Scattering theory
Exercises #8, 16.11.2007
(return by 23.11.2007)

1. Let B_1, B_2 be Banach spaces. Show that for any $T \in L(B_1, B_2)$ there is a unique operator $T^* \in L(B_2^*, B_1^*)$ defined by
$$\langle Tx, \eta \rangle = \langle x, T^* \eta \rangle, \quad x \in B_1, \eta \in B_2^*.$$
Show that the map $L(B_1, B_2) \to L(B_2^*, B_1^*), T \mapsto T^*$, is a linear isometry, which is surjective if B_2 is reflexive.

2. Let W be a closed linear subspace of a Banach space B, and define its annihilator W° by
$$W^\circ = \{\xi \in B^* \, ; \, \langle x, \xi \rangle = 0 \text{ for all } x \in W\}.$$
Show that W° is a closed linear subspace of B^*. Also, show that W^* can be isometrically identified with B^*/W°, and that W° can be identified with $(B/W)^*$.

3. Let $T \in L(B_1, B_2)$ have closed range. Show that T^* has closed range and that $\ker(T)^\circ = \operatorname{im}(T^*)$, $\operatorname{im}(T)^\circ = \ker(T^*)$.

4. Let B and B^* be the Banach spaces in the lectures. Define for $\delta \in \mathbf{R}$
$$L^2_\delta = \{f \in L^2_{\mathrm{loc}}(\mathbf{R}^n) \, ; \, \|f\|_{L^2_\delta} := \|(1 + |x|^2)^{\delta/2} f\|_{L^2} < \infty\}.$$
Show the inclusions
$$\bigcap_{\delta > 1/2} L^2_\delta \subset B, \qquad B^* \subset \bigcap_{\delta < -1/2} L^2_\delta.$$