MATS315 - Fourier Analysis

Exercises #1, 06.09.2024

(return by 20.09.2024)

Each exercise is graded 0/1/2 points. The exercises can be returned to Mikko during the lectures, to Hjørdis during the exercise sessions, or by email (hjordis.a.schluter@jyu.fi).

1. Consider two ways of writing Fourier series

$$f(x) = \sum_{k=0}^{\infty} (a_k \cos(kx) + b_k \sin(kx)), \qquad f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}.$$

   Express c_k in terms of a_k, b_k and vice versa.

2. Let f, g be smooth 2π-periodic functions, and consider a periodic solution of the wave equation

$$\begin{aligned} \partial_t^2 u(x,t) - \partial_x^2 u(x,t) &= 0 && \text{in } \mathbb{R} \times \{t > 0\}, \\ u(x,0) &= f(x) && \text{for } x \in \mathbb{R}, \\ \partial_t u(x,0) &= g(x) && \text{for } x \in \mathbb{R}. \end{aligned}$$

   Find a formal expansion for $u(x,t)$ in terms of Fourier expansions of f and g.

3. If $f \in L^1([-\pi,\pi])$ and if $P(x) = \sum_{k=-N}^{N} c_k e^{ikx}$ is a trigonometric polynomial, show that $P * f$ is a trigonometric polynomial.

4. Show that there exists a continuous periodic function whose Fourier series diverges at a point.

5. Let $f \in L^2([-\pi,\pi])$. Given N, consider all functions of the form $f_N(x) = \sum_{k=-N}^{N} c_k e^{ikx}$ with $c_k \in \mathbb{C}$. Show that there is a unique such f_N that best approximates f in the sense that $\|f - f_N\|_{L^2}$ is minimal.

6. If $f \in L^1([-\pi,\pi])$ and $\hat{f}(k) = 0$ for all k, show that $f = 0$.

7. Show that the Fejér kernel $F_N(x) = \frac{1}{N+1} \frac{\sin^2(\frac{N+1}{2}x)}{\sin^2(\frac{1}{2}x)}$ is an approximate identity.

8. (Abel summability of Fourier series) If $f \in L^1([-\pi,\pi])$, show that for any $r < 1$

$$u(re^{i\theta}) := \sum_{k=-\infty}^{\infty} \hat{f}(k) r^{|k|} e^{ik\theta} = (P_r * f)(\theta)$$

where P_r is given by

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}, \qquad 0 \le r < 1, \quad \theta \in [-\pi, \pi].$$

(P_r is called the *Poisson kernel* in the unit disc.) Define the *Abel means* of the Fourier series

$$A_r f := P_r * f, \qquad 0 \le r < 1.$$

Show that P_r is an approximate identity. If $f \in L^p([-\pi,\pi])$, $1 \le p < \infty$ or if f is continuous 2π-periodic and $p = \infty$, show that the Abel means $A_r f$ converge to f in L^p as $r \to 1$.

9. (Jordan's criterion) If $f \in L^1([-\pi,\pi])$ is a function of bounded variation (=difference of two monotonic functions) near x, show that

$$\lim_{m\to\infty} S_m f(x) = \frac{1}{2}\left[f(x-) + f(x+)\right].$$

10. If $u, v, w \in \mathscr{P}$, show that $u * v = v * u$ and $u * (v * w) = (u * v) * w$.

11. If $f \in \mathscr{P}$, show that the Fourier transform on $\mathscr{P}$ has the following properties:

$$\begin{aligned}
&(1) \quad (\tau_{x_0} u)\hat{}\,(k) = e^{-ik\cdot x_0}\hat{u}(k) && \text{(translation)}\\
&(2) \quad (e^{ik_0\cdot x} u)\hat{}\,(k) = \tau_{k_0}\hat{u}(k) && \text{(modulation)}\\
&(3) \quad (u * v)\hat{}\,(k) = \hat{u}(k)\hat{v}(k) && \text{(convolution)}\\
&(4) \quad (fu)\hat{}\,(k) = (\hat{f} * \hat{u})(k) && \text{(product)}
\end{aligned}$$

12. Let X be a vector space and let $\{\rho_N\}_{N=0}^{\infty}$ be a countable separating family of seminorms. Show that the function

$$d(u,v) = \sum_{N=0}^{\infty} 2^{-N} \frac{\rho_N(u-v)}{1 + \rho_N(u-v)}, \qquad u, v \in X,$$

is a metric on X, and that $u_j \to u$ in (X, d) iff for any N one has

$$\rho_N(u_j - u) \to 0.$$

13. If $s \in \mathbb{R}$, show that the series

$$\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{-s}$$

converges iff $s > n$.

14. (Poisson summation formula) If $f \in C^\infty(\mathbb{R}^n)$ has compact support and if $a > 0$, show that

$$\sum_{k \in \mathbb{Z}^n} \hat{f}(ak) = (2\pi/a)^n \sum_{k \in \mathbb{Z}^n} f((2\pi/a)k)$$

where we define $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} \, dx$ for $\xi \in \mathbb{R}^n$.