KSLUENTO 5

Def 1 The limits of

$$P_\eta(\xi) = \frac{P(\xi+\eta)}{\tilde{P}(\eta)}$$

are denoted by $L(P)$. $\theta \in \mathbb{R}^n \setminus \{0\}$

the limits with

$$\frac{\eta}{|\eta|} = \frac{\theta}{|\theta|}$$

is denoted by $L_\theta(P)$

The elements in $L(P)$ and $L_\theta(P)$ are called *localizations at ∞*.

Lemma 1 The sets $L(P)$ and L_θ^P are closed subsets in

$$\{ p \in Pol_{m,n} \mid \| p \|_{Pol_{m,n}} = 1 \}$$

Proof (i) The space of polynomials on degree $\leq m$ is equipped with norm

$$\|P\|_{Pol_{m,n}} = \Big(\sum_\alpha |\partial^\alpha P(0)|^2 \Big)^{1/2}$$

Clearly

$$\dim Pol_{m,n} < \infty .$$

Now if

$$P_\eta(\xi) = \frac{P(\xi+\eta)}{\tilde{P}(\eta)} , \quad \text{then}$$

$$\|P_\eta\|^2_{Pol_{m,n}} = \frac{\sum_\alpha |\partial^\alpha P(\xi+\eta)|_{\xi=0}|^2}{\tilde{P}(\eta)^2}$$

$$= 1.$$

(ii) If $\varphi_i \to \varphi$, $\varphi_i \in L(P)$, then

$$\varphi_i(\xi) = \lim_{k\to\infty} P_{\eta_i^k}(\xi) \quad \text{in } \| \ \|_{P_{m,n}}$$

Now $\varphi(\xi) = \lim_{k\to\infty} P_{\eta_k^k}(\xi)$. $\square$

Denote

$$\Lambda(P) = \{ \eta \in \mathbb{R}^n \mid P(\xi + t\eta) \equiv P(\xi) \ \forall t, \eta \}$$

which is a linear space.

Proposition If $|\xi| \to \infty$ in such way that

$$\mathrm{dist}(\xi, \Lambda(P)) \to \infty$$

then

$$\tilde{P}(\xi) \to \infty .$$

Proof Let $M \subset \mathbb{R}^n$ be such that $\tilde{P}$ is bounded in M. Write

$$P = q + r ,$$

where q is homog of degree

$$m = \deg P$$

and

$$\deg r < m .$$

Now

$$\partial^\alpha (q - P) = \partial^\alpha r = \mathrm{const.}$$

if $|\alpha| = m - 1$. Thus $\partial^\alpha q$ is bounded in M. Denote

$$N = \{ \eta \in \mathbb{R}^n \mid \partial^\alpha q(\eta) = 0, \text{ when } |\alpha| = m-1 \}$$

Now

(1) $N \subset \Lambda(q)$

since q homog $\Rightarrow$

(2) $(m-|\alpha|)\partial^\alpha q(\eta) = \eta \cdot \nabla \partial^\alpha q(\eta)$

Note: $\partial^\alpha q$ is homog of degree $m-|\alpha|$

Hence it sufficies to show (1) for $m=0$:

$$\eta \cdot \nabla q = \sum \eta_j \partial_j q$$

$$= \sum \eta_j \partial_j \sum_{|\alpha|=m} a_\alpha \eta_1^{\alpha_1} \cdots \eta_n^{\alpha_n}$$

$$= \sum_j \alpha_j q = m q$$

Now by (2) induction with decreasing α shows

$\partial^\alpha q(\eta) = 0$ for $\eta \in N$ and $|\alpha| \le m-1$

Taylor at η gives

$$q(\xi+\eta) = \underbrace{\sum_{|\alpha|\le m-1} \partial^\alpha q}_{=0} + \sum_{|\alpha|=m} \partial^\alpha q = \text{const.}$$

i.e.

$$q(\xi+\eta)=\eta \qquad \forall \xi \in \mathbb{R}^n \text{ and } \forall \eta \in N$$

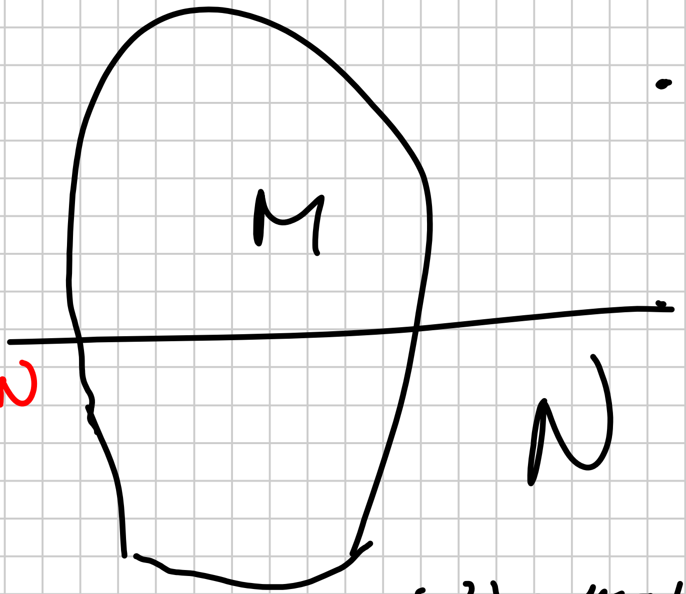
which proves (1).

Now

(3) $\quad \tilde{q}$ is bounded on M

Indeed

$\sup_{x\in M} \operatorname{dist}(x,N) < \infty$



i.e.

M is bounded mod N

This follows from

$\partial^\alpha q$ bounded in M, for $|\alpha|=m-1$

$N=\bigcap_{|\alpha|=m-1}\{\partial^\alpha q=0\}$ so in the

direction $N^\perp$, $|\partial^\alpha q|$ increases

This implies that

$\tilde{q}$ is bounded on M

$\Longrightarrow \tilde{\lambda}$ is bounded on M

Note $\quad \eta \in N \Rightarrow$

$$q(\xi+\eta) = q(\xi) \qquad \forall \xi$$

$$\Rightarrow \partial^\alpha q(\xi+\eta) = \partial^\alpha q(\xi)$$

$$\Rightarrow \quad \tilde{q}(\xi+\eta) = \tilde{q}(\xi) \qquad \forall \xi$$

Hence $\quad \tilde{r}$ is bounded in M.

Induction: We may assume

M is bounded mod $\Lambda(r)$

Now

$$\Lambda(r) \cap \Lambda(q) \subset \Lambda(P)$$

since $\quad \eta \in \Lambda(r) \cap \Lambda(q) \Rightarrow$

$$P(\xi+\eta) = q(\xi+\eta) + r(\xi+\eta) =$$

$$q(\xi) + r(\xi) = P(\xi)$$

$\Rightarrow$ M is bounded mod $\Lambda(P)$

which proves the claim. $\boxtimes$

2.5.3 **Theorem** Assume

(i) P_0 is simply characteristic

(ii) $K \subset \mathbb{C}_+$ (or $K \subset \mathbb{C}_-$) compact

(iii) $K \cap Z(P_0) = \emptyset$

(iv) $Q(D) \prec P(D)$.

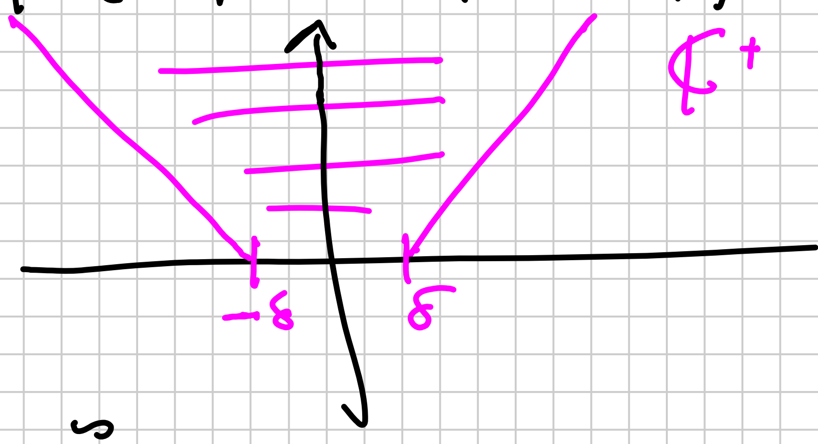
Then

$$(*) \quad \| Q(D) R_0(z) f \|_{B^*} \leq C \sup_{\xi} \frac{\tilde{Q}}{\tilde{P}_0} \|f\|_B$$

for $\hat{f} \in C_0^\infty$ and $z \in K$

Proof Denote

$$K_\delta = \{ z \in \mathbb{C}^+ \mid |\mathrm{Re}\, z| \le \delta + |\mathrm{Im}\, z| \}$$



If $0 \notin Z(P_0)$ it is enough to prove the claim for $z \in K_\delta$, for δ small enough.

Denote

$$P_{0\eta}(\xi) = \frac{P_0(\xi+\eta)}{\tilde{P}_0(\eta)}$$

and

$$\varphi(\xi) = \lim_{|\eta_j|} \frac{P_0(\xi+\eta_j)}{\tilde{P}_0(\eta_j)}, \quad |\eta_j| \to \infty$$

Note that

$$P_{0\eta}(\xi) := \frac{P_0(\xi+\eta)}{\tilde{P}_0(\eta)} = \sum_{|\alpha| \le m} \frac{a_\alpha(\eta)}{\tilde{P}_0(\eta)} \xi^\alpha$$

where $a_\alpha(\eta)$ is polyn. of $\deg a_\alpha = m$.

Since
$$\left| \frac{P_0(\xi+\eta)}{\tilde{P}_0(\eta)} \right| \leq C(1+|\xi|)^m$$

where C is independent of η, the limits
$$\lim_{j\to\infty} \frac{a_\alpha(\eta_j)}{\tilde{P}_0(\eta_j)} \quad , |\eta_j| \to \infty$$

may exist and then Q is a polynomial of degree $\leq m$.

Polynomials $P_{0\eta}$ and their limits form a compact set in
$$\mathcal{P}_{mn} = \{ P \text{ is a polyn. of degree } \leq m \text{ in } \mathbb{R}^n \}$$
when equipped with norm
$$\| P \|^2_{\mathcal{P}_{mn}} = \sum_\alpha |\partial^\alpha P(0)|^2$$

Moreover in $X = \{ \xi \,/\, |\xi| < g \}$

i) $p' = \nabla p \neq 0$ in X or

ii) $\exists\ c > 0$,

$$p(\xi) > c \ , \quad \text{in } X_-$$

Note $p(\xi) = 0$ and $\nabla p(\xi) = 0 \Rightarrow$
0 is a critical value.

Thus for $\eta = 0$, either

$$|P_\eta| \geq c > 0 \quad \text{in } X_-$$

or $\quad |\nabla P_\eta| \neq 0 \quad$ in X_-

Considering $\phi(\xi) = P(\xi + \eta)$
the claim follows for arbitrary η.
Note: ϕ and P have same critical values.

Thus for $\chi \in C_0^\infty(X_-)$

(1) $\left\| F^{-1}\left(\left(P_\eta - \frac{z}{\tilde{P}(\eta)}\right) \chi_0 \hat{f}\right) \right\|_{B^*} \leq C \|f\|_B$

for $f \in S$ and $\frac{z}{\tilde{P}_0(\eta)} \in K_\varepsilon$,

where C is independent of η.

choose $\delta = \varepsilon \min \tilde{P}$ and $\chi \in C_0^\infty(X_2)$

$$\begin{cases} \|\chi\|_{L^2} = 1 & \text{and } \chi_0 \in C_0^\infty \text{ s.t.} \\ \chi \chi_0 = \chi \end{cases}$$

use (1) with $\hat{g} = \chi(\xi)\, \varphi(\xi+\eta)\, \hat{f}(\xi+\eta)$:

Then $z \in K_\delta \Rightarrow$

$$\left| \mathcal{F}^{-1}\left(\chi(\xi) \left(\frac{P(\xi+\eta) - z}{\tilde{P}_0(\eta)} \right)^{-1} \varphi(\xi+\eta)\, \hat{f}(\xi+\eta) \right) \right|$$

$$= \tilde{P}_0(\eta) \, \left| \chi(D-\eta)\, \varphi(D)\, \hat{f}(D) \right|$$

Thus

$$\| \chi(D-\eta)\, \varphi(D) f \|_{B^*} \leq$$

(2)

$$\frac{c}{\tilde{P}(\eta)} \| \varphi(D)\, \chi(D-\eta) f \|_{B}$$

Here

$$\varphi(\xi)\, \chi(\xi-\eta) = \lambda(\xi)\, \chi(\xi-\eta)\, \tilde{\varphi}(\eta)$$

where

$$\lambda(\xi) = \frac{\varphi(\xi)\, \chi_0(\xi-\eta)}{\tilde{\varphi}(\eta)}$$

Now $\sum |\partial^\alpha \lambda(\xi)| \le C$

where C is indep of ξ (and η)

Since we have shown

$$\| \lambda(D) u \|_B \le \left(\sup \sum |\partial^\alpha \lambda(\xi)|\right) \| u \|_B$$

we get from (2)

(3) $$\| \chi(D-\eta)\varphi(D)R_0(z) f \|_{B^*} \le C \| \chi(D-\eta) f \|_B \frac{\tilde{\varphi}(\eta)}{\tilde{P}(\eta)}$$

We also have shown (interpolation)

(4) $$\int \| \chi(D-\eta) u \|_B^2 \, d\eta \le C \| u \|_B^2$$

(5) $$\| u \|_{B^*}^2 \le C \int \| \chi(D-\eta) u \|_{B^*}^2 \, d\eta$$

Thus from (3), (4) and (5) we get

$$\| \varphi(D) R_0(z) f \|_{B^*} \le C \frac{\tilde{\varphi}(\eta)}{\tilde{P}(\eta)} \| f \|_B$$

Note

(6) $\varphi(D) R_0(z) : \mathcal{B} \longrightarrow \mathcal{B}^*$ (extension) and w^* - cont w.r.t. z.

<u>Remark</u> For $\lambda \in \mathbb{R} \setminus Z(P_0)$ we must write

$$R_0(\lambda \pm i0)$$

instead of $R_0(\lambda)$. Note that

$$(*) \quad R_0(\lambda + i0) f - R_0(\lambda - i0) f = 2\pi i \, F^{-1}\left(\delta_\lambda(P_0) \hat{f} \right)$$

for $f \in \mathcal{B}$. Note

$$\langle \delta_\lambda(P_0), \varphi \rangle = \int \delta(P_0(\xi) - \lambda) \varphi(\xi)$$

$$= \int_{P_0(\xi) = \lambda} \varphi(\xi) \frac{dS}{|P'(\xi)|}$$

2.5.3 Theorem

Assume

(i) $u \in B^*$

(ii) $\lambda \in K \Subset \mathbb{R} \setminus Z(P_0)$ and

(iii) $(P_0(D) - \lambda)u = 0$.

Then

(1) $\hat{u} = v \, dS$ and $v \in L^2(M_\lambda)$

Conversely (1) $\Rightarrow$

$$(P(D) - \lambda)u = 0 \quad \text{and} \quad u \in B^*$$

Moreover

(2) $$\frac{1}{C_K} \|u\|_{B^*}^2 \leq \int_{M_\lambda} |v|^2 dS \leq C_K \|u\|_{B^*}^2$$

Finally for all $f \in B$

(3) $$\int_{M_\lambda} |\hat{f}|^2 dS \leq C_K \|f\|_B^2$$

<u>Proof</u> Now

$$(P_0(D) - \lambda)u = 0 \Rightarrow (P(\xi) - \lambda)\hat{u}(\xi) \equiv 0$$

$\Rightarrow \operatorname{supp} \hat{u} \subset M_\lambda$

Now Theorem 2.2.2 and $u \in B^* \Rightarrow$

$$\hat{u} = v \, dS, \quad v \in L^2 \quad \text{and}$$

(4) $$\|v\|_{L^2} \leq C_K \|u\|_{B^*}$$

To show

$$\|u\|^2_{B^*} \le C_K \int_{M_\lambda} |v|^2$$

it is enough to show (3). For this

(*) and Thm 2.5.2 $\Rightarrow$

(5) $\|\mathcal{F}^{-1}(\varphi\,\delta_\lambda(P_0)\hat{f})\|_{B^*} \le C \sup \frac{\tilde{\varphi}}{\tilde{P}_0} \|f\|_B$

for all $f \in B$. Thus (4) and

(5) with $\varphi = \partial^\alpha P_0$ give

$$\int_{M_\lambda} |(\partial^\alpha P_0)\hat{f}|^2 \frac{dS}{|P_0'(\xi)|^2} \le C \|f\|^2_B$$

Summing up with $|\alpha| = 1 \Rightarrow$

$$\int_{M_\lambda} |\hat{f}|^2\, dS \le C \|f\|_B$$

Thus (2) and (3) are proven.

Note that $\hat{u} = v\,dS$, $v \in L^2$

$\Rightarrow \begin{cases} u \in B^* & \text{by Theorem 2.2.1} \\ (p(\xi) - \lambda)\hat{u} = 0 \end{cases}$

Theorem is proven

<u>Remark</u> If $P_0(\xi) = +\xi^2$ and $\lambda = k^2 \pm i0$. Then Theorem 2.5.3

$$\Rightarrow \left.\begin{array}{l} u \in B^* \\ (\Delta + k^2) u = 0 \end{array}\right\} \quad \hat{u} = v\, dS, \quad v \in L^2$$

That is

$$u(x) = \int_{|\xi| = k} e^{i x \cdot \xi} v(\xi)\, dS(\xi)$$

$$= \int_{S^1} e^{i k \theta \cdot x} g(\theta)\, dS(\theta)$$

where $g \in L^2$. It is all B^*-solutions are Herglotz-waves with L^2-densities and conversely.

Finally,

$$\|g\|_{L^2(S^1)} \sim \|u\|_{B^*}$$