Complex analysis 2

Exercises 3

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Exercise 1

Show that $f(z) = \sum_{n=1}^{\infty} \frac{1}{(z-n)^2}$ is well-defined and analytic in $\mathbb{C} \setminus \mathbb{N}$.

Solution:

To show that it's well-defined we just want to show that the series converges. But by additionally showing that it converges locally uniformly, we'll get that it is analytic as well. So we just need to prove that the series converges locally uniformly.

Let $K \subset \mathbb{C} \setminus \mathbb{N}$ be any compact subset. For any z we have

$$\begin{aligned} |z-n|^2 &= z\overline{z} - n\overline{z} - nz + n^2 \\ &= |z|^2 - 2n\operatorname{Re}(z) + n^2 \\ &\geq n(n - 2\operatorname{Re}(z)) \end{aligned}$$

Since K is compact and $z \mapsto \operatorname{Re}(z)$ is continuous, there exists a $z_0 \in K$ which minimizes $-2\operatorname{Re}(z)$. Then

$$|z-n|^2 \geq n(n - 2\operatorname{Re}(z_0))$$

For this z_0 there exist an N_0 such that if $n \geq N_0$ then $n - \operatorname{Re}(z_0) \geq n/2$. Note that z_0 and hence also N_0 depend only on K. Then we have for $n \geq N_0$ that

$$|z-n|^2 \geq n^2$$

and hence

$$\frac{1}{|z-n|^2} \leq \frac{1}{n^2}.$$

Now the series
$$\sum_{n=N_0}^{\infty} \frac{1}{(z-n)^2}$$
converges uniformly and is analytic since
$$\sum_{n=N_0}^{\infty} \frac{1}{|z-n|^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$
Now we can write
$$\sum_{n=1}^{\infty} \frac{1}{(z-n)^2} = \sum_{n=1}^{N_0} \frac{1}{(z-n)^2} + \sum_{n=N_0}^{\infty} \frac{1}{(z-n)^2}$$
where the infinite series on the right is analytic and the finite sum is clearly analytic since it's a finite sum of analytic functions. In conclusion, the series
$$\sum_{n=1}^{\infty} \frac{1}{(z-n)^2}$$
is analytic on any compact set $K \subset \mathbb{C} \setminus \mathbb{N}$ and is therefore also analytic on all of $\mathbb{C} \setminus \mathbb{N}$. ■

Exercise 2

Determine the radii and disks of convergence for the following power series

a) $\sum_{n=1}^{\infty} \frac{z^n}{n}$

b) $\sum_{n=0}^{\infty} 5^n (z-i)^n$

c) $\sum_{n=0}^{\infty} (z-2)^{n!}$

d) $\sum_{n=0}^{\infty} n! z^n$

Solution:

In a) we have $a_n = 1/n$ and the radius of convergence is
$$\rho^{-1} = \limsup_{n} |a_n|^{1/n} = \inf_n \{\sup\{|a_k|^{1/k} : k \geq n\}\}.$$
As n increases the inner supremum is nonincreasing, since the set over which the supremum is taken is getting smaller. So the outer infimum is the limit of this nonincreasing sequence, which exists as long as the

sequence is bounded. To find what the limit is, it's enough to find the limit of some tail of the sequence. To do this, note that the sequence $(|a_n|^{1/n})_{n\in\mathbb{N}}$ is eventually decreasing. We can see this by considering the function $f(x) = x^{1/x}$, whose derivative is

$$f'(x) = \frac{x^{1/x}}{x^2}(1 - \ln(x))$$

which is eventually negative. This means that for some N_0 large enough $\sup\{|a_k|^{1/k} : k \geq n\} = |a_n|^{1/n}$ for all $n \geq N_0$. The sequence is also bounded below by 0, so it definitely converges. To find the limit we can use L'Hôpital, after writing

$$|a_n|^{1/n} = n^{1/n} = e^{\frac{1}{n}\ln(n)}.$$

Then we find that $\lim_{n\to\infty}|a_n|^{1/n} = 1 = \rho^{-1}$. So the radius of convergence is $\rho = 1$ and the disc of convergence is $B(0,1)$

In b) we have $a_n = 5^n$, so the radius of convergence is

$$\rho^{-1} = \inf_n\{\sup\{|a_k|^{1/k} : k \geq n\}\} = \inf\{\sup\{5\}\} = 5.$$

The disc of convergence is $B(i, \frac{1}{5})$.

In c), we have

$$a_n = \begin{cases} 1 & \text{if } n = k! \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$$

The radius of convergence is

$$\rho^{-1} = \inf_n\{\sup\{|a_k|^{1/k} : k \geq n\}\} = \inf\{\sup\{1\}\} = 1$$

and the disc of convergence $B(2,1)$

For d) we have $a_n = n!$. We can use a similar trick as when computing arithmetic progressions. If we consider the square a_n^2 then we have

$$\begin{aligned} a_n^2 = (n!)^2 &= 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \\ &\times n \cdot n - 1 \cdot n - 3 \cdot \ldots \cdot 1 \\ &= [1 \cdot n] \cdot [2(n-1)] \cdot [3(n-2)] \cdot \ldots \cdot [n \cdot 1]. \end{aligned}$$

There are n factors within brackets [ ] and each such factor is greater than n. So we get

$$a_n^2 \geq n \cdot n \cdot \ldots \cdot n = n^n.$$

Taking the $2n$-root we get

$$a_n^{1/n} \geq \sqrt{n}.$$

This show that $\sup\{|a_k|^{1/k} : k \geq n\} \geq \sup\{\sqrt{k} : k \geq n\} = \infty$ and hence the radius of convergence is $\rho = 0$. ■

EXERCISE 3

Prove(using induction, for instance) Abel's partial summation formula

$$\sum_{k=1}^{n} a_k b_k = a_n s_n + \sum_{k=1}^{n-1} (a_k - a_{k+1}) s_k, \quad a_k, b_k \in \mathbb{C},$$

where $s_j = \sum_{k=1}^{j} b_k$. (Note: there appear to be a mistake in the exercise sheet on the website)

SOLUTION:

I will not actually use induction to prove this. Instead I just expand the sum on the right-hand side and do some manipulations untill the desired result pop out. Since $s_k = s_{k+1} - b_{k+1}$ we have

$$\begin{aligned}
\sum_{k=1}^{n-1} (a_k - a_{k+1}) s_k &= \sum_{k=1}^{n-1} a_k s_k - \sum_{k=1}^{n-1} a_{k+1} s_{k+1} + \sum_{k=1}^{n-1} a_{k+1} b_{k+1} \\
&= \sum_{k=1}^{n-1} a_k s_k - \sum_{k=2}^{n} a_k s_k + \sum_{k=2}^{n} a_k b_k \\
&= a_1 s_1 - a_n s_n + \sum_{k=2}^{n} a_k b_k \\
&= -a_n s_n + \sum_{k=1}^{n} a_k b_k.
\end{aligned}$$

Moving $a_n s_n$ to the other side gives the desired equality. ■

EXERCISE 4

Use the previous exercise to show that the series

$$\sum_{n=1}^{\infty} \frac{z^n}{n}$$

converges for any $z \in \partial B(0,1)$ with $z \neq 1$, but diverges when $z = 1$.

SOLUTION:

Applying the equality in the previous exercise with $a_n = \frac{1}{n}$ and $b_n = z^n$ gives

$$\sum_{n=1}^{N} \frac{z^n}{n} = \frac{1}{N} \sum_{n=1}^{N} z^n + \sum_{n=1}^{N-1} \left(\frac{1}{n} - \frac{1}{n+1}\right) \sum_{k=1}^{n} z^k \tag{1}$$

Using the formula for geometric series and the fact that $|z^{N+1}| = 1$ we have

$$\frac{1}{N} \sum_{n=1}^{N} z^n = \frac{1}{N} \frac{z - z^{N+1}}{1-z} \to 0.$$

Again using the geometric series, we have

$$\begin{aligned}
\sum_{n=1}^{N-1} \left(\frac{1}{n} - \frac{1}{n+1}\right) \sum_{k=1}^{n} z^k &= \sum_{n=1}^{N-1} \left(\frac{1}{n} - \frac{1}{n+1}\right) \frac{z - z^{n+1}}{1-z} \\
&= \sum_{n=1}^{N-1} \frac{1}{n(n+1)} \frac{z - z^{n+1}}{1-z} \\
&= \frac{z}{1-z} \sum_{n=1}^{N-1} \frac{1}{n(n+1)} - \frac{1}{1-z} \sum_{n=1}^{N-1} \frac{1}{n(n+1)} z^{n+1}.
\end{aligned}$$

Both series on the last line converges(the one involving z^{n+1} is easily seen to converge absolutely and hence converges) and therefore

$$\sum_{n=1}^{N-1} \left(\frac{1}{n} - \frac{1}{n+1}\right) \sum_{k=1}^{n} z^k$$

also converges. Now both series on the right in (1) converges and then the same is true for the left-hand side. All of the above require $z \neq 1$ since we divide by $1 - z$. If $z = 1$ then the original series is

$$\sum_{n=1}^{\infty} \frac{z^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

which is well-known to be a divergent series. ∎

EXERCISE 5

Determine the Taylor series representations of $f(z) = \mathrm{Log}(z)$ centered at $z_0 = 1$ and $g(z) = \sin(z)$ centered at $z_0 = 0$. What are the disks of convergence?

SOLUTION:

For the logarithm we have

$$f^{(k)}(z) = (-1)^{k+1}\frac{(k-1)!}{z^k}$$

and the Taylor series around $z_0 = 1$ is

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^n.$$

Doing something similar as in Exercise 2 part a) we get a radius of convergence of 1 and the disc of convergence is $B(1,1)$.

Next we have

$$\begin{aligned} g^{(4k)}(z) &= \sin(z) \\ g^{(4k+1)}(z) &= \cos(z) \\ g^{(4k+2)}(z) &= -\sin(z) \\ g^{(4k+3)}(z) &= -\cos(z) \end{aligned}$$

so the Taylor series around $z_0 = 0$ is

$$g(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} z^{2n-1}.$$

Similar to Exercise 2 part d) we get radius of convergence ∞ so the disc of convergence is all of $\mathbb{C}$. ■

EXERCISE 6

Give an example of a nonconstant analytic function $f : B(0,1) \to \mathbb{C}$ that has infinitely many distinct zeros in $B(0,1)$. Why does this not contradict Theorem 2.22?

SOLUTION:

A standard example of a wildly oscillating function is $\sin(1/x)$ for $x \in \mathbb{R}$, $x \neq 0$. To get a function which oscillates close to the boundary $B(0,1)$, let's consider the function

$$\sin\left(\frac{1}{z-1}\right).$$

Since the (complex) sin is 0 if and only if the argument is $k\pi$ for some $k \in \mathbb{Z}$, we get

$$\sin\left(\frac{1}{z-1}\right) = 0$$

if and only if

$$z = 1 + \frac{1}{k\pi}.$$

Taking $k = -n$ for $n \in \mathbb{N}$, we get a sequence

$$z_n = 1 - \frac{1}{n\pi}. \tag{2}$$

in $B(0,1)$ converging to 1 such that $\sin(z_n) = 0$. If we instead consider positive k, then we get zeros outside of $B(0,1)$. So the zeros inside $B(0,1)$ are given precisely by (2). That is, the set of zeros of the function $B(0,1) \ni z \mapsto sin\left(\frac{1}{z-1}\right)$ has only one accumulation point and it is 1. But $1 \notin B(0,1)$, which means that the second part of Theorem 2.22 is not satisfied. If the accumulation point would be inside $B(0,1)$ then this would be a counter example to Theorem 2.22. ∎