Scattering theory
Exercises #1, 14.9.2007
(return by 21.9.2007)

The exercises will be posted on the course webpage by Friday, and the return date is the next Friday. The exercises may be returned by email (mikko.salo 'at' helsinki.fi), to the box in the mail room C334, or at the office B414. I would be happy to answer any questions, or give hints if needed.

In the following, all operators will be densely defined unbounded operators on a Hilbert space $(H, \|\cdot\|)$.

1. Show that $A : \mathscr{D}(A) \to H$ is closed if and only if $(\mathscr{D}(A), \|\cdot\|_A)$ is a Hilbert space, where $\|u\|_A^2 = \|u\|^2 + \|Au\|^2$.

2. Let $A : \mathscr{D}(A) \to H$ and $B : \mathscr{D}(A) \to H$ be unbounded operators. Prove the following.

   (i) The adjoint A^* is closed.

   (ii) If $A \subset B$, then $B^* \subset A^*$.

3. If $A : \mathscr{D}(A) \to H$, show that A is symmetric if and only if (Ax, x) is real for all $x \in \mathscr{D}(A)$.

4. (i) If $A : \mathscr{D}(A) \to H$ is closable (i.e. it has a closed extension), show that there is a unique closed operator $\bar{A} \supset A$ such that $\bar{A} \subset B$ for any closed $B \supset A$. (Hint: try to define $\bar{A}$ with domain $\mathscr{D}(\bar{A}) = \{x \in H\,;\, (x,y) \in \overline{\mathscr{G}(A)} \text{ for some } y \in H\}$, where $\mathscr{G}(A)$ is the graph of A.)

   (ii) Let $A : \mathscr{D}(A) \to H$ be symmetric. Show that A is closable, and that the closure $\bar{A}$ is also symmetric. Show that A has a self-adjoint extension if and only if $\bar{A}$ has one.

5. Let $H = L^2(I)$, $I = (-1, 1)$, and define $A : u \mapsto u''$ with domain $\mathscr{D}(A) = C_0^2(I)$ (compactly supported C^2 functions on I). Show that A is symmetric. Compute A^* and $\mathscr{D}(A^*)$. Find a self-adjoint extension of A by choosing a domain with suitable boundary conditions.