KLUENTO I

In scattering theory, the fundamental solution

$$\mathcal{F}^{-1}\left(\frac{1}{\xi^2 + k^2 \pm i0}\right)$$

to the operator $H_0 - k^2 = -\Delta + k^2$ play a special role. Thus we study distributions

$$\lim_{\varepsilon \to 0^+} \frac{1}{p(\xi) \pm i\varepsilon} = \frac{1}{p \pm i0}$$

when p is a polynomial with real coeff. In Theorem 2.4.2 we proved the existence of $\frac{1}{p \pm i0}$ in the case when p has only simple zeros.

Preparations:

1. $p \in C^k(X)$, $X \Subset \mathbb{R}^n$

(later p will be a polynomial with real coefficients)

2. the zero's of p are simple $\Leftrightarrow$

$$p(\xi) = 0 \Rightarrow \nabla p(\xi) =: p'(\xi) \neq 0$$

3. We assume

$$\frac{\partial p}{\partial \xi_1} > 0 \qquad \text{at} \quad \xi = \xi_0 \in \underline{X}$$

Then

$$F : \xi \longmapsto (p(\xi), \xi'), \quad \xi' = (\xi_2, \dots, \xi_{n-1}$$

is a diffeo (C^k - diffeo)

$$F : \underset{\substack{\rotatebox{90}{$\in$} \\ \xi_0}}{X_0} \longrightarrow X_1 \times X \qquad \text{where}$$

$$\xi_0 \in X_0 \overset{\circ}{\subset} \underline{X}$$
$$p(\xi_0) \in X_1 \subset \mathbb{R}$$
$$\xi_0' \in X' \subset \mathbb{R}^{n-1}$$

by inverse function theorem.

The inverse of F is of the form

$$(\lambda, \xi') \longmapsto (\eta(\xi'; \lambda), \xi')$$

where η is C^k-map, $\eta : X_1 \times X' \to X_0$.

Thus

$$p(\eta(\xi'; \lambda), \xi') = \lambda$$

2.4.7 Theorem Assume

(i) $p \in C^2(X)$, $X \subset \mathbb{R}^n$ and $p' \neq 0$ in X

(ii) $f \in B$ and $\chi \in C_0^\infty(\mathbb{R}^n)$.

Then

$$z \mapsto u_z := F^{-1}\left((p-z)^{-1}\chi \hat{f}\right) \in B^*,$$

$\operatorname{Im} z \neq 0$ can be extended w^*-cont functions to $\mathbb{C}^\pm = \{z \mid \pm \operatorname{Im} z \geq 0\}$.

Moreover

(1)
$$\|u_z\|_{B^*} \leq C \|f\|_B$$

and for $\varphi \in C_0^0(\mathbb{R}^n)$

(2)
$$\lim_{R\to\infty} \frac{1}{R}\int |u_{\lambda \pm i0}(x)|^2 \varphi(x/R)\, dx$$

$$= (2\pi)^{1-n} \int_{p=\lambda} |\chi \hat{f}|^2 \left(\int_{\pm t>0} \varphi(tp')\,dt\right) \frac{ds}{|p'|}$$

and

(3)
$$\lim_{R\to\infty} \frac{1}{R}\int u_{\lambda+i0}(x)\, \overline{u_{\lambda-i0}(x)}\, \varphi(x/R)\, dx = 0$$

For the proof of Theorem 2.4.4 we need

Lemma 2.4.5 Assume $p \in C^{N+2}(X_0)$, $\chi \in C_0^\infty(X_0)$ and define

(1) $$\hat{u}_z = (p - z - i0)^{-1}(\chi \hat{f}), \quad \operatorname{Im} z \geq 0$$

for $f \in S(\mathbb{R}^n)$. Then with C indep of f

(2) $$\| u_z(x_1, \cdot) \| \leq$$

$$C \int \| f(t, \cdot) \|_{L^2} \, H(x_1 - t)_+ \, (1 + |t - x_1|)^{-N} \, dt$$

where H is the Heaviside function.

When $x_1 \to \infty$ we have

(3) $$\| u_\lambda(x_1, \cdot) - e^{i x_1 \eta(D', \lambda)} u_\infty(x', \lambda) \|_{L^2}$$

$$\to 0 .$$

Here $u_\infty(x', \lambda) = 0$, if $\lambda \notin X_0$ and

$$\hat{u}_\infty(\xi', \lambda) =$$

(4) $$= i \left\{ \frac{\chi \hat{f}}{\partial p / \partial \xi_1} \right\} (\eta(\xi', \lambda), \xi')$$

Proof We have

(i) u_z is analytic in $\{\Im z > 0\}$

and

(ii) $\| u_z(x_1, \cdot)\|_{L^2} \to 0$, as $z \to \infty$.

Note for $\Im z > 0$

$$u_z(x_1, x') = F^{-1}\left(\frac{1}{p(\xi) - z}(\chi \hat{f})\right)$$

$$\int d\xi' e^{ix'\xi'} \int e^{ix_1 \cdot \xi_1} \frac{1}{p(\xi) - x - iy} (\chi \hat{f})(\xi_1, \xi') \, d\xi_1$$

and

$$\| u_z(x_1, \cdot)\|_{L^2} = \left\| \int e^{ix_1\xi_1} \frac{\chi(\xi_1)}{p(\xi_1, \cdot) - x - iy} \hat{f}(\xi_1, \cdot) \right\|_{L^2}$$

$$\leq \frac{C}{\Im z} \int \| \chi(\xi_1, \cdot) \hat{f}(\xi_1, \cdot)\|_{L^2} \, d\xi_1$$

and thus (2) follows.

By maximum principle, (1) and (2) we need to show (2) only when

$$z = \lambda \in \mathbb{R}.$$

If $\lambda \in X$, we write

$$\frac{1}{p(\xi) - \lambda - i\varepsilon} = \frac{\xi_1 - \eta}{p - \lambda}\left[\frac{1}{\xi_1 - \eta - i\varepsilon\,\frac{\xi_1 - \eta}{p - \lambda}}\right]$$

$$= \frac{q(\xi)}{\xi_1 - \eta - i\varepsilon\, q(\xi)}$$

where

$$q(\xi) = \frac{\xi_1 - \eta(\xi', \lambda)}{p(\xi) - \lambda}$$

$$= \frac{\xi_1 - \eta}{p(\xi) - p(\eta)} > 0$$

since $\frac{\partial p}{\partial \xi_1} > 0$ in X_0.

Thus

$$\hat{u}_\lambda(\xi) = \frac{g(\xi)\hat{f}(\xi)}{\xi_1 - \eta(\xi', \lambda) + i0}$$

(c.f. Lemma 6.2.2. in Hörmander I).

We need to calculate the inverse Fourier transform w.r.t. ξ_1. We write

$$\frac{g(\xi)}{\xi_1 - \eta + i0} = \frac{g(\eta, \xi')}{\xi_1 - \eta + i0} + \underbrace{\frac{g(\xi_1, \xi') - g(\eta, \xi')}{\xi_1 - \eta}}_{G},$$

where

$$(*) \quad \mathcal{F}_1^{-1}\frac{g(\eta, \xi')}{\xi_1 - \eta + i0} = i g(\eta, \xi') e^{i x_1 \eta} H(x_1) \in L^\infty$$

($\mathcal{F}_1$ is Fourier transform w.r.t. x_1)

Now $G \in C^N$ and it is thus easy to see that

$$K(x_1, \xi') = \mathcal{F}_1^{-1} G(\cdot, \xi')(x_1)$$

satisfies by Riemann - Lebesgue

$$(1+|x_1|^N) |K(x_1, \xi')| \to 0 \quad \text{as}$$

$x_1 \to \infty$.

Let now

$$\mathcal{F}_{x'} u_\lambda(x_1, \cdot) = U(x_1, \xi')$$

$$\mathcal{F}_{x'} f(x_1, \cdot) = F(x_1, \xi')$$

Then by convolution theorem

$$U(x_1, \xi') =$$

$$\int F(t, \xi') \left(i g(\eta, \xi') e^{i(x_1 - t)\eta(\xi', \lambda)} H(x_1 - t) \right.$$

$$+ \int F(t, \xi') K(x_1 - t, \xi') \, dt$$

Thus

$$\| u_\lambda(x_1, \cdot) \|_{L^2} = \| U(x_1, \cdot) \|_{L^2}$$

$$\leq C \int \|F(t,\cdot)\|_{L^2} \left(H(x_1 - t) - (1+|t_1 - x_1|)^{-N} \right) dt$$

and (2) follows for $z = \lambda$.

If $p(\xi) \neq \lambda$ in supp χ, then

$$\frac{\chi}{p - \lambda} \in C_0^{N+2}$$

and (2) is true with $H = 0$ and N replaced by $N+2$. Since

$$\left\| \frac{\chi}{p-\lambda} \right\|_{C^{N+2}} \to 0 \quad \text{as } \lambda \to \infty$$

we see that the constant C in (2) sat.

$$C = C(\lambda) \to 0, \quad \text{as } \lambda \to \infty$$

Thus (2) holds for every $\lambda \in \overline{\mathbb{C}}_+$.

Moreover

(K) $$\int F(t,\xi')\, i g(\eta,\xi') \overbrace{H(x_1 - t)}^{e^{-it\eta}} dt$$

$$\to i g(\eta,\xi') \hat{f}(\eta,\xi')$$

by dominated convergence.

Finally

$$\mathcal{F}_{x'}\left[u_\lambda(x_1,\cdot) - e^{i x_1 \eta(\cdot,\lambda)} u_\infty(\cdot,\lambda)\right]$$

$$= \int F(t,\cdot)\, i g(\eta,\cdot)\, e^{i(x_1-t)\eta} H(x_1-t)\, dt$$

$$- e^{i x_1 \eta(\cdot,\lambda)}\, \hat{u}_\infty(\cdot,\lambda)$$

$$= \int F(t,\cdot)\, g(\eta,\cdot)\, H(x_1-t)\, e^{-it\eta}$$

$$- \frac{\hat{f}\chi}{\partial p/\partial \xi_1}(\eta,\xi')$$

But

$$g(\eta,\xi') = \frac{\chi}{\partial p/\partial \xi_1}(\eta,\xi')$$

and (3) follows. $\square$