Complex analysis 2

Exercises 4

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Exercise 1

Determine the Laurent series representation of $f(z) = \frac{1}{z(1-z)}$ in the sets

1. $0 < |z| < 1$,
2. $1 < |z| < \infty$.

Solution:

When $|z| < 1$ we have $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ and the convergence is absolute. So for $0 < |z| < 1$ in 1 we have

$$f(z) = \frac{1}{z}\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-1} = \frac{1}{z} + 1 + z + z^2 + \dots$$

For part 2, let $w = \frac{1}{z}$. Then $|w| < 1$ and

$$\frac{1}{1-w} = \frac{-z}{1-z}$$

so we can write

$$\begin{aligned} f(z) &= -\frac{1}{z^2} \cdot \frac{-z}{1-z} = -w^2 \frac{1}{1-w} = -w^2 \sum_{n=0}^{\infty} w^n \\ &= -\sum_{n=0} \frac{1}{z^{n+2}} = -\frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \dots \end{aligned}$$

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Exercise 2

Determine the Laurent series representation of $f(z) = \frac{1}{(z-1)(z-2)}$ in the sets

1. $|z| < 1$,
2. $1 < |z| < 2$,
3. $|z| > 2$.

Solution:

By partial fractions we can write

$$f(z) = \frac{1}{1-z} - \frac{1}{2}\frac{1}{1-\frac{z}{2}}.$$

In part 1 both $|z| < 1$ and $|z/2| < 1$, so we get

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \qquad \frac{1}{1-\frac{z}{2}} = \sum_{n=0}^{\infty} \frac{z^n}{2^n}$$

and we get

$$f(z) = \sum_{n=0}^{\infty} (1 - \frac{1}{2^{n+1}}) z^n.$$

In part 2 we set $w = \frac{1}{z}$ to get $|w| < 1$ and get as in the previous exercise that

$$\frac{1}{1-z} = -w \frac{1}{1-w} = -w \sum_{n=0}^{\infty} w^n = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$$

and conclude that

$$f(z) = -\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}.$$

In part 3 we still have

$$\frac{1}{1-z} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$$

and to deal with the other term we set $w = \frac{2}{z}$ to get $|w| < 1$ and

$$\frac{1}{1-\frac{z}{2}} = -w\frac{1}{1-w} = -\frac{2}{z}\sum_{n=0}^{\infty}\frac{2^n}{z^n}$$

and conclude that

$$f(z) = -\sum_{n=0}^{\infty}\frac{1+2^{n+1}}{z^{n+1}}$$

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Exercise 3

Determine the Laurent series representation of $f(z) = \frac{\sin(\pi z)}{(z-1)^3}$ in the set $0 < |z-1| < \infty$.

Solution:

The Taylor series of $\sin(\pi z)$ around $z_0 = 1$ is

$$\sin(\pi z) = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}\pi^{2n+1}}{(2n+1)!}(z-1)^{2n+1}$$

so we get

$$\begin{aligned} f(z) &= \frac{1}{(z-1)^3}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\pi^{2n+1}}{(2n+1)!}(z-1)^{2n+1} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}\pi^{2n+1}}{(2n+1)!}(z-1)^{2(n-1)} \\ &= -\frac{\pi}{(z-1)^2} + \frac{\pi^3}{3!} - \frac{\pi^5}{5!}(z-1) + \frac{\pi^7}{7!}(z-1)^2 - \frac{\pi^9}{9!}(z-1)^3 + \dots \end{aligned}$$

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Exercise 4

What type of singularity do the following functions have at $z = 0$?

1. $\frac{z}{\sin(z)}$
2. $\frac{\cos(z)}{z(z^2-1)}$
3. $\frac{1}{1-\cos(z)}$
4. $e^{-\frac{1}{z^2}}$

Solution:

For part 1 we can use Riemann's removable singularity theorem on the function. Using a Taylor series, we have

$$z\frac{z}{\sin(z)} = z\frac{1}{1+\sum_{k=1}^{\infty}\frac{(-1)^k z^{2k}}{(2k+1)!}} \to 0 \text{ as } z \to 0,$$

so 0 is a removable singularity.

In part 2 we don't have a removable singularity, because

$$z\frac{\cos(z)}{z(z^2-1)} = \frac{\cos(z)}{(z^2-1)} \to -1 \text{ as } z \to 0.$$

But since $\frac{\cos(z)}{(z^2-1)}$ is analytic away from ± 1, Theorem 3.6 says that 0 is a pole of order 1 of $\frac{\cos(z)}{z(z^2-1)}$.

In part 3 we have by using Taylor expansion that

$$\begin{aligned}\frac{1}{1-\cos(z)} &= \frac{1}{\sum_{k=1}^{\infty}\frac{(-1)^k z^{2k}}{(2k)!}} = \frac{1}{z^2}\frac{1}{\sum_{k=1}^{\infty}\frac{(-1)^k z^{2(k-1)}}{(2k)!}} \\ &= \frac{1}{z^2}\frac{1}{1+\sum_{k=2}^{\infty}\frac{(-1)^k z^{2(k-1)}}{(2k)!}}\end{aligned}$$

and we see that 0 is a pole of order 2.

In part 4 just note that e^z and $-z^2$ are entire(holomorphic/analytic everywhere) and then the same is true for their composition e^{-z^2}. Now Theorem 3.11 says that the composition of any entire function which is not a polynomial and $\frac{1}{z}$ has an essential singularity at the origin. Since $e^{-\frac{1}{z^2}}$ is the composition of e^{-z^2} and $\frac{1}{z}$ it follows from the theorem that 0 is an essential singularity. ∎

EXERCISE 5

Determine the residue $\mathrm{Res}(f, 0)$ for each of the functions f in the previous exercise.

SOLUTION:

For a removable singularity the residue is 0, so $\mathrm{Res}(\frac{z}{\sin(z)}, 0) = 0$.

For parts 2,3 we use Remark 3.7, which says that if a function f has a pole of order k and can be written $f(z) = (z - z_0)^k g(z)$ with g analytic, then

$$\mathrm{Res}(f, z_0) = \frac{g^{(k-1)}(z_0)}{(k-1)!}.$$

In part 2 we get

$$\mathrm{Res}(\frac{\cos(z)}{z(z^2-1)}, 0) = \frac{\cos(0)}{(0^2-1)} = -1$$

and in part 3 we get

$$\mathrm{Res}(\frac{1}{1-\cos(z)}, 0) = 0.$$

To see this, recall that

$$\frac{1}{1-\cos(z)} = \frac{1}{z^2} \frac{1}{1 + \sum_{k=2}^{\infty} \frac{(-1)^k z^{2(k-1)}}{(2k)!}}$$

and since the power series has only even powers, its derivative has odd powers and therefore evaluate to 0 at 0.

From the proof of Theorem 3.11 we see that the residue of a function $g(z) = f(1/z)$ where f is entire is equal to the first term in the Taylor expansion of $f(z)$. In this case $f(z) = e^{-z^2}$ so the first term in the Taylor series is $f'(0) = 0$. So we get

$$\mathrm{Res}(e^{-\frac{1}{z^2}}, 0) = 0.$$

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Exercise 6

Let $P(z) = (z - z_1)^{k_1} \cdot \ldots \cdot (z - z_n)^{k_n}$ where $z_1, \ldots, z_n \in \mathbb{C}$ are distinct and $k_1, \ldots, k_n \geq 1$. Show that

$$\frac{1}{P(z)} = \sum_{j=1}^{n} \sum_{m=1}^{k_j} \alpha_{k,m}(z - z_j)^{-m}$$

for some $\alpha_{j,m} \in \mathbb{C}$.

Solution:

For any $z \in \mathbb{C} \setminus \{z_1, \ldots, z_n\}$, we can define the function

$$z \mapsto \frac{1}{P(z)} = \frac{1}{\prod_{i=1}^{n}(z - z_i)^{k_i}} =: Q(z)$$

Since $P(z_i) = 0$ then Q has singularities at z_i, $i \in \{1, \ldots, n\}$. Next we write

$$Q(z) = \frac{1}{(z - z_i)^{k_i}} \frac{1}{\prod_{\substack{j=1 \\ j \neq i}}^{n}(z - z_j)^{k_j}}$$

and since $z \mapsto \frac{1}{\prod_{\substack{j=1 \\ j \neq i}}^{n}(z - z_j)^{k_j}}$ is analytic close to z_i, it follows from Theorem 3.6 that z_i is a pole of order k_i of Q. This means that Q has a Laurent series with a prinicpal part containing finitely many terms,

$$Q(z) = \sum_{l=1}^{k_i} \frac{\alpha_{i,l}}{(z - z_i)^l} + \sum_{l=0}^{\infty} \beta_l (z - z_i)^l.$$

Now $S(z) = \sum_{l=0}^{\infty} \beta_l (z - z_i)^l$ is analytic close to z_i, so

$$\lim_{z \to z_i} (z - z_i)\Big(Q(z) - \sum_{l=1}^{k_i} \frac{\alpha_{i,l}}{(z - z_i)^l}\Big) = \lim_{z \to z_i} (z - z_i)S(z) = 0 \cdot S(z_i) = 0.$$

This shows that z_i is a removable singularity of $Q(z) - \sum_{l=1}^{k_i} \frac{\alpha_{i,l}}{(z-z_i)^l}$. Also, note that while Q still has singularities at the other z_j, $j \in \{1, \ldots, n\}$, $j \neq i$, the sum $\sum_{l=1}^{k_i} \frac{\alpha_{i,l}}{(z-z_i)^l}$ does not. Repeating the above procedure for all $i \in \{1, \ldots, n\}$ we get coefficients $\alpha_{i,j}$ for $i \in \{1, \ldots, n\}$ and $l \in$

$\{1, \ldots, k_i\}$ and can consider the sum

$$\sum_{i=1}^{n} \sum_{l=1}^{k_i} \frac{\alpha_{i,l}}{(z - z_i)^l}.$$

Next we define the function $T : \mathbb{C} \setminus \{z_1, \ldots, z_n\} \to \mathbb{C}$ by

$$T(z) = Q(z) - \sum_{i=1}^{n} \sum_{l=1}^{k_i} \frac{\alpha_{i,l}}{(z - z_i)^l}$$

then T is analytic away from $z_1, \ldots, z_n$. From the above discussion we also conclude that these singularities $z_1, \ldots, z_n$ are removable and we can therefore extend T to the entire complex plane $\mathbb{C}$. Now T is entire(holomorphic/analytic in the whole plane $\mathbb{C}$). By continuity, T is bounded in any compact set. To see that it is bounded everywhere, consider z for $|z| > R$ for large enough R that $|z_i| \leq R$ for all $i \in \{1, \ldots, n\}$. Then we get by using the reverse triangle inequality that

$$R - |z_i| < |z| - |z_i| = ||z| - |z_i|| \leq |z - z_i|$$

and from this we conclude that

$$|T(z)| \leq C\Big(\prod_{i=1}^{n} \frac{1}{(R - |z_i|)^{k_i}} + \sum_{i=1}^{n} \frac{1}{(R - |z_i|)^{k_i}}\Big).$$

From this we see that $T(z) \to 0$ as $|z| \to \infty$ but for now we need in particular that it shows that $T(z)$ is bounded for large $|z|$. So T is a bounded and entire function, so by using Corollary 5.3.4 in the notes(Liouville's Theorem) we get that $T(z)$ is constant. Since $T(z) \to 0$ as $|z| \to \infty$ we conclude that this constant is 0, so $T(z) = 0$ which we can rewrite as

$$\frac{1}{P(z)} = Q(z) = \sum_{i=1}^{n} \sum_{l=1}^{k_i} \frac{\alpha_{i,l}}{(z - z_i)^l}.$$

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