KSLVENT03

Proof of Theorem 2.4.9.

By Theorem 2.4.8 (i.e. $\|u_z\|_{B^*} \le C\|u\|_B$) we may assume

$$f \in S \text{ and } \varphi \in C_0^\infty.$$

(Note that $v \in B^* \Rightarrow v \in L^2_{loc}$, and $\forall R$ $\int_{|x|\le R} |v(x)|^2 dx / R \sim \|v\|_{B^*}$)

For Corollary 2.4.7

we have already proven the first claim if χ has a small support.

Note that χ can have infinitely many components but only countably many.

If $\operatorname{supp} \chi_j \cup \operatorname{supp} \chi_k$ is sufficiently small we obtain by polarization of this result (polarization = "what is true for norms is true for inner product")

$$\int u^j_{\lambda+i0}\, \overline{u^k_{\lambda+i0}}\, \varphi(\cdot/R)\, dx/R \longrightarrow$$

$$(2\pi)^{1-n} \int_{p=\lambda} \chi_1 \overline{\chi_2} |\hat{f}|^2 \left(\int_0^\infty \varphi(sp)\, ds \right) \frac{dS}{|p'|}$$

Here $u^j_z = F^{-1}((p-z)^{-1} \chi_j \hat{f})$

Summing up this yields the claim for + sign. Replacing p by $-p$ yields the claim for − sign.

The second claim follows from

(1) $\quad u_{\lambda \pm i0}(x_1, 0) \to 0$ in $L^2(\mathbb{R}^{n-1})$

as $x_1 \to \mp\infty$.

To see why (1) is true recall

(2) $\| u_{\lambda+i0}(x_1, \cdot) \|_{L^2} \leq C \int \| f(t) \|_{L^2} \left(H(x_1 - t) + (1+|t-x_1|)^{-N} \right) dt$

$\to 0$ as $x_1 \to -\infty$ if $N > 0$.

For the claim for $N = 0$ one needs a sharpening of (2) where $(1+|t-x_1|)^{-N}$ is replaced by $o(|t-x_1|)$. This is obtained by using Riemann-Lebesgue.

Note that in real applications p is polynomial and $p \in C^N$, $\forall N$.

Thus (1) is true and hence the second claim is shown for χ with small support. The extension to arbitrary support follows again by polarization.

$\square$

2.4.10 Corollary The following are equivalent for fixed $\lambda \in \mathbb{R}$

(i) $\quad u_{\lambda+i0} = u_{\lambda-i0}$

(ii) $\quad u_{\lambda+i0} \in \mathring{B}^*$

(ii') $\quad u_{\lambda-i0} \in \mathring{B}^*$

(iii) $\quad \chi \hat{f} = 0$ when $p = \lambda$

Proof (i) $\Leftrightarrow$ (iii):

$$u_{\lambda+i0} - u_{\lambda-i0} =$$

$$\lim_{\varepsilon \to 0^+} F^{-1}\left(\left(\frac{1}{p-\lambda-i0} - \frac{1}{p-\lambda+i0}\right)\chi \hat{f}\right)$$

$$= 2\pi i F^{-1}\left(\chi\, \delta(p-\lambda)\, \hat{f}\right)$$

Thus (i) $\Leftrightarrow$ (iii).

To show (ii) $\Leftrightarrow$ (iii) assume $\varphi(0) > 0$ and $\varphi \geq 0$ in Theorem 2.4.9.

Then

$$u_{\lambda+i0} \in \overset{\circ}{B}{}^{*} \Leftrightarrow$$

$$0 = \int_{p=\lambda} |\hat{xf}|^2 \Big/ \underbrace{\Big(\int_{s>0} \varphi(sp')\, ds \Big)}_{> c > 0 \text{ in } \{p=\lambda\}} \, dS / |p'|$$

≥ 0

which shows (ii) $\Leftrightarrow$ (iii).

$\square$

If (i) – (iii) is satisfied then

$$u_\lambda(x) \sim |x f(x)| \quad \text{at } |x| \to \infty$$

4.2.11 Theorem Assume $p \in C^{N+2}$, $X \in \mathbb{R}^n$, $p(x) \in \mathbb{R}$, $p'(x) \neq 0$ in X.

If $\hat{xf} = 0$ when $p = \lambda$, then for

$$u_{\lambda \pm i0} = F^{-1}\left(\frac{1}{p - \lambda \pm i0} \hat{xf} \right) \quad \text{it holds}$$

(i) $\quad u_{\lambda+i0} = u_{\lambda-i0}$

(ii) $\quad \|\mu u\|_{B^*} \leq C_N \|\mu f\|_B$

for any radial μ with

(iii) $(1+t)\,\mu'(t) \le N\mu(t)$

Proof We have seen before that

(iii) $\Rightarrow$

$$\mu(s) \le \mu(t)\,(1+|t-s|)^N$$

Hence

$$\mu(|x_1|)\,(1+|t-x_1|)^{-N} \le \mu(|t|)$$

For $x_1 \le 0$ we have

$$\mu(|x_1|) \le \mu(t), \quad \text{if } t \le x_1 \le 0$$

Hence from Lemma 2.4.7 it follows

$$(*) \quad \mu(x_1)\,\|u_{\lambda+i0}(x_1,\cdot)\|_{L^2}$$

$$\le C\int \|f(t,\cdot)\|_{L^2}\,\underbrace{(1+|t-x_1|)^{-N}\mu(x_1)}_{\le \mu(|t|)}\,dt$$

for $x_1 \le 0$. By using $u_{\lambda-i0}$ we see that $(*)$ is valid also for $x_1 \ge 0$. From the $L^1(L^2)$ and $L^\infty(L^2)$ estimates for B and B^* norms we get from $(*)$ that

$$\|\mu(|x_1|)u\|_{B^*} \le C\|\mu(x_1)f\|_{L^1(L^2)}$$

$$\le C\|\mu f\|_{L^1(L^2)} \le C\|\mu f\|_B$$

If $\varepsilon > 0$ is small then in X

$$\frac{\partial p}{\partial x_1} > 0 \Rightarrow \frac{\partial p}{\partial y_1} > 0$$

when $(y_1, y_2, \dots, y_n)$ are new coordinates

given $(y_1, y_2, \dots, y_n) = (x_1 + \varepsilon x_j, x_2, \dots, x_n)$

Clearly

$$|x| \le C \max\{|x_1|, |x_1 + \varepsilon x_2|, \dots, x_1 + \varepsilon x_n\}$$

thus

$$\mu(|x|) \le$$

$$C^N \max\{\mu(|x_1|), \mu(|x_1 + \varepsilon x_2|), \dots, \mu(|x_1 + \varepsilon x_n|)$$

Finally, for some j

$$\|\mu(|x|)u_\lambda\|_{B^*} \le C^N \|\mu(|x_1 + \varepsilon x_j|)\|_{B^*}$$

$$\le C\|\mu f\|_B$$

$\square$