# MATS315 - Fourier Analysis

Exercises #3, 04.10.2024

(return by 24.10.2024)

Each exercise is graded 0/1/2 points. The exercises can be returned to Mikko during the lectures, to Hjørdis during the exercise sessions, or by email (hjordis.a.schluter@jyu.fi).

1. Let $f \in \mathscr{S}(\mathbb{R}^2)$. The *X-ray transform* of $f$ is the function $Xf : [0, 2\pi) \times \mathbb{R} \to \mathbb{C}$, defined as follows. For $\theta \in [0, 2\pi)$ and $t \in \mathbb{R}$, write $e_\theta := (\cos\theta, \sin\theta)$, and let

$$\ell_\theta(t) := \{x \in \mathbb{R}^2 \, : \, x \cdot e_\theta = t\}$$

be the line perpendicular to $e_\theta$ at "height $t$". Define $[Xf](\theta, t)$ to be the line integral of $f$ over $\ell_\theta(t)$:

$$[Xf](\theta, t) := \int_{\ell_\theta(t)} f(s) \, \mathrm{d}s, \quad (\theta, t) \in [0, 2\pi) \times \mathbb{R}.$$

The function $t \mapsto [Xf](\theta, t) =: [Xf]_\theta(t)$ represents an X-ray image of the density function $f$.

(a) Show that $[Xf]_\theta \in \mathscr{S}$ for every $\theta \in [0, 2\pi)$, and $\widehat{[Xf]_\theta}(\xi) = \hat{f}(\xi e_\theta)$ for all $\xi \in \mathbb{R}$.

(b) Use the formula from (a) and the Fourier inversion theorem, to find the following *inversion formula for the X-ray transform*:

$$f(x) = \frac{1}{2} \int_0^{2\pi} \widehat{F_\theta}(x \cdot e_\theta) \, \mathrm{d}\theta, \quad x \in \mathbb{R}^2. \tag{1}$$

Here $F_\theta \in L^1(\mathbb{R})$ is defined by $F_\theta(\xi) := |\xi| \cdot \widehat{[Xf]_\theta}(\xi)$. (Hint: Integrate using polar coordinates $\int_{\mathbb{R}^2} g(x) \, \mathrm{d}x = \int_0^{2\pi} \int_0^\infty r \cdot g(re_\theta) \, \mathrm{d}r \, \mathrm{d}\theta$)

*The point of the formula (1) is that it allows you to reconstruct $f$ if you only know the values of $Xf$. This is the mathematical theorem behind CT-scanners.*

2. Below, $T \in \mathscr{S}'$ is a tempered distribution.

   (a) Let $x \in \mathbb{R}^d$, write $e_x(y) = e^{ix\cdot y}$, and define the linear maps $e_xT$, $\tau_xT : \mathscr{S} \to \mathbb{C}$ by

   $$[e_xT](\phi) := T(e_x\phi) \quad \text{and} \quad [\tau_xT](\phi) := T(\tau_{-x}\phi), \quad \phi \in \mathscr{S}.$$

   where $(\tau_x\phi)(y) := \phi(y-x)$. Show that $e_xT, \tau_xT \in \mathscr{S}', \widehat{\tau_xT} = e_{-x}\hat{T}$, and $\widehat{e_xT} = \tau_x\hat{T}$.

   (b) Let $P(x) = \sum_{n=-N}^{N} c_n e^{inx}$ be a trigonometric polynomial on $\mathbb{R}$, $N \geq 0, c_n \in \mathbb{C}$. Check that $T_P$ is a tempered distribution, and compute the Fourier transform $\widehat{T_P} \in \mathscr{S}'$.

3. Let $\{\psi_\delta\}_{\delta>0}$ be an approximate identity of the form $\psi_\delta := \delta^{-d}\psi(x/d)$, where $\psi \in C^\infty(\mathbb{R}^d)$ is a fixed, non-negative, compactly supported function with $\int \psi(x)\,\mathrm{d}x = 1$. Prove that if $\phi \in \mathscr{S}$, then $\phi * \psi_\delta \to \phi$ in $(\mathscr{S}, d)$. (Hint: It may be more comfortable to prove the convergence of Fourier transforms in $(\mathscr{S}, d)$. Why is this good enough?)

4. Let $T \in \mathscr{S}'$, and $\phi, \psi \in \mathscr{S}$.

   (a) Show that $(T * \phi)(\psi) = T(\tilde{\phi} * \psi)$, where $\tilde{\phi}(x) = \phi(-x)$.

   (b) Show that $(T * \phi) * \psi = T * (\phi * \psi)$.

   (c) Show that if $\{\psi_\delta\}_{\delta>0}$ is an approximate identity as in the previous exercise, and $T_\delta := T * \psi_\delta \in C^\infty(\mathbb{R}^d)$, then $T_\delta(\phi) \to T(\phi)$ for all $\phi \in \mathscr{S}$.

   The convergence encountered in exercise 3(c) is called *weak* convergence of tempered distributions*. To be precise, a sequence $\{T_j\}_{j\in\mathbb{N}} \subset \mathscr{S}'$ *weak* converges to $T \in \mathscr{S}'$ in $\mathscr{S}'$* if

   $$\lim_{j\to\infty} T_j(\phi) = T(\phi), \quad \phi \in \mathscr{S}.$$

   Weak* convergence is denoted by $T_j \rightharpoonup T$.

5. (a) Prove that weak* convergence is weaker than $L^p$-convergence for any $1 \leq p \leq \infty$: If $f_j \in L^p(\mathbb{R}^d)$, and $f_j \to f \in L^p$ in the $L^p$-norm, then $T_{f_j} \rightharpoonup T_f$.

(b) Prove by example that weak* convergence is strictly weaker than convergence in the $L^p$-norm: Find a diverging sequence $\{f_j\}_{j\in\mathbb{N}} \subset L^p(\mathbb{R}^d)$ such that $T_{f_j} \rightharpoonup 0$.

(c) Prove that $T_j \rightharpoonup T$ if and only if $\widehat{T_j} \rightharpoonup \hat{T}$.

6. Let $f \in L^p(\mathbb{R}^d), 1 \le p \le \infty$, and consider the functions $g_R : \mathbb{R}^d \to \mathbb{C}$,

$$g_R(\xi) := \int_{B(0,R)} f(x) e^{-ix\cdot\xi}\,\mathrm{d}x, \quad \xi \in \mathbb{R}^d.$$

Show that $T_R := T_{g_R} \in \mathscr{S}'$, and $T_R \rightharpoonup \widehat{T_f}$ as $R \to \infty$.

7. Find the Fourier transform of the tempered distribution $T(f) = (\frac{d^n}{dx^n}\delta)(f)$.

8. Let $a > 0$. With $H(t)$ denoting the Heaviside function, show that

$$\mathscr{F}\left[H(t)e^{-at}\right] = \frac{1}{a + i\xi}.$$

What is $\mathscr{F}[H(-t)e^{at}]$?

9. (a) Show that for $\xi \in \mathbb{R}$,

$$\mathscr{F}^{-1}\left[\frac{1}{1+\xi^2}\right] = ce^{-|x|};$$

determine $c$. (One can use the previous exercise).

(b) Show that for $\xi \in \mathbb{R}^3$

$$\mathscr{F}^{-1}\left[\frac{1}{1+|\xi|^2}\right] = \frac{c}{|x|}e^{-|x|};$$

with $c = \frac{1}{4\pi}$. (One may observe that the function is the unique solution $v$ in $\mathscr{S}'$ of $(1-\Delta)v = \delta$).

10. Let $a \in \mathbb{C}$ and show that the distribution $u = e^{-ax}H(x)$ is a solution of the differential equation

$$(\partial_x + a)u = \delta \quad \text{in } \mathscr{D}'(\mathbb{R}).$$

Can we show this by Fourier transformation?