1.6 ANOTHER SPECTRAL THEOREM

1.6.1 Theorem (spectral theorem – functional calculus form)

Let A be self-adjoint in H.
Then there exist a unique homom. $\alpha : L^\infty_B(\mathbb{R}) \longrightarrow L(H)$ s.t.

(i) $\alpha(\bar{\varphi}) = \alpha(\varphi^*)$

(ii)' $\|\alpha(\varphi)\| \leq \|\varphi\|_\infty$

(ii)'' $\varphi_j \longrightarrow \varphi$ pointwise and boundedly

$\Longrightarrow \|\alpha(\varphi_j)u - \alpha(\varphi)u\| \to 0, \ \forall u \in H$

(iii) $\alpha(\varphi_1) = A\,\alpha(\varphi)$, if $\varphi_1(\lambda) = \lambda\varphi(\lambda)$

Moreover $u \in D_A \iff$

$$\sup_{\substack{0 \leq \varphi \leq 1 \\ \text{supp}\,\varphi \text{ compact}}} \|\alpha(x\varphi)\| < \infty$$

and in this case

$$Au = \lim_{n \to \infty} \alpha(\lambda\varphi^{(n)})u$$

where $0 \leq \varphi_n \leq 1$ have compact support and $\to 1$ everywhere.

Remarks 1) $L^\infty_B(\mathbb{R})$ is the space of bounded Borell - measurable functions equipped with $\| \ \|_\infty$ - norm.

2) From (ii)' it follows
$$\| \alpha(\varphi_j) - \alpha(\varphi) \| \to 0, \text{ if}$$
$$\varphi_j \to \varphi \quad \text{uniformly}$$

Sketch of proof: From Spectral Theorem PVM - form, $\exists$ PVM,
$$E : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(H) \quad \text{s.t.}$$
$$A = \int \lambda \, dE_\lambda$$
If $\varphi = \sum_{i=1}^n c_i \chi_{\Omega_i}$, where Ω_i are disjoint define
$$\alpha(\varphi) = \sum_{i=1}^n c_i E_{\Omega_i}$$
Since $E_{\cup_i \Omega_i}$ is a projection we have
$$\| E_{\cup_i \Omega_i} \| = 1 \qquad \text{and}$$

$$\|\alpha(\varphi)x\|^2 = \left\|\sum_{i=1}^{n} c_i (E_{\Omega_i} x)\right\|^2$$

$$\leq \max|c_i|^2 \sum_{i=1}^{n} \|E_{\Omega_i} x\|^2$$

$$= \max|c_i|^2 \|E_{\cup_i \Omega_i} x\|^2 \leq$$

$$\max|c_i|^2 \|x\|^2$$

Thus

$$\|\alpha(\varphi)\| \leq \|\varphi\|_{L^\infty}$$

for φ a step-function.

By continuity we can extend the definition of α to $LB(\mathbb{R})$.

The other properties are easy to check

$\blacksquare$

<u>Remark</u> We denote

$$\alpha(\varphi) = \varphi(A)$$

$$= \int \varphi(\lambda)\, dE_\lambda$$

1.7. ABSOLUTELY CONTINUOUS SPECTRUM

Let μ be a Borell measure in $\mathbb{R}$.

We let

$$P = \{x \mid \mu\{x\} \neq 0\}.$$

1.7.1 Lemma P is countable

Proof Since μ is Borell, $\mu(K) < \infty$ for all compact $K \subset \mathbb{R}$.

Define

$$P_n = \{x \in P \mid |\mu\{x\}| > 1/n\}.$$

Assume P is not countable $\Rightarrow$ $\exists\, n$ s.t. P_n is not count. $\Rightarrow$ P_n has an accumulation point $x_0 \in \mathbb{R}$. Thus $\exists\, x_i \in P$ s.t. $x_i \longrightarrow x_0$, $x_i \neq x_j$

Now

$$P\{x \mid |x - x_0| < 1\}$$

$$\geq \sum_{\substack{j \\ |x_j - x| < 1}} P(x_j) = \infty \qquad \lightning$$

$\square$

If $x \in P$ we say that x is a pure point of μ.

We define a measure μ_{pp} on $\mathbb{R}$

$$\mu_{pp}(A) = \mu(P \cap A)$$
$$= \sum_{x \in P \cap A} \mu\{x\}$$

We also define $\mu_{cont} = \mu - \mu_{pp}$.

Then μ_{pp} and μ_{cont} are measures and

$$\mu_{cont}\{x\} = 0$$

for all $x \in \mathbb{R}$. Moreover we have shown

1.7.2 Theorem Any Borel-measure can be composed uniquely as

$$\mu = \mu_{pp} + \mu_{cont}$$

We recall that μ is absolutely continuous w.r.t. Lebesgue measure if

$$d\mu = f\,dx ,$$

with $f \in L^1_{loc}(\mathbb{R}, dx)$.

We also say that μ is singular w.r.t. Lebesgue measure if $\exists\, S \subset \mathbb{R}$

$$\mu(S) = 0 \quad \text{and} \quad m(\mathbb{R} \setminus S) = 0$$

Lebesgue measure

1.7.3 Theorem (Lebesgue decomp. theorem)
If μ is Borell in $\mathbb{R}$, then $\exists!$

$$\mu = \mu_{ac} + \mu_{sing}$$

1.7.4 Corollary $\forall$ Borell measure μ in $\mathbb{R}$ can be decomposed uniquely as

$$\mu = \mu_{pp} + \mu_{ac} + \mu_{sing}$$

where μ_{pp} is pure point, μ_{ac} is absolutely cont and μ_{sing} is continuous and singular.

<u>Proof</u>