Complex analysis 2

Exercises 1

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Exercise 1

Let $G \subset \mathbb{C}$ be a nonempty open set. For any $z \in G$, define

$$C_z = \{w \in G \colon \exists \text{ path } \gamma \colon [a, b] \to G \text{ with } \gamma(a) = z \text{ and } \gamma(b) = w\}.$$

Prove that C_z is the maximal connected open subset of G containing z. Show that for $z \neq w$ one has either $C_z = C_w$ or $C_z \cap C_w = \emptyset$. Finally, show that G is the disjoint union of its maximal connected open subsets (called the *components* of G).

Solution:

Note that now after I have written these solutions, I realize that the terms path connected and connected might be used interchangably in this course. That does make sense since even if these concepts do have different definitions in general spaces, they are equivalent in Euclidean spaces. If they are indeed used interchangably then my solution might look a bit unnecessarily difficult, since I also prove the relation between path connectedness and connectedness.

To show that C_z is a connected component, I want to use that C_z is both open and closed to show that C_z contains a connected component and then prove by contradiction that C_z is equal to that connected component.

- To see that it is open, take any $w \in C_z$. There exists an open ball $B(w, \varepsilon) \in G$ since G is open. The ball is convex and clearly path connected. Then $B(w, \varepsilon) \subseteq C_z$, showing that C_z is open.

- To see that it is closed, take $w \in \partial C_z$ on the boundary. Then the open ball $B(w, \varepsilon)$ has a nonempty and open intersection with C_z. But since $B(w, \varepsilon)$ is path-connected, this shows that there exists paths from w to points inside C_z which means that $w \in C_z$. So C_z is both open and closed.

- Next I want to show that if a connected component Y has nonempty intersection with C_z then $Y \subseteq C_z$. Since C_z is both open and closed, $Y \cap C_z$ is both open and closed closed in Y. Then $Y \cap C_z$ and $Y \setminus C_z$ are two open sets in Y that disconnect Y, which contradicts the connectedness of Y. This shows that $Y \subseteq C_z$.

- Finally, we conclude that $Y = C_z$ by proving that also $C_z \subseteq Y$, and we do this by showing that C_z is connected. If to the contrary C_z is not connected, let $U, V \subset C_z$ be two nonempty open subsets that form a partition and take $a \in U$, $b \in V$. Let $\gamma\colon [0,1] \to X$ be a path between a, b. Then $[0,1] = \gamma^{-1}(U) \cup \gamma^{-1}(V)$ is a partition of $[0,1]$ into relatively open subsets, which contradicts the connectedness of $[0,1]$. So C_z must be connected and hence $C_z \subseteq Y$. We conclude that C_z is a maximal connected set(or a connected component of G), which is the first part of the exercise.

For the second part, C_z is clearly a path-connected set since all points have a path to z. It is also a maximal path-connected set(or a path component of G). To see this, first note that any maximal path-connected set whose intersection with C_z is nonempty must contain all of C_z. So there's a maximal path-connected set X with $C_z \subseteq X$. But X contains z so any point in X has a path to z, so $X \subseteq C_z$ and we get $X = C_z$. So C_z is indeed a path-component of G. Now for two sets C_z, C_w we obviously have either $C_z \cap C_w = \emptyset$ or $C_z \cap C_w \neq \emptyset$. In the latter case, we must have $C_z = C_w$ by the maximality of both C_z, C_w and the fact that they are both path-connected. This proves the second part.

For the last part, note that any singleton set $\{p\}$ is path connected. So G is in a trivial way the union of path connected sets. Since every path connected set is contained in a path component we get that G is a union of path components. And we saw in the second part of the exercise that these path components are disjoint, so G a disjoint union of path components. But we established above that the path components are also connected components and that they are all open so we conclude that G is a disjoint union of open connected components, which proves the last part of the exercise. ■

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Exercise 2

Let $\sigma = (\gamma_1, \gamma_2, \gamma_3)$, where $\gamma_1(t) = 3e^{it}$, $\gamma_2(t) = 1 + e^{-it}$ and $\gamma_3(t) = -1 + e^{-it}$ for $0 \leq t \leq 2\pi$. Determine the components of $\mathbb{C} \setminus |\sigma|$ and the possible values $\eta(\sigma, z)$ for $z \in \mathbb{C} \setminus |\sigma|$. Is σ null-homologous in $B(0,4)$, in $B(0,4) \setminus \{-1\}$ or in $B(0,4) \setminus \{-1,1\}$?

SOLUTION:

Let $B(z,r)$ be the open ball of radius r centered at z. Then the components are $\mathbb{C} \setminus \overline{B(0,3)}$, $B(1,1)$, $B(-1,1)$ and $B(0,3) \setminus (\overline{B(1,1)} \cup \overline{B(-1,1)})$ and the corresponding winding numbers are

$$\begin{aligned} \eta(\sigma, \mathbb{C} \setminus \overline{B(0,3)}) &= 0, \\ \eta(\sigma, B(1,1)) &= 0, \\ \eta(\sigma, B(-1,1)) &= 0, \\ \eta(\sigma, B(0,3) \setminus (\overline{B(1,1)} \cup \overline{B(-1,1)})) &= 1. \end{aligned}$$

σ is null-homologous in $B(0,4)$, $B(0,4) \setminus \{-1\}$ and $B(0,4) \setminus \{-1,1\}$. ■

EXERCISE 3

Let f be analytic in $G \subset \mathbb{C}$, and for $z, w \in G$ define

$$g(z,w) = \begin{cases} \frac{f(w)-f(z)}{w-z}, & z \neq w, \\ f'(z), & z = w. \end{cases}$$

Show that $g(z,w)$ is continuous in $G \times G$.

SOLUTION:

To show continuity, let (z_0, w_0) be arbitrary and (z_n, w_n) a sequence converging to (z_0, w_0). We have to consider the cases

$$z_0 \neq w_0 \text{ and } w_n \neq z_n, \tag{1}$$
$$z_0 = w_0 \text{ and } w_n \neq z_n, \tag{2}$$
$$z_0 = w_0 \text{ and } w_n = z_n. \tag{3}$$

Since $z_0 \neq w_0$ in case (1), we have for large enough n that $z_n \neq w_n$. So the case $z_0 \neq w_0$ and $w_n = z_n$ doesn't need to be handled. In the case

(1) we get by adding and subtracting $f(w_n) + f(z_n)$ in the first term,

$$
\begin{aligned}
|g(z_0, w_0) - g(z_n, w_n)| &= |\frac{f(w_0) - f(z_0)}{z_0 - w_0} - \frac{f(w_n) - f(z_n)}{z_n - w_n}| \\
= |\frac{f(w_0) - f(w_n) + f(z_n) - f(z_0)}{z_0 - w_0} &+ \frac{f(w_n) - f(z_n)}{z_0 - w_0} - \frac{f(w_n) - f(z_n)}{z_n - w_n}| \\
\leq \frac{|f(w_0) - f(w_n)| + |f(z_n) - f(z_0)|}{|z_0 - w_0|} &+ |\frac{f(w_n) - f(z_n)}{z_0 - w_0} - \frac{f(w_n) - f(z_n)}{z_n - w_n}|
\end{aligned}
$$

Since f is continuous,

$$
\frac{|f(w_0) - f(w_n)| + |f(z_n) - f(z_0)|}{|z_0 - w_0|} \leq \frac{2\varepsilon}{|z_0 - w_0|}
$$

for n large enough. Similarly,

$$
\begin{aligned}
|\frac{f(w_n) - f(z_n)}{z_0 - w_0} - \frac{f(w_n) - f(z_n)}{z_n - w_n}| &\leq |f(w_n) - f(z_n)||\frac{1}{z_0 - w_0} - \frac{1}{z_n - w_n}| \\
&\leq C|\frac{1}{z_0 - w_0} - \frac{1}{z_n - w_n}| \\
&\leq C\varepsilon
\end{aligned}
$$

and C doesn't depend on n since the sequence $f(w_n) - f(z_n)$ is convergent. So we have

$$
|g(z_0, w_0) - g(z_n, w_n)| \leq C\varepsilon.
$$

For case (2), first note that

$$
f(w_n) - f(z_n) = \int_{[w_n, z_n]} f'(s)\, ds
$$

so that

$$
\begin{aligned}
|g(z_0, w_0) - g(z_n, w_n)| &= |f'(z_0) - \frac{f(w_n) - f(z_n)}{z_n - w_n}| \\
&\leq |f'(z_0) - \frac{\int_{[w_n, z_n]} f'(s)\, ds}{z_n - w_n}| \\
&\leq |\frac{\int_{[w_n, z_n]} f'(z_0) - f'(s)\, ds}{z_n - w_n}| \\
&\leq \frac{\int_{[w_n, z_n]} ds\, \|f'(z_0) - f'(s)\|_{L^\infty([w_n, z_n])}}{|z_n - w_n|} \\
&\leq \|f'(z_0) - f'(s)\|_{L^\infty([w_n, z_n])}.
\end{aligned}
$$

Now note that since z_n, w_n converges to $z_0 = w_0$, the entire line segment $[w_n, z_n]$ gets arbitrarily close to the point z_0 as n gets large. By continuity of f' there exists n large enough that $|f'(z_0) - f'(s)| \leq \varepsilon$ for any $s \in [w_n, z_n]$ and we conclude that

$$|g(z_0, w_0) - g(z_n, w_n)| \leq \varepsilon.$$

In case (3) we have from continuity of f' that for large enough n,

$$|g(z_0, w_0) - g(z_n, w_n)| = |f'(z_0) - f'(z_n)| \leq \varepsilon.$$

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Exercise 4

If $D \subset \mathbb{C}$ is a bounded domain and $\mathbb{C} \setminus D$ is path-connected, show that D is simply connected. Show that the conclusion may fail if one drops the assumpion that D is bounded.

Solution:

By Corollay 5.2.8 in the notes for complex analysis 1, the winding number $\eta(\gamma, z)$ is constant in any connected open subset $U \subset \gamma^*$ and that if in addition U is unbounded then $\eta(\gamma, z) = 0$, where γ^* is the image/trace of the closed path. Since for any closed path γ in D we have $\mathbb{C} \setminus D \subseteq \mathbb{C} \setminus \gamma^*$ and since $\mathbb{C} \setminus D$ is unbounded,

$$\eta(\gamma, z) = 0 \quad \forall z \in \mathbb{C} \setminus D.$$

So any closed path in D is null-homologous in D and D is therefore simply connected.

If the boundedness of D is dropped, then we still assume it to be a connected open set(domains are connected). So if we take $D = \mathbb{C} \setminus \overline{B(0,1)}$ then it is connected, open and unbounded, while $\mathbb{C} \setminus D = B(0,1)$ is connected, open and bounded. If $\gamma(t) = 2e^{it}$ for $t \in [0, 2\pi]$ then γ is a path in D with $\eta(\gamma, z) = 1$ for any $z \in B(0,1)$. So γ is not null-homologous in D and D is not simply connected. ■

Exercise 5

Let $\sigma = (\gamma_i, \gamma_0, \gamma_{-i})$ where $\gamma_z = z + \frac{e^{it}}{2}$, $0 \leq t \leq 2\pi$. Evaluate $\int_\sigma \frac{e^z}{z^3+z}\, dz$.

Solution:

Before evaluating any integrals, let's write

$$\frac{1}{z^3+z} = \frac{1}{z} - \frac{1}{2(z-i)} - \frac{1}{2(z+i)}$$

so that the integral is split into 3 integrals

$$\int_\sigma \frac{e^z}{z^3+z}\, dz = \int_\sigma \frac{e^z}{z}\, dz - \frac{1}{2}\int_\sigma \frac{e^z}{z-i}\, dz - \frac{1}{2}\int_\sigma \frac{e^z}{z+i}\, dz.$$

The function $\frac{e^z}{z}$ is analytic on the interior of γ_i and γ_{-i}, so the integrals over those two paths are equal to 0. So we get

$$\int_\sigma \frac{e^z}{z}\, dz = \int_{\gamma_0} \frac{e^z}{z}\, dz.$$

Similarly,

$$\int_\sigma \frac{e^z}{z-i}\, dz = \int_{\gamma_i} \frac{e^z}{z-i}\, dz$$

and

$$\int_\sigma \frac{e^z}{z+i}\, dz = \int_{\gamma_{-i}} \frac{e^z}{z+i}\, dz$$

We can evaluate each of these integrals using the Cauchy integral formula,

$$\int_{\gamma_0} \frac{e^z}{z}\, dz = 2\pi i e^0$$

$$\int_{\gamma_i} \frac{e^z}{z-i}\, dz = 2\pi i e^i$$

$$\int_{\gamma_{-i}} \frac{e^z}{z+i}\, dz = 2\pi i e^{-i}$$

and conclude that

$$\int_\sigma \frac{e^z}{z^3+z}\, dz = \pi i(2 - e^i - e^{-i}).$$

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Exercise 6

Use the previous exercise and the global Cauchy theorem to evaluate

$$\int_\gamma \frac{e^z}{z^3+z}\,dz, \quad \text{where } \gamma(t) = 2e^{it},\ 0 \le t \le 2\pi.$$

Solution:

Let $\sigma = (\overleftarrow{\gamma}_i, \overleftarrow{\gamma}_0, \overleftarrow{\gamma}_{-i}, \gamma)$. Then

$$\begin{aligned}\int_\gamma \frac{e^z}{z^3+z}\,dz &= \int_\sigma \frac{e^z}{z^3+z}\,dz + \sum_{p\in\{i,-i,0\}} \int_{\gamma_p} \frac{e^z}{z^3+z}\,dz \\ &= \int_\sigma \frac{e^z}{z^3+z}\,dz + \pi i(2 - e^i - e^{-i}).\end{aligned}$$

To evaluate the integral over σ, I want to conclude that σ is null-homologous somewhere and the use Cauchy's integral formula. Let $B_r = B(i,r) \cup B(0,r) \cup B(-i,r)$ be the union of the three open balls around $i, -i, 0$ of radius r. For any $z \in B_{1/2}$, $\eta(\gamma, z) = 1$ and exactly one of $\eta(\overleftarrow{\gamma}_i, z), \eta(\overleftarrow{\gamma}_i, z), \eta(\overleftarrow{\gamma}_i, z)$ is equal to -1 and the other two are equal to 0. So for any $z \in B_{1/2}$, $\eta(\sigma, z) = 0$. This does not, however, show that σ is null-homologous in $\mathbb{C} \setminus B_{1/2}$ since this is a closed set and null-homologous was defined for open sets. $\mathbb{C} \setminus \overline{B}_{1/4}$ is closed and $\eta(\sigma, z) = 0$ for any $z \in \overline{B}_{1/4}$, so σ is null-homologous in $\mathbb{C} \setminus \overline{B}_{1/4}$. It follows from the global Cauchy theorem that

$$\int_\sigma \frac{e^z}{z^3+z}\,dz = 0$$

since $\frac{e^z}{z^3+z}$ is analytic in $\mathbb{C} \setminus \overline{B}_{1/4}$ and we conclude that

$$\int_\gamma \frac{e^z}{z^3+z}\,dz = \pi i(2 - e^i - e^{-i}).$$

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