Complex analysis 2

Exercises 2

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April 12, 2024

Exercise 1

If $\sum_{n=1}^{\infty} z_n$ converges absolutely, show that $\sum_{n=1}^{\infty} z_n$ converges and

$$\left|\sum_{n=1}^{\infty} z_n\right| \leq \sum_{n=1}^{\infty} |z_n|.$$

Solution:

For $k > l$ then

$$\left|\sum_{n=1}^{k} z_n - \sum_{n=1}^{l} z_n\right| = \left|\sum_{n=l}^{k} z_n\right| = \sum_{n=l}^{k} |z_n| \leq \varepsilon$$

so $\sum_{n=1}^{\infty} z_n$ indeed converges. Next

$$\begin{aligned}
\left|\sum_{n=1}^{\infty} z_n\right| &\leq \left|\sum_{n=1}^{\infty} z_n - \sum_{n=1}^{k} z_n\right| + \left|\sum_{n=1}^{k} z_n\right| \\
&\leq \left|\sum_{n=1}^{\infty} z_n - \sum_{n=1}^{k} z_n\right| + \sum_{n=1}^{k} |z_n| \\
&\leq \underbrace{\left|\sum_{n=1}^{\infty} z_n - \sum_{n=1}^{k} z_n\right|}_{<\varepsilon} + \underbrace{\left|\sum_{n=1}^{k} |z_n| - \sum_{n=1}^{\infty} |z_n|\right|}_{<\varepsilon} + \sum_{n=1}^{\infty} |z_n| \\
&\leq 2\varepsilon + \sum_{n=1}^{\infty} |z_n|.
\end{aligned}$$

Since this is true for any $\varepsilon > 0$ we conclude that

$$\left|\sum_{n=1}^{\infty} z_n\right| \le \sum_{n=1}^{\infty} |z_n|.$$

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Exercise 2

Let $s_k : G \to \mathbb{C}$ be continuous functions converging to some $s : G \to \mathbb{C}$ locally uniformly in an open G. Show that if $\gamma : [a, b] \to G$ is a piecewise C^1-path then

$$\lim_{k\to\infty} \int_\gamma s_k(z)\, dz = \int_\gamma s(z)\, dz.$$

Solution:

Since γ is continuous and $[a, b]$ compact, $\gamma^* = \gamma([a, b]) \subset G$ is compact. Then $\|s_k - s\|_{C(\gamma^*)} \to 0$ by the assumption. From Corollary 4.2.12 in the lecture notes from Complex analysis 1 we get

$$\left|\int_\gamma s_k(z)\, dz - \int_\gamma s(z)\, dz\right| \le \|s_k - s\|_{C(\gamma^*)} \operatorname{length}(\gamma) \to 0.$$

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Exercise 3

Prove that the Riemann ζ-function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

is well-defined and analytic in the half-plane $\{z \in \mathbb{C} : \operatorname{Re}(z) > 1\}$.

Solution:

Write $z = x + iy$ where $x > 1$. Then

$$|n^z| = |e^{(x+iy)\ln(n)}| = |e^{x\ln(n)} e^{iy\ln(z)}| = |e^{x\ln(n)}| = n^x.$$

For $n \geq 2$,

$$\frac{1}{n^x} = \int_{n-1}^{n} \frac{1}{n^x}\, dt \leq \int_{n-1}^{n} \frac{1}{t^x}\, dt = \frac{1}{1-x}\left(\frac{1}{n^{x-1}} - \frac{1}{(n-1)^{x-1}}\right)$$

and

$$\sum_{n=2}^{k} \frac{1}{n^x} \leq \sum_{n=2}^{k} \frac{1}{1-x}\left(\frac{1}{n^{x-1}} - \frac{1}{(n-1)^{x-1}}\right)$$
$$= \frac{1}{x-1} - \frac{1}{x-1}\frac{1}{k^{x-1}} \to 0 \text{ as } k \to \infty.$$

So by using

$$M_n = \frac{1}{1-x}\left(\frac{1}{n^{x-1}} - \frac{1}{(n-1)^{x-1}}\right)$$

in the Weierstrass M-test, we get that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^z}$$

converges uniformly and hence $\zeta(z)$ is well-defined. Since each function term in the series is analytic, it follows from Theorem 2.13 that the limit $\zeta(z)$ is analytic. ■

EXERCISE 4

Prove that the distasnce between a compact $K \subset \mathbb{C}$ and a closed $F \subset \mathbb{C}$ with $K \cap F \neq \emptyset$ is positive.

SOLUTION:

If it is not true, then for any $r > 0$ there exists $z_r \in K$ and $w_r \in F$ such that

$$|z_r - w_r| \leq r.$$

Taking $r = 1/n$ we get sequences z_n, w_n such that $|z_n - w_n| \leq 1/n$. Since K is compact, the sequence z_n is bounded and hence has a convergent subsequence, $z_{n_k} \to z \in K$. Then

$$|w_{n_k} - z| \leq |w_{n_k} - z_{n_k}| + |z_{n_k} - z| \to 0$$

so $w_{n_k} \to z$. But since F is closed this implies $z \in F$, which contradicts $K \cap F \neq \emptyset$. ■

Exercise 5

Prove the Heine-Borel property: if $K \subset \mathbb{C}$ is compact then any open cover of K has a finite subcover.

Solution:

Suppose to the contrary that there exists an open cover $G = \{U_\alpha\}_{\alpha \in A}$ of K which does not have a finite subcover. From this we create a sequence that will lead to a contradiction.

- First pick any $U_{\alpha_1} \in G$ and pick $x_1 \in U_{\alpha_1}$. Then $K \setminus U_{\alpha_1}$ is again compact and doesn't have a finite subcover.
- I claim that there exist an open ball $B \subset \mathbb{C}$ that intersects $K \setminus U_{\alpha_1}$ and such that $B \subset \bigcup_{\alpha \in A} U_\alpha$ but does not have a finite subcover in G. This follows by noting that $K \setminus U_{\alpha_1}$ can be covered by finitely many balls with any fixed but small enough radius. If each of those finitely many balls had a finite subcover in G then their finite union would also have a finite subcover, which is a contradiction. I will however not prove that $K \setminus U_{\alpha_1}$ can be covered by finitely many balls in such a way, but I believe it is fairly intuitive for $\mathbb{C}$ or even $\mathbb{R}^n$.
- Let B_1 be such a ball with radius at most 1 and which intersects $K \setminus U_{\alpha_1}$.
- Pick $x_2 \in B_1$ and let $U_{\alpha_2} \in G$ be a set containing x_2 and let $B_2 \subset B_1$ be another ball whose radius is less than $1/2$ and such that it intersects $K \setminus \bigcup_{j=1}^{2} U_{\alpha_j}$.
- In general, pick $x_n \in B_{n-1}$ and let $U_{\alpha_n} \in G$ be a set containing x_n. Then there still must exist an open ball $B_n \subset B_{n-1}$ of radius less than $1/n$ and such that B_n intersects $K \setminus \bigcup_{j=1}^{n} U_{\alpha_j}$.
- Next note that, by construction,

  $$B_1 \supset B_2 \supset \ldots \supset B_n \supset \ldots$$

  From this we get that the sequence x_n is Cauchy, since for any $N \in \mathbb{N}$ and any $n > m \geq N$ we have $x_n, x_m \in B_m \subset B_N$ and then

  $$|x_n - x_m| \leq \frac{2}{N}.$$

  So there exist a limit $x \in K$, $x_n \to x$.

- Since x_n converges to x, for any $\varepsilon > 0$ there exist a ball $B(x, \varepsilon)$ centered at x and of radius ε which contains all but at most finitely many x_n. Since G is a cover and $x \in K$, there exist $U_{\alpha_0} \in G$ containing x and since U_{α_0} is open, any small enough ball around x is contained in U_{α_0}. Let $\varepsilon_0 > 0$ be such ball $B(x, \varepsilon_0) \subset U_{\alpha_0}$.

- Next note that
$$x \in \bigcap_{n=1}^{\infty} \overline{B}_n \subset \overline{B}_k$$
for any $k \in \mathbb{N}$. Since B_k has radius at most $1/k$, it is contained in $B(x, \frac{2}{k})$ and the closure $\overline{B}_k$ is contained at least in $B(x, \frac{3}{k})$. For $k > \frac{3}{\varepsilon_0}$, $B(x, \frac{3}{k}) \subset B(x, \varepsilon_0)$. This implies that for such k,
$$B_k \subset B(x, \varepsilon_0) \subset U_{\alpha_0}$$
so that B_k is covered by $U_{\alpha_0} \in G$, which is a contradiction.

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EXERCISE 6

If $U \subset \mathbb{C}$ is open and $f_n \to f$ uniformly on any closed ball in U, show that $f_n \to f$ locally uniformly in U.

SOLUTION:

Let $K \subset U$ be any compact set. Since U is open, for each $x \in K$ there is a radius r_x such that the closed ball $\overline{B}(x, r_x)$ is contained in U. With such radii, we can get a trivial open cover of K, namely,
$$K \subset \bigcup_{x \in K} B(x, r_x).$$
By the previous exercise, there is a finite subcover $\{B(x_1, r_1), \ldots B(x_k, r_k)\}$. For each $\varepsilon > 0$ and each $j \in \{1, \ldots, k\}$ there is an N_j such that if $n \geq N_j$
$$\|f_n - f\|_{C(\overline{B}(x_j, r_j))} < \varepsilon.$$
Choosing $N = \max\{N_1, \ldots, N_k\}$ we have for $n \geq N$ that
$$\|f_n - f\|_{C(K)} \leq \max\{\|f_n - f\|_{C(\overline{B}(x_j, r_j))} : j \in \{1, \ldots, k\}\} < \varepsilon.$$
which shows that $f_n \to f$ locally uniformly since K was an arbitrary compact subset. ■