Complex analysis 2
Exercises 6

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Exercise 1

Let $D \neq \mathbb{C}$ be a simply connected domain and let $f\colon \mathbb{C} \to D$ be analytic. Show that f is a constant function.

Solution:

Since D is simply connected, Theorem 4.19(Riemann mapping theorem) ensure the existence of an analytic bijection $g\colon D \to B(0,1)$. Then $g \circ f\colon \mathbb{C} \to B(0,1)$ is everywhere analytic(entire). By Liouville's theorem (Theorem 5.3.4 in the notes from Complex analysis 1), $g \circ f$ is constant, so $g(f(z)) = g(f(w))$ for all $z, w \in \mathbb{C}$. Since g is a bijection, $g(f(z)) = g(f(w))$ implies $f(z) = f(w)$ so also f is constant. ■

Exercise 2

Show that $\mathcal{F} = \{e^{nz}\colon n \geq 1\}$ is a normal family in $G = \{z\colon \operatorname{Re}(z) < 0\}$ but not in any domain larger than G.

Solution:

As with many compactness questions in function spaces, we could use Arzelà-Ascoli (Theorem 4.23). But it turns out to be easier to use Montel's theorem (Theorem 4.25), which of course is using Arzelà-Ascoli behind the scenes.

Let's first show that G is the largest possible domain where $\mathcal{F}$ is normal. By the "only if" part of Montel's theorem, we see that $\mathcal{F}$ cannot be normal if the family is unbounded. But for z with $\operatorname{Re}(z) > 0$,

$$|e^{nz}| = e^{n \operatorname{Re}(z)}$$

which is unbounded in $n \in \mathbb{N}$. So $\mathcal{F}$ cannot be normal in $\{\text{Re}(z) > 0\}$. For z with $\text{Re}(z) = 0$ we instead have $|e^{nz}| = 1$ for all n, so we have do deal with this set in a different manner. But by the definition of normal families(Definition 4.22), the concept of normal family only make sense on open sets(domains to be precise). So if $\mathcal{F}$ is normal on a set which contains part of $\{\text{Re}(z) = 0\}$ then it must by continuity also be normal on at least a small subset of $\{\text{Re}(z) > 0\}$ but we saw above that this is not possible. So $\mathcal{F}$ cannot be normal on any set which intersects $\mathbb{C} \setminus G$.

To see that $\mathcal{F}$ is normal on G all we need to do is verify that $\mathcal{F}$ is bounded on arbitrary compact subsets of G. Then the result follows from Montel's theorem. Let K be any such compact subset. Then we have for any $z \in K$ and $n \in \mathbb{N}$ that

$$|e^{nz}| = e^{n\,\text{Re}(z)} \leq e^{\text{Re}(z)}.$$

Since K is compact, there exist $z_0 \in K$ which maximizes $K \ni z \mapsto e^{\text{Re}(z)}$. Then

$$|e^{nz}| \leq e^{\text{Re}(z_0)}$$

and the result follows from Montel's theorem. ■

EXERCISE 3

Let $D \subset \mathbb{C}$ be an unbounded simply connected domain. Without using the Riemann mapping theorem, show that D can be conformally mapped onto a bounded domain if

i) $\mathbb{C} \setminus D$ contains a ball, or

ii) $D \neq \mathbb{C}$.

SOLUTION:

We start with part i), so let $z_0 \in \mathbb{C} \setminus D$ and $r_0 > 0$ be such that $B(z_0, r_0) \subset \mathbb{C} \setminus D$. Then $|z - z_0| \geq r_0$ for all $z \in D$ and it follows that $f(z) := \frac{1}{z-z_0}$ maps D to $B(0, \frac{1}{r_0})$. So f maps D to a bounded set and f is analytic in D since $z_0 \notin D$. Furthermore, f is conformal(by Theorem 4.4) since $f'(z) = -\frac{1}{(z-z_0)^2} \neq 0$ in D. So f maps D conformally to a bounded set.

Now let's prove part ii), and we first show that we may without loss of generality assume that $0 \notin D$. Let $z_0 \in \mathbb{C} \setminus D$ be arbitrary and define $f(z) = z - z_0$. Then f is conformal since it is analytic and $f'(z) \neq 0$. f therefore maps D conformally onto the set $f(D)$ which contains $0 = f(z_0)$. So the case $0 \neq z_0 \in \mathbb{C} \setminus D$ is reduced to the case

$0 \in \mathbb{C} \setminus D$ and we may therefore assume without loss of generality that $0 \in \mathbb{C} \setminus D$.

Assume that $0 \notin D$. I want to find an analytic branch of the square root in D. Let $h(z) = z^2$. Then h is conformal in D since it is analytic and $h'(z) \neq 0$ in D. Then the image $h(D)$ is simply connected, by Theorem 4.20. Since $h(D)$ is simply connected and does not contain 0, Theorem 1.15 ensures existence of an analytic branch of the logarithm in $h(D)$, that is, there exist

$$\operatorname{Log}_{h(D)} \colon h(D) \to \mathbb{C}$$

which is analytic and satisfies

$$e^{\operatorname{Log}_{h(D)}(z)} = z \quad \forall z \in h(D).$$

Then

$$g(z) := e^{\frac{1}{2}\operatorname{Log}_{h(D)}(z)}$$

is an analytic function satisfying

$$g(z)^2 = z \quad \forall z \in h(D).$$

That is, $g \colon h(D) \to \mathbb{C}$ is an analytic branch of the square root. Also note that $g'(z) \neq 0$, since both exponential and logarithm have nonzero derivatives, which implies that g is conformal. Now $g \circ h$ maps D conformally to $g(h(D))$. If $\mathbb{C} \setminus g(h(D))$ contains a ball then we're done, after invoking part i).

Now let $w \in g(h(D))$ be any nonzero element. I want to conclude that $-w \notin g(h(D))$. Suppose to the contrary that $-w \in g(h(D))$. Then there exist unique $z_w, z_{-w} \in h(D)$ such that

$$g(z_w) = w$$
$$g(z_w)^2 = z_w,$$

and

$$g(z_{-w}) = -w$$
$$g(z_{-w})^2 = z_{-w}.$$

But from this we get

$$z_w = g(z_w)^2 = w^2 = (-w)^2 = g(z_{-w})^2 = z_{-w}.$$

This in turn gives

$$-w = g(z_{-w}) = g(z_w) = w,$$

which contradicts $w \neq 0$. So we conclude that if $w \in g(h(D))$ is nonzero then $-w \notin g(h(D))$. Since $g(h(D))$ is open, there exist a ball $B(z_0, r) \subseteq g(h(D))$ around any $z_0 \in g(h(D))$. Then $B(-z_0, r) \nsubseteq g(h(D))$. In other words, $B(-z_0, r) \subseteq \mathbb{C} \setminus g(h(D))$. So $g \circ h$ maps D conformally into a set whose complement contains a ball. By composing $g \circ h$ with the conformal map in part i) we can conformally map D into a bounded set.

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EXERCISE 4

Fix $a \in B(0,1)$ and let $\varphi_a \colon B(0,1) \to \mathbb{C}$, $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$. Prove that

$$\varphi_a(B(0,1)) \subset B(0,1).$$

Moreover, prove that $\varphi_a \colon B(0,1) \to B(0,1)$ is a conformal bijection whose inverse function is $\varphi_a^{-1}(z) = \varphi_{-a}(z)$.

SOLUTION:

Proving that $\varphi_a(B(0,1)) \subseteq B(0,1)$ was done in Exercise 6 in the first exercises in Complex analysis 1, see the model solutions. Theorem 5.16(with $\lambda = 1$ and $z_0 = a$) says that φ_a is a conformal bijection. So we only need to verify that $\varphi_a^{-1} = \varphi_{-a}$:

$$\begin{aligned}
\varphi_a(\varphi_{-a}(z)) &= \frac{\varphi_{-a}(z) - a}{1 - \bar{a}\varphi_{-a}(z)} \\
&= \frac{\frac{z+a}{1+\bar{a}z} - a}{1 - \bar{a}\frac{z+a}{1+\bar{a}z}} \\
&= \frac{z + a - a - |a|^2 z}{1 + \bar{a}z - \bar{a}z - |a|^2} \\
&= \frac{z - |a|^2 z}{1 - |a|^2} \\
&= z.
\end{aligned}$$

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Exercise 5

Find the image of the upper half plane $\{z \in \mathbb{C} \colon \operatorname{Im}(z) \geq 0\} \cup \{\infty\}$ under the Möbius transformation $f(z) = \frac{z-i}{z+i}$.

Solution:

Let's write as in Theorem 5.6,

$$f(z) = -2i\frac{1}{z+i} + 1.$$

Note that the map $re^{i\theta} = z \mapsto \frac{1}{z} = r^{-1}e^{-i\theta}$ reflects a points across the real axis and inverts the modulus. So $z \mapsto \frac{1}{z+i}$ first shift along the imaginary axis, then reflects across the real axis and inverts the modulus. Also, $z \mapsto -2iz$ rotates 90° clockwise and then doubles the modulus.

Doing this step-by-step we first map $\{\operatorname{Im}(z) \geq 0\}$ under $z \mapsto z + i$,

$$\{\operatorname{Im}(z) \geq 0\} \mapsto \{\operatorname{Im}(z) \geq 1\}.$$

Finding the image of $\{\operatorname{Im}(z) \geq 1\}$ under $z \mapsto \frac{1}{z}$ turns out to not be so easy by using the above intuition of first reflecting across the real axis and then inverting the modulus. Instead we can try to inspect the proof of Theorem 5.7 to figure out what this image look like. Note that $\{\operatorname{Im}(z) \geq 1\} = \bigcup_{R\geq 1}\{\operatorname{Im}(z) = R\}$ and $\{\operatorname{Im}(z) = R\}$ is a horizontal line in the complex plane(or a generalized circle in the language of the extended complex plane). The condition $\operatorname{Im}(z) = R$ is equivalent to

$$\frac{i}{2}(\bar{z} - z) = R,$$

and by setting $B = -\frac{i}{2}$ this reads

$$Bz + \bar{B}\bar{z} - R = 0,$$

which according to the proof of theorem 5.7 is the equation of a straight line in the complex plane. In the proof it's concluded that a line as the above is mapped by $z \mapsto \frac{1}{z}$ to a circle centered at

$$-\frac{B}{-R} = -\frac{-i/2}{-R} = -\frac{i}{2R}$$

and radius

$$\frac{|B|}{|R|} = \frac{1}{2R}.$$

In other words, $z \mapsto \frac{1}{z}$ maps $\{\mathrm{Im}(z) = R\}$ to $B(-\frac{i}{2R}, \frac{1}{2R})$ and the image of $\{\mathrm{Im}(z) \geq 1\}$ is $\bigcup_{R\geq 1} B(-\frac{i}{2R}, \frac{1}{2R}) = B(-\frac{i}{2}, \frac{1}{2}) \setminus \{0\}$,

Next we rotate $90°$ and double the modulus, $z \mapsto -2iz$

$$B(-\frac{i}{2}, \frac{1}{2}) \setminus \{0\} \mapsto B(-1, 1) \setminus \{0\}$$

and finally shift along the real axis, $z \mapsto z + 1$

$$B(-1, 1) \setminus \{0\} \mapsto B(0, 1) \setminus \{1\}.$$

Finally, $f(\infty) = 1$, so the image of $\{\mathrm{Im}(z) \geq 0\} \cup \{\infty\}$ is $B(0, 1)$. ■

EXERCISE 6

Find the images of the following sets under the Möbius transformation $f(z) = \frac{z+1}{z}$:

1. the imaginary axis,
2. the right half plane $\{z\colon \mathrm{Re}(z) > 0\}$,
3. the unit circle $\partial B(0, 1)$,
4. the unit disk $B(0, 1)$.

SOLUTION:

First note that the map $re^{i\theta} = z \mapsto \frac{1}{z} = r^{-1}e^{-i\theta}$ reflects a points across the real axis and then inverts the modulus. So the function

$$f(z) = \frac{z+1}{z} = \frac{1}{z} + 1$$

first reflects across the real axis, then inverts the modulus and finally shifts along the real axis. So we get the following images

1. $\{1 + iy\colon y \in \mathbb{R}\}$,
2. $\{z\colon \mathrm{Re}(z) > 1\}$,
3. $\partial B(1, 1)$,
4. $\mathbb{C} \setminus \overline{B(1, 1)}$.

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