sum_{j=1}^{N} 2^{-2j}\right)^{1/2} \left(\sum_{j=1}^{N} \|2^j v_j\|_{L^2}^2\right)^{1/2}$$

$$\le 2\|v\|_B$$

Thus

$$v \in L^2. \qquad \square$$

Remark 1

Note that the above proof $\Rightarrow$

$$\sum v_j \longrightarrow v$$

both in L^2-norm and in B-norm.

Remark 2

$$v \in B \Leftrightarrow 2^j v_j \in l^1(L^2)$$

$$\Leftrightarrow \|2^j v_j\|_{L^2} \in l^1$$

$$\Leftrightarrow \sum \|2^j v_j\|_{L^2} < \infty$$

2.3.3 Lemma The dual space B^* of B is given by

$$B^* = \{u \in L^2_{loc} \mid \|u\|_{B^*} = \sup_{j>0} (2^{-j} \|u_j\|^2_{L^2})^{1/2} < \infty\}$$

Proof $v \in B \Leftrightarrow \{v_j\} \in \ell^1(L^2)$

$\Rightarrow \quad u \in B^* \Leftrightarrow \{u_j\} \in \ell^\infty(L^2)$

$\square$

2.3.4 Lemma $\forall\, u \in B^*$ we have

$$\|u\|^2_{B^*} \le \sup_{R>1} \frac{1}{R} \int_{|x|\le R} |u|^2 dx \le 4 \|u\|^2_{B^*}$$

Proof

$$\|u\|^2_{B^*} = \sup_{j>0} 2^{-j} \|u_j\|^2_{L^2}$$

$$\le \sup_{R>0} R^{-1} \int_{|x|<R} |u(x)|^2 dx =$$

$$\le \sup_{N>0} 2^{1-N} \int_{|x|\le 2^N} |u(x)|^2$$

$$\le 2 \sup_{N>0} 2^{-N} \sum_{j=1}^{N} \|u_j\|^2_{L^2}$$

$$\le 2 \sup_{N>0} 2^{-N} \sum_{j=1}^{N} \|u_j\|^2_{L^2} 2^{-j} 2^j$$

$$\le 2\|U\|_{B^*}^2 \sup_{N\geq 0} 2^{-N} \sum_{j=1}^{N} 2^j$$

$$\le 4\|U\|_{B^*}^2 \qquad \square$$

Remark

$$v \in B^* \iff \sup_{R>0} \frac{1}{R} \int_{|x|<R} |u|^2 \, dx < \infty$$

2.3.5 Theorem Let M be a C^1 hypersurface in $\mathbb{R}^n$ and $K \subset M$ compact. Then

$$S \ni v \longmapsto \hat{v}|_K \in L^2_K$$

extends to a $\overset{\text{cont.}}{\text{surjective}}$ map $T: B \to L^2_K$

Proof Theorem 2.2.1 gives for $\hat{u} = u_0 \, ds$, $\operatorname{supp} v_0 \subset K$

$$|(u_0, \hat{v}|_K)| = |(u, v)| \le \|U\|_{B^*} \|v\|_B$$

$$\le C \|u_0\|_{L^2} \|v\|_B$$

which proves the continuity.

The adjoint of T is given by

$$L^2_k \ni u_0 \mapsto \hat{F}(u_0 ds) \in B^*$$

since

$$(u_0, Tv) = (u_0, \hat{v}) = (\hat{F}(u_0 ds), v).$$

By Theorem 2.2.3, with $\varrho \geq 0$ and $\varrho(0) > 0$ T^* is injective with closed range. Thus T is surjective by the Lemma 2.3.6 below. $\square$

<u>2.3.6 Lemma</u> Let $T \in L(B_1, B_2)$ where B_1 and B_2 are Banach spaces. Then for the transpose T^* of T it holds

(i) $\operatorname{Im} T$ is closed $\Leftrightarrow$ $\operatorname{Im} T^*$ is closed

(ii) $(\operatorname{Ker} T)^\circ = \operatorname{Im} T^*$, if $\operatorname{Im} T$ is closed

(iii) $(\operatorname{Im} T)^\circ = \operatorname{Ker} T^*$, if $\operatorname{Im} T$ is closed

Moreover T is surjective if (i) holds and T^* is injective.

Remarks: 1°) The annihilator W° of a subset $W \subset B$ is defined as

$$\xi \in W^\circ \subset B^* \iff \langle \xi, x \rangle = 0 \ , \ \forall x \in W$$

2°) $\operatorname{Im} T$ is closed $\Rightarrow$ $\operatorname{Im} T^*$ is closed is easy to prove. The converse is difficult.

3) The last claim follows from (ii) since, then

$$\operatorname{Im} T = (\operatorname{Ker} T)^\circ = \{0\}^\circ = B_1^*$$

4) Finally note that $T^* : B_2^* \to B_1^*$ is defined by

$$(*) \quad \langle Tx, \eta \rangle_2 = \langle x, T^*\eta_1 \rangle \ , \ x \in B_1 \ , \ \eta \in B_2^*$$

It is an easy exercise to see that (*) defines a bounded linear operator $T^* : B_2^* \to B_1^*$

Remark Since

$$B = \oplus_{j\,\ell^1} L^2(X_j)$$

where

$$X_j = \{x \mid 2^{j-1} \le |x| \le 2^j\}$$

it is clear that B is not reflexive (ℓ^1 is not reflexive).

Let

$$\mathring{B}^*$$

be the closure of L^2_{comp} in B^*.

Clearly S is dense in $\mathring{B}^*$.

It is easy to see (Ex) that

$$\mathring{B}^* = \{v \in L^2_{loc} \mid \frac{1}{R}\int_{|x|<R} |u|^2 dx \to 0\}$$

Moreover

$$B = (\mathring{B}^*)^*.$$

Compare $(c_0)^* = \ell^1$ and $(\ell^1)^* = \ell^\infty$.

Thus

$$B \subset B^{**}$$

has finite codimension.