Scattering theory
Exercises #5, 12.10.2007
(return by 19.10.2007)

Let A be a self-adjoint operator in a Hilbert space H. In Exercises 4, the discrete spectrum $\sigma_{\mathrm{d}}(A)$ was defined as the set of isolated eigenvalues with finite multiplicity. The *essential spectrum* is $\sigma_{\mathrm{ess}}(A) = \sigma(A) \setminus \sigma_{\mathrm{d}}(A)$. The purpose of these exercises is to prove Weyl's criterion: $\lambda \in \sigma_{\mathrm{ess}}(A)$ iff there is a sequence $(u_j) \subset \mathscr{D}(A)$, $\|u_j\| = 1$, such that $u_j \to 0$ weakly and $\|(A - \lambda)u_j\| \to 0$. (Such a sequence is called a *Weyl sequence*.)

1. Assuming Weyl's criterion, show that $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(A + K)$ for any self-adjoint compact operator K on H. (The stability of $\sigma_{\mathrm{ess}}(A)$ is a major reason why the essential spectrum is interesting.)
2. If $\lambda \in \sigma_{\mathrm{ess}}(A)$ and dim ker$(A-\lambda) = \infty$, show that there is an orthonormal Weyl sequence.
3. If $\lambda \in \sigma_{\mathrm{ess}}(A)$ and dim ker$(A-\lambda) < \infty$, show that $A_\lambda = A-\lambda|_{\mathscr{D}(A)\cap\ker(A-\lambda)^\perp}$ is injective with dense range but A_λ^{-1} is not bounded (use the previous Exercises), and find a Weyl sequence.
4. If there is a Weyl sequence for λ, show that dim ker$(A - \lambda) = \infty$ or A_λ^{-1} is unbounded, and prove that $\lambda \in \sigma_{\mathrm{ess}}(A)$.