Complex analysis 2

Exercises 5

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Exercise 1

Find the poles of $f(z) = \frac{z^2+1}{(z^2-4)(z^4-1)}$ and determine the corresponding residues.

Solution:

By factorizing the polynomials we have

$$f(z) = \frac{z^2 + 1}{(z+2)(z-2)(z^2+1)(z+1)(z-1)}.$$

From this we see that the singular points are ± 2,± 1,$\pm i$. Since, for $z_0 = \pm i$, $\lim_{z \to z_0}(z - z_0)f(z) = 0$, $z_0 = \pm i$ are removable singularities by Theorem 3.4.

The remaining singularities are poles of order 1. Note that the functions

$$g_{-2}(z) = \frac{z^2+1}{(z-2)(z^2+1)(z+1)(z-1)}$$
$$g_{2}(z) = \frac{z^2+1}{(z-2)(z^2+1)(z+1)(z-1)}$$
$$g_{-1}(z) = \frac{z^2+1}{(z-2)(z^2+1)(z+1)(z-1)}$$
$$g_{1}(z) = \frac{z^2+1}{(z-2)(z^2+1)(z+1)(z-1)}$$

are analytic in some neighborhood of -2,2,-1 and 1 respectively. The it follows from Theorem 3.6 that $z_0 \in \{-2, 2, -1, 1\}$ are poles or order 1 since we can write

$$f(z) = \frac{1}{z - z_0} g_{z_0}(z).$$

As noted in Remark 3.7, we can use g_{z_0} to compute the residue at z_0,

$$\begin{aligned}
\operatorname{Res}(f,-2) &= g_{-2}(-2) = -\frac{1}{12} \\
\operatorname{Res}(f,2) &= g_2(2) = \frac{1}{12} \\
\operatorname{Res}(f,-1) &= g_{-1}(-1) = \frac{1}{6} \\
\operatorname{Res}(f,1) &= g_1(1) = -\frac{1}{6}
\end{aligned}$$

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EXERCISE 2

Let $f(z)$ be as in Exercise 1. Evaluate $\int_\gamma f(z)\,dz$ when $\gamma_r(t) = re^{it}$, $0 \le t \le 2\pi$, and r takes the values $1/2$, $3/2$, and $5/2$.

SOLUTION:

First note that the paths are null-homologous and that we have the following winding numbers

$$\begin{aligned}
&\eta(\gamma_{1/2}, z_0) = 0, && z_0 \in \{\pm 1, \pm 2\} \\
&\eta(\gamma_{3/2}, 1) = \eta(\gamma_{3/2}, -1) = 1 && \\
&\eta(\gamma_{3/2}, 2) = \eta(\gamma_{3/2}, -2) = 0 && \\
&\eta(\gamma_{5/2}, z_0) = 1, && z_0 \in \{\pm 1, \pm 2\}.
\end{aligned}$$

Then it follows from the residue theorem(Theorem 3.12), using the residues calculated in Exercise 1, that

$$\int_{\gamma_{1/2}} f(z)\,dz = \int_{\gamma_{3/2}} f(z)\,dz = \int_{\gamma_{5/2}} f(z)\,dz = 0.$$

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EXERCISE 3

Apply the residue theorem to evaluate $\int_{-\infty}^{\infty} \frac{1}{x^2+x+1}\,dx$.

SOLUTION:

With the path $\gamma_r(t) = re^{it}$, $0 \le t \le \pi$, $r > 0$, define the closed path

$\sigma_r = [-r, r] * \gamma_r$. Then we can write

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{x^2+x+1}\,dx &= \lim_{r\to\infty} \int_{-r}^{r} \frac{1}{x^2+x+1}\,dx \\ &= \lim_{r\to\infty} \int_{\sigma_r} \frac{1}{z^2+z+1}\,dz - \int_{\gamma_r} \frac{1}{z^2+z+1}\,dz,\end{aligned} \tag{1}$$

provided all involved integrals exist. The integral over σ_r can be computed using the residue theorem and the integral over γ_r can be shown to converge to 0 as $r \to \infty$.

To apply the residue theorem, let's first find the poles. We have

$$\frac{1}{z^2+z+1} = \frac{1}{z - \frac{-1+\sqrt{3}i}{2}} \frac{1}{z - \frac{-1-\sqrt{3}i}{2}}.$$

From Theorem 3.6 we get that $\frac{-1\pm\sqrt{3}i}{2}$ are poles of order 1 and the residue at $\frac{-1+\sqrt{3}i}{2}$ is $-\frac{i}{\sqrt{3}}$. For large enough $r > 0$, $\frac{-1+\sqrt{3}i}{2}$ is inside σ_r so the winding number $n(\sigma_r, \frac{-1+\sqrt{3}i}{2}) = 1$ and $\frac{-1-\sqrt{3}i}{2}$ is always outside σ_r so its winding number is 0, $n(\sigma_r, \frac{-1-\sqrt{3}i}{2}) = 0$. We now get from the residue theorem(Theorem 3.12) that

$$\int_{\sigma_r} \frac{1}{z^2+z+1}\,dz = 2\pi i \frac{-i}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}}. \tag{2}$$

To deal with the integral over γ_r, first note that

$$\left| \int_{\gamma_r} \frac{1}{z^2+z+1}\,dz \right| \leq \int_{\gamma_r} \frac{1}{|z^2+z+1|}\,d|z|.$$

From the reverse triangle inequality($|\|a\| - \|b\|| \leq \|a \pm b\|$) followed by the triangle inequality we get

$$\frac{1}{|z^2+z+1|} \leq \frac{1}{|z|^2 - |z+1|} \leq \frac{1}{|z|^2 - |z| - 1}.$$

Now we have

$$\begin{aligned}\left| \int_{\gamma_r} \frac{1}{z^2+z+1}\,dz \right| &\leq \int_{\gamma_r} \frac{1}{|z|^2 - |z| - 1}\,d|z| \\ &= \int_0^{\pi} \frac{r}{r^2 - r - 1}\,dt \\ &= \frac{\pi}{r - 1 - \frac{1}{r}}.\end{aligned}$$

From this we see that

$$\lim_{r\to\infty}\int_{\gamma_r}\frac{1}{z^2+z+1}\,dz=0. \tag{3}$$

From (1),(2) and (3) we conclude that

$$\int_{-\infty}^{\infty}\frac{1}{x^2+x+1}\,dx=\frac{2\pi}{\sqrt{3}}.$$

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Exercise 4

Apply the residue theorem to evaluate $\int_0^{2\pi}\frac{1}{2-\sin(t)}\,dt$.

Solution:

By using the path $\gamma(t)=e^{it}$, $0\le t\le 2\pi$ and $\sin(t)=\frac{e^{it}-e^{-it}}{2i}$ we have

$$\frac{1}{2-\sin(t)}=\frac{1}{2-\frac{e^{it}-e^{-it}}{2i}}=\frac{\gamma'(t)}{i\gamma(t)\big(2-\frac{\gamma(t)-\gamma(t)^{-1}}{2i}\big)}$$

and this allow us to transform the integral into a complex path integral:

$$\begin{aligned}\int_0^{2\pi}\frac{1}{2-\sin(t)}\,dt&=\int_0^{2\pi}\frac{\gamma'(t)}{i\gamma(t)\big(2-\frac{\gamma(t)-\gamma(t)^{-1}}{2i}\big)}\,dt\\&=\int_\gamma\frac{1}{iz\big(2-\frac{z-z^{-1}}{2i}\big)}\,dz\\&=\int_\gamma\frac{-2}{(z-(2-\sqrt{3})i)(z-(2+\sqrt{3})i)}\,dz.\end{aligned}$$

Using Theorem 3.6 we find that $(2-\sqrt{3})i$ is a pole of order 1 and the residue of the integrand in this pole is $-\frac{i}{\sqrt{3}}$(again calculated as in Remark 3.7). From the residue theorem we now get that

$$\int_0^{2\pi}\frac{1}{2-\sin(t)}\,dt=\frac{2\pi}{\sqrt{3}}.$$

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Exercise 5

Let f be analytic in a domain that contains $A = \{1 \leq |z| \leq 2\}$. Assume that $f(z) \neq 0$ whenever $z \in S_1 = \partial B(0,1)$ or $z \in S_2 = \partial B(0,2)$. Prove that

$$\frac{1}{2\pi} \int_{S_2} \frac{f'(z)}{f(z)}\, dz - \frac{1}{2\pi} \int_{S_1} \frac{f'(z)}{f(z)}\, dz$$

is equal to the sum of the orders of the zeros of f in A.

Solution:

By the exercise assumption, there exist a domain D which contains A. Since f is analytic in D the any accumulation points of the set of zeros must be on the boundary ∂D, by Theorem 2.22. A doesn't intersect ∂D since A is a closed subset and it follows that A can contain at most finitely many zeros of f. By a similar reasoning, there exist a neighborhood B of A which contains only the zeros of f which belong to A.

Next consider the paths $\gamma_2 = 2e^{it}$, $\gamma_1 = e^{it}$ for $0 \leq t \leq 2\pi$. For both of these paths we have the winding number 1 at the origin

$$n(\gamma_1, 0) = n(\gamma_2, 0) = 1.$$

Hence the cycle $\sigma = (\overleftarrow{\gamma_1}, \gamma_2)$ is nullhomologous in B. Next let $a_1, \ldots, a_N \in A$ be all zeros of f in A and let $k_1, \ldots, k_N$ be the corresponding multiplicities. Then we can now apply the argument principle(Theorem 4.5) to coclude that

$$\frac{1}{2\pi i} \int_{\sigma} \frac{f'(z)}{f(z)}\, dz = \sum_{j=1}^{N} k_j n(\sigma, a_j)$$

If we write the integral as two integrals over γ_1, γ_2 and use $n(\sigma, a_j) = 1$ then we get

$$\frac{1}{2\pi i} \int_{\gamma_2} \frac{f'(z)}{f(z)}\, dz - \frac{1}{2\pi i} \int_{\gamma_1} \frac{f'(z)}{f(z)}\, dz = \sum_{j=1}^{N} k_j.$$

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Exercise 6

Let D be a domain, and suppose that f_j, $j \geq 1$, are analytic injections converging locally uniformly in D to a function f. Prove that f is either a constant function or injective in D.

Solution:

Let $w \in D$ and define the sequence

$$g_j(z) = f_j(z) - f_j(w).$$

Clearly g_j is analytic in D and it converges locally uniformly to $f(z) - f(w)$. Since f_j is injective, we know that $g_j(z) \neq 0$ for $z \in D \setminus \{w\}$. Now we get from Hurvitz's theorem(Theorem 4.9) that either

$$g(z) = 0, \quad \forall z \in D \setminus \{w\}$$

or

$$g(z) \neq 0, \quad \forall z \in D \setminus \{w\}.$$

In the former case we conclude that f is constant(equal to $f(w)$ for all $z \in D$) and in the latter that f is injective(since w is arbitrary). ■