14.4.2 Theorem V is dent range iff

i) $\forall y \in R^n$ $\bigcup V(x+y, D)\{u \in C_0^\infty(\Omega) \mid \|P_0(D)u\|_{L^2} \le 1\}$, $\Omega \subset$ unite ball in R^n

is precompact set in L^2.

ii)

(1) $\|V(\cdot+y, D)u\|_{L^2} \le M_j \|P_0(D)u\|_{L^2}$, $(u \in C_0^\infty(\Omega), y \in X_j)$

and

(2) $\sum 2^j M_j < \infty$

Moreover $C^\infty \cap B^*_{P_0}$ is dense in $B^*_{P_0}$ and $V(x,D)$ extens to a compact operator

$$V(x,D) : B^*_{P_0} \to B$$

Proof "V dent range $\Rightarrow$ (i)":

For every α

(3) $\|P_0^{(\alpha)}(D)u\|_{L^2} \le C \|P_0(D)u\|_{L^2}$

for $u \in C_0^\infty(\Omega)$.

The estimate (3) follows from existence of the regular fundamental solution, that is formulated in the below theorem.

10.3.7 Theorem If $\Omega \Subset \mathbb{R}^n$ is bounded, $\exists$ bounded lin.

$$E : L^2(\Omega) \to L^2(\Omega) \quad \text{s.t.}$$

a) $P_0(D) E f = f \quad , f \in L^2$

b) $E P_0(D) u = u \quad$, if $u \in \mathcal{E}'(\Omega)$, $P_0(D) u \in L^2(\Omega)$

c) $Q(D) E : L^2(\Omega) \to L^2(\Omega)$ is bounded

if $Q \prec P_0$

Proof cf. [Hö].

Now (3) follows as:

$$\| P_0^{(\alpha)}(D) u \|_{L^2} \overset{b)}{=} \| P_0^{(\alpha)}(D) E P_0(D) u \|_{L^2} \overset{c)}{\leq} C \| P_0(D) u \|_{L^2}$$

for $u \in C_0^\infty(\Omega)$. Now

$$\{ u \in C_0^\infty(\Omega) \mid \| P_0(D) u \|_{L^2} \leq 1 \}$$

$$\subset \{ u \in C_0^\infty(\Omega) \mid \sum_\alpha \| P_0^{(\alpha)}(D) u \|_{L^2} \leq C \}$$

By the definition of B^*-norm (i) follows:

Let $\tau_y u(x) = u(x+y)$. Then

$V(x+y, D) u = \tau_{-y} V(x, D) \tau_y u$ and

$$\| V(x+y, D) u \|_{L^2} \sim \| V(x, D) \tau_y u \|_{L^2} \sim \| V(x, D) \tau_y u \|_B$$

"Short range ⇒ (2)"

Assume M_j is the smallest constant s.t. (1) is true. Choose $y_j \in X_j$ and $u_j \in C_0^\infty(\Omega)$ with

(4) $\| P_0(D) u_j \|_{L^2} = 1$ and

(5) $\| V(\cdot + y_j) u_j \|_{L^2} > M_j/2$

Set

(6) $$\begin{cases} u = \sum u_{2j}(\cdot - y_{2j}) R_{2j}^{1/2} \\ v = \sum u_{2j+1}(\cdot - y_{2j+1}) R_{2j+1} \end{cases}$$

and note that the terms have disjoint supports in both series. Note that

$$\operatorname{supp} u_j(\cdot - y_j) \subset X_{j-1} \cup X_j \cup X_{j+1}$$

since

$$u_j(x - y_j) \neq 0 \Rightarrow |x - y_j| \leq 1 .$$

Thus $u, v \in B^*_{P_0}$.

Note

$$\| u \|_{B^*_{P_0}} = \sum \| P_0^{(\alpha)}(D) u \|_{B^*}$$

and

$$\| P_0^{(\alpha)}(D) u \|^2_{B^*} = \sup_j R_j^{-1} \| P_0^{(\alpha)}(D) u \|^2_{L^2(X_j)}$$

$$\leq c \sup_j \| P_0^{(\alpha)}(D) u_j \|_{L^2(X_j)}$$

$$\leq c \| P_0(D) u_j \|_{L^2} \leq C < \infty$$

If v is short range, then

$$V(x,D) u \in B \quad \text{and}$$

$$V(x,D) v \in B$$

But

$$V(x,D) u = \sum_{j \in 2\mathbb{N}} V(x,D) u_j(x - y_j) R_j^{1/2}$$

and

$$\infty > \| V(x,D) u \|_B = \sum_{j \in 2\mathbb{N}} R_j^{1/2} \| V(x,D) u_j(x-y_j) \|_{L^2} R_j^{1/2}$$

$$= \sum_{j \in 2\mathbb{N}} R_j^{1/2} \| V(x+y_j, D) u_j(x) \|_{L^2} R_j^{1/2}$$

$$\geq \sum_{j \in 2\mathbb{N}} R_j M_j$$

Similar analysis for v gives

$$\sum_{j \in 2\mathbb{N}+1} R_j M_j < \infty$$

and thus (2) is proven.