2.5 THE RESOLVENT OF THE (GENERALIZED) FREE HAMILTONIAN

Let $P_0 : \mathbb{R}^n \to \mathbb{R}$ be polynomial ($P_0(\xi) = \xi^2$ in classical scattering)

Let further

$$Z(P_0) = \{ \lambda \in \mathbb{R} : \exists\, \xi \in \mathbb{R}^n \text{ s.t. } P_0(\xi) = \lambda \text{ and } \nabla P_0(\xi) = 0 \}$$

The set $Z(P_0)$ is called the the set of critical values of P_0.

In case $P_0(\xi) = \xi^2$,

$Z(P_0) = \{0\}$ since

$$\nabla P_0(\xi) = 2\xi = 0 \iff \xi = 0$$

and

$$P_0(0) = 0$$

In general $Z(P_0)$ is a finite set.
We do not prove this, since it is obvious in the classical case. It follows from two facts

(i) $Z(P_0)$ is semi-algebraic

(ii) $Z(P_0)$ is of measure 0

The first claim follows from Seidenberg-Tarski-theorem. We prove the second claim, since it is of general interest:

2.5.1 Theorem Let $K \Subset \Omega \subset \mathbb{R}^n$ and $f: \Omega \to \mathbb{R}$ C^∞-map. Then the set

$$A = \{f(x) \mid x \in \Omega, \ \nabla(x) = 0\}$$

is of measure 0.

Proof We prove the claim by induction w.r.t. n. So we assume the claim is true for $n-1$. Denote

$$C_j = \{x \in \Omega \mid f'(x) = 0, \dots, f^{(j)}(x) = 0\}$$

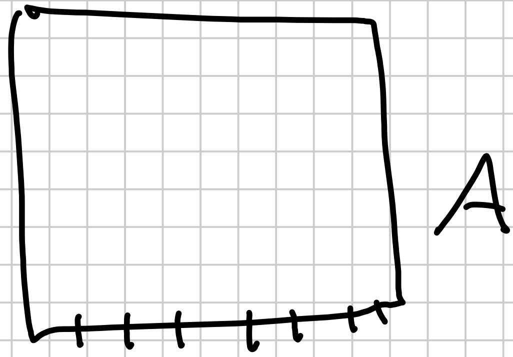
We prove first

(1) $m(f(C_j)) = 0$

For this it suffices to show

$$m(f(C_j) \cap K) = 0$$

where K is a cube of side A, say.



Divide the edges to ν equal parts

$\Rightarrow$ ν^n small cubes I of side $\varepsilon = A/\nu$.

Let $I_1, \dots, I_N$ be those cubes with

$$I_j \cap C_j \neq \emptyset .$$

Choose $x_k \in I_j \cap C_j$. Then Taylor

$\Rightarrow$

$$|f(x+x_k) - f(x)| \le B|x|^{j+1} \le B\varepsilon^{j+1}$$

if $x + x_k \in I_k$.

When $j \ge n$ then this implies

$$m(f(K \cap C_j)) \le \sum_1^N m(f(I_k))$$

$$\le \sum_{1}^{N} B\, \varepsilon^{j+1} \le B \sum_{1}^{N} \varepsilon^{n+1} \le B \sum_{1}^{N} m(I_k)\, \varepsilon$$

$$\le B\, A^n\, \varepsilon = B\, m(K)\, \varepsilon$$

$$\Rightarrow \quad m(f(K \cap C_j)) = 0$$

Especially the whole claim is proven for $n = 1$.

We continue with general n.

Denote

$$E_k = C_k / C_{k+1} .$$

It is enough to show that

$$m(f(E_k)) = 0 \,, \text{ if}$$

$$m(f(C_{k+1})) = 0,$$

since it would imply

$$m(f(C_1)) = 0 .$$

To this end, assume $x_0 \in E_k$

$\Rightarrow \quad \exists\, \alpha, |\alpha| = k$ s.e. $\partial^\alpha f(x_0) = 0$

and

$$\nabla \partial^\alpha f(x_0) \neq 0$$

Denote $\quad g(x) = \partial^\alpha (x)$

Now by implicit function theorem

$$S = \{x \mid g(x) = 0\}$$

is a $n-1$ dimensional manifold in the nbhood of x_0 and

$$C_n \subset S.$$

If $x_1 \in C_n$, then $\nabla f(x_1) = 0$

and if

$$\varphi : \mathbb{R}^{n-1} \to S$$

is local diffeo. then

$$\nabla (f \circ \varphi)(y_1) = 0, \quad y_1 = \varphi(x_1)$$

By induction

$$\{ f(x_1) \mid \nabla(f \circ \varphi)(y_1) = 0, \ y_1 = \varphi(x_1) \}$$

is of Lebesgue measure 0 and hence

$$m(f(E_n)) = 0.$$

Thus

$$m(f(C_1)) \leq \sum_{k=1}^{n-1} m(f(E_k)) + m(f(C_n)) = 0$$

$\square$

Remark 1) Theorem 2.5.1 is called Morse - Sard - theorem.

2) It is valid for functions $f : \Omega \to \mathbb{R}^m$, as well, in the following form:
If $\Omega \subset \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^m$ is C^1, then a point $x \in \Omega$ is called critical point, if

$$f'(x) : \mathbb{R}^n \to \mathbb{R}^m$$

is surjective. In this case $f(x)$ is called a critical value.

Morse - Sard:

$$m\{y \mid y \text{ is critical value of } f\} = 0$$

3) The smoothness assumption can be reduced to

$$f \in C^k(\Omega, \mathbb{R}^m)$$

where

$$k > \max\{0, n-m\},$$

and there is a counterexample of $k = n-m$.

When $\lambda \notin Z(P_0)$ the level set

$$M_\lambda = \{\xi \mid P_0(\xi) = \lambda\}$$

is a C^∞-submanifold of $\mathbb{R}^n$.

Case $P_0(\xi) = \xi^2 \Rightarrow$

$$M_\lambda = \emptyset \,, \text{ if } \lambda < 0$$

$M_\lambda = \{\xi \mid \xi^2 = \lambda\}$ is a sphere of radius $\sqrt{\lambda}$, when $\lambda > 0$.

For any polynomial P denote

$$\tilde{P}(\xi) = \Big(\sum_\alpha |\partial^\alpha P(\xi)|^2\Big)^{1/2}$$

Taylor's formula ($m = \deg P$)

(1) $$\tilde{P}(\xi+\eta) \leq (1 + C|\xi|)^m \, \tilde{P}(\eta)$$

<u>Proof</u> Denote $Q = \partial^\alpha P$. Now

$$Q(\xi+\eta) = \sum_{|\beta| \leq m - |\alpha|} \xi^\beta \, \partial^\beta Q(\eta)$$

$$\leq (1 + C|\xi|^m)\, \tilde{P}(\eta)$$

$\square$

2.5.2 Definition P_0 is *simply characteristic* if

$$\tilde{P}_0(\xi) \le C\Big(\sum_{|\alpha|\le 1} |\partial^\alpha P_0(\xi)| + 1\Big)$$

Examples: 1) $P_0(\xi) = \xi^2$

$$\tilde{P}_0(\xi)^2 = \sum_{|\alpha|\le 2} |\partial^\alpha \xi^2|^2 = |\xi|^4 + \sum_{j=1}^{n} |2\xi_j|^2 + 4n$$

$$\le 4n\Big(\sum_{|\alpha|\le 1} |\partial^\alpha \xi^2|^2 + 1\Big)$$

ok

2) All elliptic operators and all hypoelliptic operators

Def $P(D)$ is hypoelliptic if for $\alpha \neq 0$

$$P^{(\alpha)}(\xi)\big/P(\xi) \longrightarrow 0 \quad \text{as } |\xi| \to \infty.$$

For hypoelliptic operators

$$\operatorname{sing\,supp} u = \operatorname{sing\,supp} P(D)u$$

Example: ① elliptic operators are hypoelliptic as $P(\xi) = \xi^2 + \sum_{|\alpha| \le 1} a_\alpha \xi^\alpha$

Assume now $\operatorname{Im} z \neq 0$. Denote the resolvent

$$R_0(z) = (P_0(D) - zI)^{-1},$$

(1) $$R_0(z)f = \mathcal{F}^{-1}\left(\frac{\hat{f}(\xi)}{P_0(\xi) - z}\right), \quad f \in L^2$$

By Theorem 2.4.8 we may extend for $\hat{f} \in C_0^\infty$ the def. (1) to $z \in \mathbb{C}^\pm \setminus Z(P_0)$. The extension is however different for different signs.

Comparison of operators:

Def If

$$\tilde{Q}(\xi) / \tilde{P}(\xi) < C < \infty$$

$\forall \xi \in \mathbb{R}^n$ we say that $Q(D)$ is weaker than $P(D)$ and denote

$$Q \prec P$$

2.5.3 Theorem Assume

(i) P_0 is simply characteristic

(ii) $K \subset \mathbb{C}_+$ (or $K \subset \mathbb{C}_-$) compact

(iii) $K \cap Z(P_0) = \emptyset$

(iv) $Q(D) \prec P(D)$.

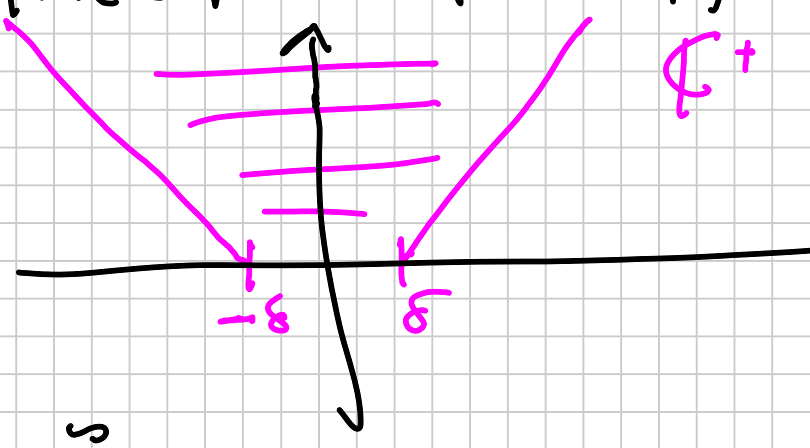
Then

$$(*) \quad \| Q(D) R_0(z) f \|_{B^*} \leq C \sup_{\xi} \frac{\tilde{Q}}{\tilde{P}_0} \|f\|_B$$

for $\hat{f} \in C_0^\infty$ and $z \in K$

Proof Denote

$$K_\delta = \{ z \in \mathbb{C}^+ \mid |\mathrm{Re}\, z| \le \delta + |\mathrm{Im}\, z| \}$$



If $0 \notin Z(P_0)$ it is enough to prove the claim for $z \in K_\delta$, for δ small enough.

Denote

$$P_{0\eta}(\xi) = \frac{P_0(\xi+\eta)}{\tilde{P}_0(\eta)}$$

and

$$P_1(\xi) = \lim_{|\eta_j|} \frac{P_0(\xi+\eta_j)}{\tilde{P}_0(\eta_j)}, \quad |\eta_j| \to \infty$$

Note that

$$\frac{P_0(\xi+\eta)}{\tilde{P}_0(\eta)} = \sum_{|\alpha| \le m} \frac{a_\alpha(\eta)}{\tilde{P}_0(\eta)} \xi^\alpha$$

where $a_\alpha(\eta)$ is polyn. of $\deg a_\alpha = m$.

Since

$$\left| \frac{P_0(\xi+\eta)}{\tilde{P}_0(\eta)} \right| \le C(1+|\xi|)^m$$

where C is independent of η, the limits

$$\lim_{j\to\infty} \frac{a_\alpha(\eta_j)}{\tilde{P}_0(\eta_j)} \quad , |\eta_j| \to \infty$$

may exist and then P_1 is a polynomial of degree $\le m$. Indeed for any sequence η_j, $|\eta_j| \to \infty$, there is a subsequence ν_j, $|\nu_j| \to \infty$ s.t.

$$\lim_{j\to\infty} \frac{P_0(\xi+\nu_j)}{\tilde{P}_0(\nu_j)}$$

is a polynomial of degree $\le m$.