Complex analysis 2

Supplementary exercises[1]

1. Draw the images of cycles (ω), (θ), $\sigma = (\gamma, \eta)$ and $\tau = (\theta, \overleftarrow{\eta})$ when

$$\begin{aligned} &\gamma(t) = 2 - e^{it}, && \eta(t) = -2 + e^{it}, \\ &\omega(t) = 4\cos(t) + i\cos(t), && \theta(t) = 4\cos(t) + i2\sin(t), \qquad 0 \le t \le 2\pi. \end{aligned}$$

2. Let the cycles (ω), (θ), σ and τ be as in Exercise 1. Let $D = \mathbb{C} \setminus \{-2, 2\}$. Show that the following pairs of cycles are homologous in D:
(i) (ω) and (θ),
(ii) σ and (θ),
(iii) τ and (γ).

3. Prove that the formula

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{2^n + z^n}$$

defines an analytic function in $B(0,1)$.

4. Let

$$f(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} z^{2k} \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} \frac{(-1)^k k}{2^{k-1}} z^{2k-1}.$$

Find the radii of convergence of both power series.

5. Evaluate the power series f, g in Exercise 4 (i.e. find simpler expressions that do not involve sums for f, g).

6. What kind of isolated singularity do the following functions f have at $z = 0$?
(i) $f(z) = z^2 e^{1/z}$
(ii) $f(z) = (e^z - ez)/z^4$
(iii) $f(z) = \cos(z + \pi/2)/z$

7. Find the residue of $z \mapsto \frac{\cos(z)}{\operatorname{Log}(z)}$ at $z = 1$.

[1] You can submit the solutions to at most 12 exercises to David Johansson by 20 May. These will be counted toward the bonus points (maximum 5) in the course exam.

8. Let
$$f(z) = \frac{z}{z^2+1} \quad \text{and} \quad \gamma(t) = 3e^{it},\ 0 \le t \le 2\pi.$$
(i) Find the poles of f and evaluate the corresponding residues.
(ii) Apply the residue theorem to evaluate $\int_\gamma f(z)\,dz$.

9. Fix $a > 0$. Apply the residue theorem to evaluate
$$\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}\,dx.$$

10. Fix $a > 1$. Apply the residue theorem to evaluate
$$\int_{-\infty}^{\infty} \frac{\cos(x)}{a+\cos(x)}\,dx.$$

11. Let $f\colon B(0,1) \to \mathbb{C}$ be analytic and satisfy $f(i2^{-n}) = -2^{-n}$ for all $n \in \mathbb{N}$. Prove that $f(z) = iz$ for all $z \in B(0,1)$.

12. Suppose $\mathcal{F}$ consists of those analytic functions $f\colon B(0,1) \to \mathbb{C} \setminus \{1\}$ for which
$$\left|\frac{f(z)}{1-f(z)}\right| \le |z| \quad \text{for all } z \in B(0,1).$$
Prove that $\mathcal{F}$ is a normal family in $B(0,1)$.

13. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers that satisfies $\limsup_{n\to\infty} a_n^{1/n} = 1$, and let $\mathcal{F}$ be a family of functions $f\colon B(0,1) \to \mathbb{C}$ that are analytic and satisfy
$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \quad \text{where } |c_n| \le a_n \text{ for all } n \in \mathbb{N}.$$
Prove that $\mathcal{F}$ is a normal family in $B(0,1)$.

14. Prove or disprove: there is a conformal surjection from the punctured disk $B^*(0,1) = B(0,1) \setminus \{0\}$ onto the annulus $B(0,2) \setminus \overline{B}(0,1)$.

15. Find a Möbius transformation $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ for which $f(1) = -1$, $f(0) = i$ and $f(\infty) = -i$.

16. With f as in Exercise 15, find the images $f(\mathbb{R})$ and $f(\mathbb{H})$ of the real line and the half-plane
$$\mathbb{H} := \{z \in \mathbb{C} \colon \operatorname{Im}(z) > 0\}.$$