MATS315 - Fourier Analysis

Exercises #2, 20.09.2024

(return by 04.10.2024)

Each exercise is graded 0/1/2 points. The exercises can be returned to Mikko during the lectures, to Hjørdis during the exercise sessions, or by email (hjordis.a.schluter@jyu.fi).

1. Prove Lemma 2.4.6.

2. Show that any $f \in L^2(\mathbb{T}^n)$ may be expressed as

$$f = (1 - \Delta)^N u$$

for a continuous 2π-periodic function u and some N. Here $\Delta = \sum_{j=1}^n \partial_{x_j}^2$.

3. (Poincaré inequality) If u is a C^1 function in $\mathbb{R}^n$ with period 2π satisfying $\int_Q u(x)\,dx = 0$, show that

$$\int_Q |u(x)|^2\,dx \leq \sum_{j=1}^n \int_Q |\partial_j u(x)|^2\,dx$$

where $Q = [-\pi, \pi]^n$. When does equality hold?

4. For regions in $\mathbb{R}^2$ bounded by simple closed C^1 curves, show that equality holds in the isoperimetric inequality iff the curve is a circle.

5. (Riemann zeta values) If $\zeta(s) = \sum_{n=1}^\infty n^{-s}$ is the Riemann zeta function defined for $s > 1$, show that

$$\zeta(2m) = \frac{(-1)^{m+1}(2\pi)^{2m}B_{2m}}{2(2m)!}, \qquad m \in \mathbb{Z}_+,$$

where the Bernoulli numbers are defined via

$$\frac{z}{e^z - 1} = \sum_{n=0}^\infty \frac{B_n}{n!} z^n, \qquad |\mathrm{Im}\,(z)| < 2\pi.$$

6. (Weyl's equidistribution criterion) Show that a sequence $(x_n)_{n=1}^{\infty} \subset [0,1)$ is equidistributed iff for any integer $k \neq 0$,
$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k x_n} \to 0$$
as $N \to \infty$.

7. (Eigenfunctions of Laplacian on the torus) If u is a nonzero L^2 function on the torus satisfying $-\Delta u = \lambda u$ in the weak sense for some $\lambda > 0$, show that $\lambda = |k|^2$ for some $k \in \mathbb{Z}^n$ and describe all L^2 solutions u for this λ.

8. (Exercise 12 from exercise sheet 1) Let X be a vector space and let $\{\rho_N\}_{N=0}^{\infty}$ be a countable separating family of seminorms. Show that the function
$$d(u,v) = \sum_{N=0}^{\infty} 2^{-N} \frac{\rho_N(u-v)}{1+\rho_N(u-v)}, \qquad u, v \in X,$$
is a metric on X, and that $u_j \to u$ in (X,d) iff for any N one has
$$\rho_N(u_j - u) \to 0.$$

9. Show that there exists a linear functional $T : \mathscr{S} \to \mathbb{C}$ which is not continuous.

10. Let $u(x) = 0$ for $-\infty < x < 0$ and $u(x) = x$ for $0 < x < \infty$. Compute the distributional derivatives u' and u''.

11. Prove that for each fixed number of $a \in (0,\infty)$ the function $f(x) = e^{-a|x|^2}$ ($x \in \mathbb{R}^n$) belongs to $\mathscr{S}(\mathbb{R}^n)$. Therefore $C_c^\infty(\mathbb{R}^n) \subsetneq \mathscr{S}(\mathbb{R}^n) \subsetneq C^\infty(\mathbb{R}^n)$. (Note: $e^{-|x|}$ is not in the Schwartz space since it is not C^∞ near the origin.)

12. Prove that the function $f(x) := e^{i|x|^2}$ ($x \in \mathbb{R}^n$) belongs to $\mathscr{O}_M(\mathbb{R}^n)$ and that for each $s \in \mathbb{R}$ the function $f(x) := \langle x \rangle^s$ ($x \in \mathbb{R}^n$) belongs to $\mathscr{O}_M(\mathbb{R}^n)$.

13. (Poisson kernel revisited) Let $P_t(x) = c_n \frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}}$ ($t > 0, x \in \mathbb{R}^n$) be the Poisson kernel where $c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}$. We say that f is a bounded tempered distribution if $f * \phi \in L^\infty(\mathbb{R}^n)$ for every Schwartz function ϕ. Show that for any bounded tempered distribution f we have $P_t * f \to f$ in $\mathscr{S}'(\mathbb{R}^n)$ as $t \to 0$.
Hint: Fix a smooth function ϕ whose Fourier transform is equal to 1 in a neighborhood of zero. Show that $P_t * (\phi * f) \to \phi * f$ in $\mathscr{S}'(\mathbb{R}^n)$ and that $\hat{P}_t(1-\hat{\phi})\hat{f} \to (1-\hat{\phi})\hat{f}$ in $\mathscr{S}'(\mathbb{R}^n)$ as $t \to 0$.