Scattering theory
Exercises #6, 29.10.2007
(return by 09.11.2007)

1. Recall that an operator $T \in L(B_1, B_2)$, where B_1 and B_2 are Banach spaces, is called Fredholm if $\ker(T)$ and $\operatorname{coker}(T) := B_2/\operatorname{im}(T)$ are finite dimensional. Show that any Fredholm operator has closed range, and that T_2T_1 is Fredholm whenever $T_1 \in L(B_1, B_2)$ and $T_2 \in L(B_2, B_3)$ are Fredholm.

2. Prove the Fredholm part of Theorem 1.8.2 in the lectures: If B_1, B_2 are Banach spaces, $T \in L(B_1, B_2)$ is Fredholm, and $S \in L(B_1, B_2)$ with $\|S\|$ sufficiently small, show that $T + S$ is Fredholm with
$$\operatorname{ind}(T+S) = \operatorname{ind} T,$$
$$\dim \ker(T+S) \leq \dim \ker(T).$$

3. Prove that as distributions on $\mathbf{R}$, one has the identity
$$\frac{1}{x - i0} - \frac{1}{x + i0} = 2\pi i \delta_0.$$
If A is a self-adjoint operator on H and $\varphi \in C_c(\mathbf{R})$, show that this implies the following variant of Stone's formula:
$$\varphi(A) = \lim_{\varepsilon \to 0+} \frac{1}{2\pi i} \int_{\mathbf{R}} [R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon)]\varphi(\lambda)\, d\lambda.$$

4. Let $V = V_1 + V_2$ where $V_1 \in L^\infty(\mathbf{R}^3)$ and $V_2 \in L^2(\mathbf{R}^3)$ are real. Show that $H : u \mapsto (-\Delta + V)u$ with domain $\mathscr{D}(H) = H^2(\mathbf{R}^3) \subset L^2(\mathbf{R}^3)$ is self-adjoint. (You may assume that $-\Delta$ is self-adjoint on $H^2(\mathbf{R}^3)$, proof is the same as in Ex. 3, Problem 3. Use the Fourier transform to show that $\forall\, \varepsilon > 0\ \exists\, t > 0 : \|(-\Delta + it)^{-1} f\|_{L^\infty} \leq \varepsilon \|f\|_{L^2}$, and use Kato's theorem.)