Complex analysis 2

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(AI-assisted English translation by Mikko Salo)

Lecture notes for Spring 2024

Foreword

The following pages contain lecture notes for the Complex Analysis 2 course. These have been mostly adapted from lecture notes written by Tero Kilpeläinen to fit the current 30-hour lecture series. The proof of the Riemann mapping theorem is based on the lecture notes Complex Analysis II by Kari Astala and Eero Saksman.

Anna Tuhola wrote the first version of these notes in LaTeX.

The course covers complex integration in very general regions of the complex plane, power series representations of analytic functions, residue calculus, singularities, and conformal mappings.

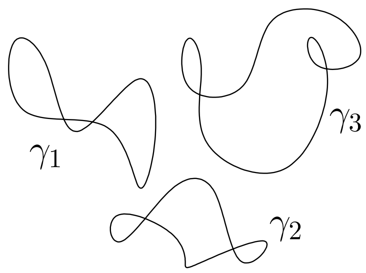
All references in the form CA1.xx.xx point to section xx.xx of the Complex Analysis 1 (Tero Kilpeläinen, ed. 2015) lecture notes.

Sisällys

1. Cauchy's theorem and integral formula — homological versions

To prove the general (homological) form of Cauchy's theorem, we need some additional concepts and a "reminder" of iterated integrals.

1.1. Definition. A *cycle* in the complex plane is a finite sequence $\sigma = (\gamma_1, \gamma_2, \ldots, \gamma_p)$, where γ_k are closed paths for all $k = 1, 2, \ldots, p$.



1.2. Remark. The order of paths γ_k is irrelevant. Often, we identify, for example, cycles $(\gamma, \beta, \alpha, \overleftarrow{\gamma}, \overleftarrow{\gamma}, \beta)$ and $(\overleftarrow{\gamma} * \gamma, \beta * \beta, \overleftarrow{\gamma}, \alpha)$. We also identify a closed path γ with the cycle $\sigma = (\gamma)$.

If $\sigma = (\gamma_1, \gamma_2, \ldots, \gamma_p)$ is a cycle, then

$$|\sigma| = |\gamma_1| \cup |\gamma_2| \cup \ldots \cup |\gamma_p|,$$

and furthermore, σ is a *cycle in the set* A, if $|\sigma| \subset A$. Note! The *trace* $|\sigma|$ of the cycle σ is compact, as it is a finite union of compact sets.

If $\sigma = (\gamma_1, \gamma_2, \ldots, \gamma_p)$ is a cycle and $f : |\sigma| \to \mathbf{C}$ is continuous, then

$$\int_\sigma f(z)\, dz = \sum_{k=1}^{p} \int_{\gamma_k} f(z)\, dz.$$

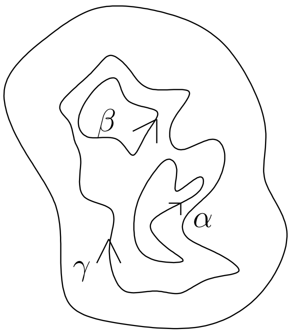
In particular, the winding number of the cycle σ around the point $z_0 \notin |\sigma|$ is

$$n(\sigma, z_0) := \frac{1}{2\pi i} \int_\sigma \frac{dz}{z - z_0} = \sum_{k=1}^{p} n(\gamma_k, z_0).$$

1.3. Definition. Let σ be a cycle in an open set $G \subset \mathbf{C}$. The cycle σ is *null-homologous* in the set G, if

$$n(\sigma, z) = 0 \quad \text{for all } z \in \mathbf{C} \setminus G\,.$$

A closed path γ in an open set G is *null-homologous* in the set G, if the cycle $\sigma = (\gamma)$ is null-homologous in the set G.



Two cycles $\sigma_0 = (\gamma_1, \gamma_2, \ldots, \gamma_p)$ and $\sigma_1 = (\beta_1, \beta_2, \ldots, \beta_q)$ in the open set G are *homologous* in the set G if the cycle

$$\sigma = (\gamma_1, \gamma_2, \ldots, \gamma_p, \overleftarrow{\beta_1}, \overleftarrow{\beta_2}, \ldots, \overleftarrow{\beta_q})$$

is null-homologous in the set G.

The paths λ_0 and λ_1 in the open set G are *homologous* in the set G if they have common starting and ending points and the closed path $\lambda = \lambda_0 * \overleftarrow{\lambda_1}$ is null-homologous in the set G.

1.4. Remark. Cycles σ_0 and σ_1 in the (open) set G are homologous in the set G if, and only if,

$$n(\sigma_0, z) = n(\sigma_1, z) \quad \text{for all } z \in \mathbf{C} \setminus G\,.$$

This follows directly from the definition and the following calculation (notations as above):

$$\begin{aligned} n(\sigma, z) &= \sum_{k=1}^{p} n(\gamma_k, z) + \sum_{m=1}^{q} n(\overleftarrow{\beta_m}, z) \\ &= \sum_{k=1}^{p} n(\gamma_k, z) - \sum_{m=1}^{q} n(\beta_m, z) = n(\sigma_0, z) - n(\sigma_1, z)\,, \end{aligned}$$

where $z \in \mathbf{C} \setminus G$.

1.5. Remark. All cycles in the disk $B \subset \mathbf{C}$ are null-homologous in the set B. Let $\sigma = (\gamma_1, \gamma_2, \ldots, \gamma_p)$ be a cycle in the disk B. Then $|\sigma| \subset B$ and thus $\mathbf{C} \setminus B$ is contained in the unbounded component of $\mathbf{C} \setminus |\gamma_k|$ for all $k = 1, 2, \ldots, p$. By Lemma CA1.5.4, for all $z \in \mathbf{C} \setminus B$, we have

$$n(\sigma, z) = \sum_{k=1}^{p} n(\gamma_k, z) = 0\,.$$

We need an auxiliary result on the interchange of order of integration in iterated (complex) integrals. If $R = \{z = x + iy : a \le x \le b,\ c \le y \le d\}$ is a closed rectangle and $h : R \to \mathbf{C}$ is continuous, then by Fubini's theorem,

$$\int_c^d \Big(\int_a^b h(t,s)dt \Big) ds = \int_a^b \Big(\int_c^d h(t,s)ds \Big) dt. \tag{1.1}$$

Furthermore,

1.6. Lemma. *Let $\gamma : [a,b] \to \mathbf{C}$ and $\beta : [c,d] \to \mathbf{C}$ be paths and $g : |\gamma| \times |\beta| \to \mathbf{C}$ continuous.*[1] *Then*

$$\int_\beta \Big(\int_\gamma g(z,\zeta)dz \Big) d\zeta = \int_\gamma \Big(\int_\beta g(z,\zeta)d\zeta \Big) dz.$$

Proof: Since g is continuous on the compact set $|\gamma| \times |\beta|$, g is uniformly continuous. Thus, the mappings

$$\zeta \mapsto \int_\gamma g(z,\zeta)dz \quad \text{and} \quad z \mapsto \int_\beta g(z,\zeta)d\zeta$$

are continuous (exercise). Hence, the integrals are well-defined.

For simplicity, let's assume that β and γ are continuously differentiable — the general case can be obtained by partitioning $[a,b]$ and $[c,d]$ into subintervals and summing. Now, define the function h as follows,

$$h(t,s) := g(\gamma(t), \beta(s))\gamma'(t)\beta'(s),$$

[1] Remember: $|\gamma| \times |\beta| = \{(z,w) \in \mathbf{C}^2 : z \in |\gamma|, w \in |\beta|\}$.

which is continuous on the rectangle $R = [a,b] \times [c,d]$. Therefore, from equation (1.1), it follows that

$$\int\limits_{\beta} \Big(\int\limits_{\gamma} g(z,\zeta) dz \Big) d\zeta = \int\limits_{c}^{d} \Big(\int\limits_{a}^{b} g(\gamma(t),\beta(s))\gamma'(t) dt \Big) \beta'(s) ds$$
$$= \int\limits_{c}^{d} \Big(\int\limits_{a}^{b} g(\gamma(t),\beta(s))\gamma'(t)\beta'(s) dt \Big) ds = \int\limits_{a}^{b} \int\limits_{c}^{d} h(t,s) ds\, dt$$
$$= \int\limits_{a}^{b} \Big(\int\limits_{c}^{d} g(\gamma(t),\beta(s))\beta'(s) ds \Big) \gamma'(t) dt = \int\limits_{\gamma} \Big(\int\limits_{\beta} g(z,\zeta) d\zeta \Big) dz.$$

□

1.7. Theorem (Global Cauchy theorem)**.** *Let σ be a cycle in an open set G. Then*

$$\int\limits_{\sigma} f(z)\, dz = 0 \qquad \text{for all analytic functions } f : G \to \mathbf{C}$$

if and only if σ is null-homologous in G.

Proof: First, let's prove the necessity of the condition. Let $z_0 \in \mathbf{C} \setminus G$. Then

$$f(z) = \frac{1}{z - z_0}$$

is analytic in G, so

$$0 = \frac{1}{2\pi i} \int\limits_{\sigma} f(z)\, dz = \frac{1}{2\pi i} \int\limits_{\sigma} \frac{dz}{z - z_0} = n(\sigma, z_0).$$

Thus, σ is null-homologous in G.

Let's then tackle the sufficiency of the condition. Let σ be a null-homologous cycle in the set G. Define

$$V = \{z \in \mathbf{C} \setminus |\sigma| : n(\sigma, z) = 0\}.$$

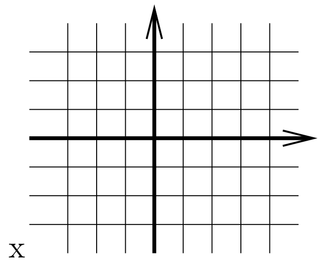
Since $n(\sigma, \cdot)$ is constant on the components of the set $\mathbf{C} \setminus |\sigma|$ (details exercise—remember Lemma CA1.5.4), V is a union of components of the set $\mathbf{C} \setminus |\sigma|$. Thus,

V is open. Moreover, by Lemma CA1.5.4, V contains an unbounded component of $\mathbf{C} \setminus |\sigma|$. By assumption, $\mathbf{C} \setminus G \subset V$. Hence, $K := \mathbf{C} \setminus V \subset G$ is closed and bounded, i.e., compact. Furthermore, $|\sigma| \subset K$.

Let $0 < \delta < \operatorname{dist}(K, \partial G)$. Then $B(z, \delta) \subset G$ for all $z \in K$. Partition $\mathbf{C}$ into disjoint coordinate-axis-aligned (closed) squares, whose edges lie on the lines

$$x = \frac{n\delta}{2} \quad \text{and} \quad y = \frac{n\delta}{2}, \qquad n \in \mathbf{Z},$$

i.e., each square's side length is $\frac{\delta}{2}$ and the four corners are at the origin.



The bounded set K intersects only finitely many of these squares. Let them be $Q_1, Q_2, \ldots, Q_r$. From the construction, if z_j is the center of square Q_j, then $B_j := B(z_j, \frac{\delta}{2}) \subset G$, since otherwise $B(z, \delta) \cap \partial G \neq \emptyset$ for all $z \in Q_j \cap K$, which is a contradiction. Furthermore, $Q_j \subset B_j$.

Now let f be analytic in the set G.
Claim:

$$\int_\sigma f(z)\, dz = 0\,.$$

Let $k = 1, \ldots, r$ and fix $z \in \operatorname{int}(Q_k)$. Now apply the local Cauchy integral formula (Theorem CA1.5.5) in the disk B_k. We obtain

$$f(z) = \underbrace{n(\partial Q_k, z)}_{=1} f(z) = \frac{1}{2\pi i} \int_{\partial Q_k} \frac{f(\zeta) d\zeta}{\zeta - z}\,,$$

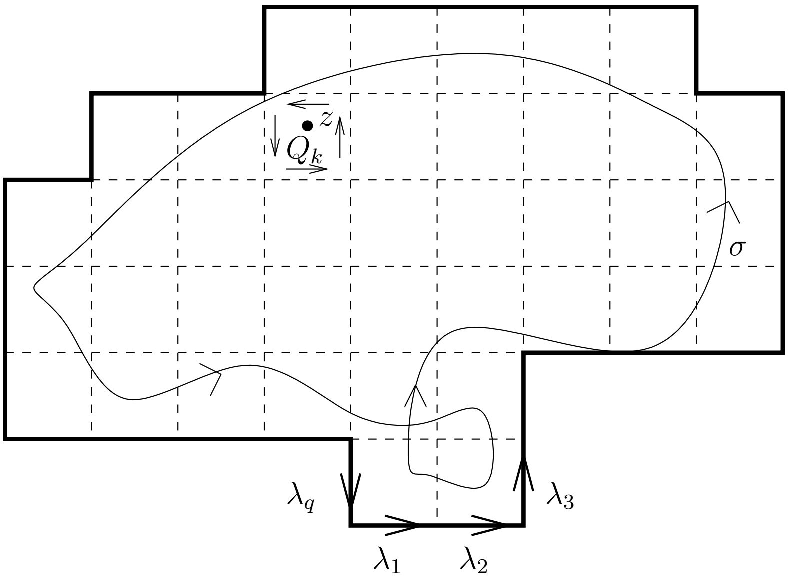
where the boundary is oriented counterclockwise. If $j \neq k$, then from the local Cauchy theorem (case $z \notin B_j$) or the integral formula (case $z \in B_j$) in the disk B_j, it follows that

$$0 = \underbrace{n(\partial Q_j, z)}_{=0} f(z) = \frac{1}{2\pi i} \int_{\partial Q_j} \frac{f(\zeta) d\zeta}{\zeta - z}.$$

Thus,

$$(*) \qquad f(z) = \frac{1}{2\pi i} \sum_{j=1}^{r} \int_{\partial Q_j} \frac{f(\zeta)d\zeta}{\zeta - z}$$

for all $z \in \text{int}(Q_k)$. Equation $(*)$ holds for all k, thus it holds for all $z \in \bigcup_{j=1}^{r} \text{int}(Q_j)$.



Let λ now be one of the directed edges forming the boundary ∂Q_j. Now either $|\lambda| \cap K = \emptyset$ or $|\lambda| \cap K \neq \emptyset$.

If $|\lambda| \cap K \neq \emptyset$, then $|\lambda|$ is also a side of another square Q_k. In this case, the integral on the right-hand side of equation $(*)$ traverses the edges λ and $\overleftarrow{\lambda}$, which cancel each other out when integrated. Thus, for all $z \in \bigcup_{j=1}^{r} \text{Int } Q_j$,

$$(**) \qquad f(z) = \frac{1}{2\pi i} \sum_{k=1}^{q} \int_{\lambda_k} \frac{f(\zeta)d\zeta}{\zeta - z},$$

where $\lambda_1, \ldots, \lambda_q$ are the directed boundary edges of squares Q_j whose traces do not intersect set K. By Lemma CA1.5.6, we obtain that

$$H(z) = \frac{1}{2\pi i} \int_{\lambda_k} \frac{f(\zeta)d\zeta}{\zeta - z}$$

defines an analytic function, specifically a continuous function in $\mathbb{C} \setminus |\lambda_k|$. Thus, the right-hand side of equation $(**)$ defines a continuous function in the set

$$\mathbb{C} \setminus \bigcup_{k=1}^{q} |\lambda_k| .$$

Note that

$$K \subset \mathbb{C} \setminus \bigcup_{k=1}^{q} |\lambda_k| .$$

Since the left-hand side of equation $(**)$ is continuously analytic in the set G, $(**)$ also holds on those parts of the edges of squares Q_j that do not belong to the traces of edges λ_j. In particular, $(**)$ holds for all $z \in |\sigma|$.

Finally, in the proof, let $\sigma = (\gamma_1, \gamma_2, \dots, \gamma_p)$. Now,

$$\begin{aligned} \int_\sigma f(z)\, dz &\overset{(**)}{=} \int_\sigma \left(\sum_{k=1}^{q} \frac{1}{2\pi i} \int_{\lambda_k} \frac{f(\zeta) d\zeta}{\zeta - z} \right) dz \\ &= \frac{1}{2\pi i} \sum_{l=1}^{p} \sum_{k=1}^{q} \int_{\gamma_l} \left(\int_{\lambda_k} \frac{f(\zeta) d\zeta}{\zeta - z} \right) dz \end{aligned}$$

and since the integrand is continuous in the set

$$\bigcup_{k,l} |\gamma_l| \times |\lambda_k| ,$$

by Lemma 1.6, the above is equal to

$$\begin{aligned} &\frac{1}{2\pi i} \sum_{k=1}^{q} \sum_{l=1}^{p} \int_{\lambda_k} \left(\int_{\gamma_l} \frac{f(\zeta)}{\zeta - z} \right) dz\, d\zeta \\ &\qquad = \sum_{k=1}^{q} \int_{\lambda_k} f(\zeta) \left(\sum_{l=1}^{p} \frac{1}{2\pi i} \int_{\gamma_l} \frac{dz}{\zeta - z} \right) d\zeta \\ &\qquad = - \sum_{k=1}^{q} \int_{\lambda_k} f(\zeta) \left(\sum_{l=1}^{p} \frac{1}{2\pi i} \int_{\gamma_l} \frac{dz}{z - \zeta} \right) d\zeta \\ &\qquad = - \sum_{k=1}^{q} \int_{\lambda_k} f(\zeta) n(\sigma, \zeta)\, d\zeta = 0, \end{aligned}$$

since $n(\sigma, \zeta) = 0$ for all $\zeta \in \lambda_k$, which follows from the fact that since $|\lambda_k| \cap K = \emptyset$, then $|\lambda_k| \subset V$. $\square$

1.8. Corollary. *Let f be analytic in an open set G, and let σ_0 and σ_1 be homologous cycles or paths in G. Then*

$$\int_{\sigma_0} f(z)\,dz = \int_{\sigma_1} f(z)\,dz.$$

PROOF: If $\sigma_0 = (\gamma_1, \dots, \gamma_p)$ and $\sigma_1 = (\beta_1, \dots, \beta_q)$ are cycles and homologous in G, then we apply Cauchy's theorem in G to the null homologous cycle $\sigma = (\gamma_1, \dots, \gamma_p, \overleftarrow{\beta_1}, \dots, \overleftarrow{\beta_q})$, where

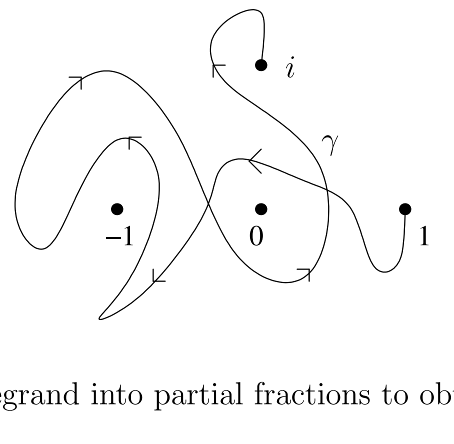
$$\int_{\sigma_0} f(z)\,dz - \int_{\sigma_1} f(z)\,dz = \int_{\sigma} f(z)\,dz = 0.$$

If σ_0 and σ_1 are paths, the claim follows similarly by applying Cauchy's theorem to the null homologous cycle $(\sigma_0 * \overleftarrow{\sigma_1})$. □

Example. Calculate the integral

$$\int_{\gamma} \frac{z^2+z+1}{z^3+z^2}\,dz$$

along the path γ depicted in the figure below.



We decompose the integrand into partial fractions to obtain

$$\int_{\gamma} \frac{z^2+z+1}{z^3+z^2} dz = \int_{\gamma} \left(\frac{1}{z^2} + \frac{1}{z+1} \right) dz = \int_{\gamma} \frac{dz}{z^2} + \int_{\gamma} \frac{dz}{z+1}.$$

Since $F(z) = -z^{-1}$ is a primitive of the function $f(z) = z^{-2}$ in $\mathbb{C} \setminus \{0\}$, the first integral is easy to compute:

$$\int_\gamma \frac{dz}{z^2} = \frac{1}{z}\Big|_1^i = 1 + i.$$

Since γ is homologous to the line segment $\beta = [1, i]$ in $\mathbb{C} \setminus \{-1\}$, where the function $g(z) = (z+1)^{-1}$ is also analytic, Corollary 1.8 yields

$$\int_\gamma \frac{dz}{z+1} = \int_\beta \frac{dz}{z+1} = \mathrm{Log}(z+1)|_1^i = \ln\sqrt{2} + \frac{\pi i}{4} - \ln 2 = -\frac{\ln 2}{2} + \frac{\pi i}{4}.$$

Here, we utilize the fact that $G(z) = \mathrm{Log}(z+1)$ is a primitive of the function g in $\mathbb{C} \setminus (-\infty, -1]$, which contains the line segment $\beta = [1, i]$. Thus, we obtain

$$\int_\gamma \frac{z^2 + z + 1}{z^3 + z^2} dz = 1 - \ln\sqrt{2} + i(1 + \frac{\pi}{4}).$$

1.9. Theorem (Cauchy's integral formula)**.** *Let f be analytic in the set G, and σ be a null-homologous cycle in the set G. Then*

$$n(\sigma, z) f(z) = \frac{1}{2\pi i} \int_\sigma \frac{f(\zeta) d\zeta}{\zeta - z}$$

for all $z \in G \setminus |\sigma|$.

Proof: Compare with the proof of Cauchy's local version (Theorem CA1.5.5).

Let $z \in G \setminus |\sigma|$ and define $g : G \to \mathbf{C}$,

$$g(\zeta) = \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z}, & \text{when } \zeta \neq z \\ f'(z), & \text{when } \zeta = z. \end{cases}$$

Then g is continuous in G and analytic in $G \setminus \{z\}$. Now by Theorem CA1.5.10, it follows that f is analytic in G. Hence, by Cauchy's theorem,

$$\begin{aligned} 0 = \int_\sigma g(\zeta) d\zeta &= \int_\sigma \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta \\ &= \int_\sigma \frac{f(\zeta)}{\zeta - z} d\zeta - 2\pi i\, n(\sigma, z) f(z). \end{aligned}$$

□

1.10. Theorem. *Let f be analytic and σ be a null-homologous cycle in G. Then for all $k = 1, 2, \ldots$*

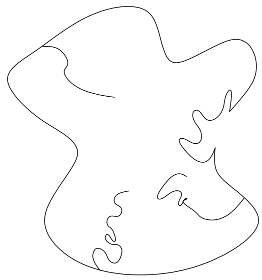
$$n(\sigma, z) f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\sigma} \frac{f(\zeta)}{(\zeta - z)^{k+1}} \, d\zeta \,,$$

when $z \in G \setminus |\sigma|$.

Proof: Omitted (analogous in the proof of Theorem CA1.5.11). □

Recall that a *domain* (or *region*) is a nonempty connected open set in **C**.

1.11. Definition. A domain $D \subset \mathbf{C}$ is *simply connected* if every closed curve γ (and thus every cycle) in the set D is null-homologous in D.



1.12. Remark. The assertions of Cauchy's theorem and integral formula hold for all cycles in simply connected domains.

1.13. Remark. A bounded domain D is simply connected if and only if $\mathbb{C} \setminus D$ is connected (exercise).

An unbounded domain $D \neq \mathbb{C}$ is simply connected if and only if all components of the set $\mathbb{C} \setminus D$ are unbounded (exercise).

1.14. Theorem. *Let $D \subset \mathbb{C}$ be a domain. Every analytic function $f : D \to \mathbb{C}$ has a primitive in the domain D, if and only if D is simply connected.*

Proof: Let D be a simply connected domain and $f : D \to \mathbb{C}$ be analytic. Then from Cauchy's theorem it follows that

$$\int_{\gamma} f(z) \, dz = 0 \quad \text{for all closed paths } \gamma \,, \quad |\gamma| \subset D \,.$$

Now by Theorem CA1.4.5 function f has a primitive in the domain D (in other words, there exists an analytic function $F : D \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in D$).

Now let γ be a closed path in the set D and $z_0 \in \mathbb{C} \setminus D$. Now the function

$$f(z) = \frac{1}{z - z_0}$$

is analytic in the domain D, thus it has a primitive there. Hence by Theorem CA1.3.15

$$0 = \frac{1}{2\pi i} \int_\gamma f(z)\, dz = n(\gamma, z_0).$$

The path γ is therefore null-homologous, hence D is simply connected. □

1.15. Theorem. *Let D be a simply connected domain with $0 \notin D$. There is an analytic function g in D such that $e^{g(z)} = z$ for $z \in D$ (i.e. there exists a branch of the logarithm in D).*

PROOF: From Theorem 1.14, it follows that the function $\frac{1}{z}$ has a primitive f in the domain D. Let $z_0 \in D$ be fixed and set

$$\begin{aligned} g(z) &= f(z) - f(z_0) + \operatorname{Log} z_0 \\ F(z) &= z e^{-g(z)} \end{aligned}$$

Now g and F are analytic and g is a primitive of the function $\frac{1}{z}$ since $g'(z) = \frac{1}{z}$. Furthermore,

$$F'(z) = e^{-g(z)}(1 - z\frac{1}{z}) = 0.$$

Thus F is constant and since

$$F(z_0) = z_0 e^{-g(z_0)} = z_0 e^{-\operatorname{Log} z_0} = \frac{z_0}{z_0} = 1,$$

we have $F(z) = 1$ for all $z \in D$. In other words, $z = e^{g(z)}$ for all $z \in D$, i.e., g is a branch of logarithm in the domain D. □

1.1. Alternative proof of Cauchy's integral formula

There are several different proofs of Cauchy's theorem and integral formula. Here we give another one, which proceeds by showing Cauchy's integral formula using several of the theorems proved in CA1.

Proof: Let f be analytic in G and let σ be a null-homologous cycle in G. We wish to prove Cauchy's integral formula

$$n(\sigma, z)f(z) = \frac{1}{2\pi i} \int_\sigma \frac{f(w)}{w - z}\, dw$$

for all $z \in G \setminus |\sigma|$. Recalling the definition of the winding number, this is equivalent to proving that

$$h(z) := \int_\sigma g(z, w)\, dw = 0, \qquad z \in G \setminus |\sigma|,$$

where $g(z, w)$ is defined for $z, w \in G$ by

$$g(z, w) = \begin{cases} \frac{f(w)-f(z)}{w-z}, & z \neq w, \\ f'(z), & z = w. \end{cases}$$

The function $g(z, w)$ is continuous in $G \times G$ (exercise) and hence $h(z)$ is a well defined function in G.

Let us prove that h is analytic in G. We first show that h is continuous in G. Let $z \in G$ and let $z_n \to z$ in G. Then

$$h(z_n) = \int_\sigma g(z_n, w)\, dw \to \int_\sigma g(z, w)\, dw = h(z).$$

Here we used that $g(z_n, \cdot\,)$ converges uniformly to $g(z, \cdot\,)$ in the compact set $|\sigma|$, which follows from the uniform continuity of g on compact sets (exercise). Thus h is continuous in G. We now apply Morera's theorem: if Δ is a triangle in G, then by Fubini's theorem (Lemma 1.6)

$$\int_\Delta h(z)\, dz = \int_\Delta \left[\int_\sigma g(z, w)\, dw \right] dz = \int_\sigma \left[\int_\Delta g(z, w)\, dz \right] dw.$$

For any fixed $w \in G$, the function $z \mapsto g(z, w)$ is analytic in G. (If $z \neq w$ this follows from the definition of g, and if $z = w$ this follows from the theorem of analytic continuation to a point, Theorem CA1.5.10.) Then the inner integral vanishes by Cauchy's theorem for triangles. Thus $\int_\Delta h(z)\,dz = 0$ for any triangle $\Delta \subset G$, which implies that h is analytic in G by Morera's theorem.

Next we wish to prove that there is an analytic function φ in $\mathbf{C}$ with $\varphi|_G = h$. To do this, let $G_1 = \{z \in \mathbf{C} \setminus |\sigma| \ : \ n(\sigma, z) = 0\}$. Motivated by the statement of Cauchy's integral formula, we define

$$h_1(z) := \int\limits_\sigma \frac{f(w)}{w-z}\,dw, \qquad z \in G_1.$$

Then h_1 is analytic in G_1 by Lemma CA1.5.6. If $z \in G \cap G_1$, the definition of G_1 ensures that $h(z) = h_1(z)$. We may thus define

$$\varphi(z) = \begin{cases} h(z), & z \in G, \\ h_1(z), & z \in G_1. \end{cases}$$

Since G_1 contains the unbounded component of $\mathbf{C} \setminus |\sigma|$, it also contains $\mathbf{C} \setminus G$. It follows that φ is an analytic function in $\mathbf{C}$ with $\varphi|_G = h$.

Finally, we have

$$\lim_{[z|\to\infty} \varphi(z) = \lim_{[z|\to\infty} h_1(z) = 0.$$

Thus φ is a bounded analytic function in $\mathbf{C}$ (it is bounded in $|z| \geq R$ by the above limit, and in $|z| \leq R$ by continuity). Liouville's theorem implies that φ is a constant function, and the above limit ensures that $\varphi \equiv 0$. This gives that $h(z) = 0$ for $z \in G$, which concludes the proof. □

2. Power series representation of analytic functions

2.1. On complex series

2.1. Definition. Let $z_n \in \mathbb{C}$, $n = 1, 2, \ldots$. A complex series

$$\sum_{n=1}^{\infty} z_n$$

converges (*to the number* $s \in \mathbb{C}$) if there exists a limit

$$\lim_{k\to\infty} \sum_{n=1}^{k} z_n \quad (= s \in \mathbf{C}).$$

In this case, we denote

$$s = \sum_{n=1}^{\infty} z_n$$

and say that s is the sum of the series $\sum_{n=1}^{\infty} z_n$. In other words, if

$$s_k = \sum_{n=1}^{k} z_n\,,$$

then

$$\text{the series} \quad \sum_{n=1}^{\infty} z_n \quad \text{converges,}$$

if and only if the sequence of partial sums $(s_k)_{k=1}^{\infty}$ converges.

If the series $\sum_{n=1}^{\infty} z_n$ does not converge, it *diverges.*

We say that the series $\sum_{n=1}^{\infty} z_n$ *converges absolutely* if the series

$$\sum_{n=1}^{\infty} |z_n| \quad \text{converges.}$$

2.2. Remark (A few exercises)**.**

- If $\sum_{n=1}^{\infty} z_n$ converges absolutely, then $\sum_{n=1}^{\infty} z_n$ converges.
- If $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} w_n$ converge and $c \in \mathbb{C}$, then $\sum_{n=1}^{\infty}(cz_n + w_n)$ converges to the number $c\sum_{n=1}^{\infty} z_n + \sum_{n=1}^{\infty} w_n$.
- If $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} w_n$ converge absolutely and $c \in \mathbb{C}$, then $\sum_{n=1}^{\infty}(cz_n+w_n)$ converges absolutely to the number $c\sum_{n=1}^{\infty} z_n + \sum_{n=1}^{\infty} w_n$.

In the future, we will need other types of indexing besides $\sum_{n=1}^{\infty}$. These, for example $\sum_{n=-k}^{\infty}$, are defined in a natural way.

Example. Let's examine the geometric series $\sum_{n=0}^{\infty} z^n$, where $z \in \mathbb{C}$. The partial sum of the series is

$$s_k = 1 + z + \cdots + z^k .$$

Note! In this context and thereafter, we denote $0^0 = 1$.

If $z = 1$, then $s_k = k + 1 \to \infty$ and the series diverges.

If $z \neq 1$, then from the formula

$$1 - z^{k+1} = (1 - z)(1 + z + \cdots + z^k)$$

we get

$$s_k = \frac{1 - z^{k+1}}{1 - z} = \frac{1}{1 - z} - \frac{z^{k+1}}{1 - z}.$$

If $|z| < 1$, then the last term

$$\frac{z^{k+1}}{1 - z} \to 0 .$$

If $|z| > 1$, then

$$\left|\frac{z^{k+1}}{1 - z}\right| \to \infty .$$

If $|z| = 1$, $z \neq 1$, then the last term rotates around the circle of radius $1/|1-z|$. Thus, the limit does not exist when $|z| = 1$, $z \neq 1$.

Therefore

$$\sum_{n=0}^{\infty} z^n \quad \text{converges if and only if} \quad |z| < 1$$

and in this case

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}\,.$$

As an exercise, you can prove that the convergence is absolute.

Sequences of functions. In order to develop the theory of infinite series, we need some facts regarding the (uniform) convergence of sequences of functions. We will later study partial sums

$$s_k(z) = \sum_{n=1}^{k} f_n(z)\,,$$

where $f_n : A \to \mathbf{C}$ for each $k \in \mathbf{N}$. First we consider general sequences (s_k) of functions $s_k : A \to \mathbf{C}$.

2.3. Definition. Let $A \subset \mathbf{C}$ and $s_k : A \to \mathbf{C}$ for each $k \in \mathbf{N}$. A function sequence s_k converges *uniformly* on the set A (towards the function $s : A \to \mathbf{C}$) if for every $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that

$$|s_k(z) - s(z)| < \epsilon$$

for all $z \in A$, when $k \geq N$. Note! The number ϵ does NOT depend on the point $z \in A$.

2.4. Remark. If $s_k : A \to \mathbf{C}$ are continuous and $s_k \to s$ uniformly on the set A, then $s : A \to \mathbf{C}$ is continuous.

2.5. Remark. *Remember the Cauchy criterion for uniform convergence:*
A sequence $s_k : A \to \mathbf{C}$ converges uniformly on the set A (towards the function $s : A \to \mathbf{C}$) if and only if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbf{N}$ such that

$$\sup_{z \in A} |s_k(z) - s_m(z)| < \epsilon$$

when $k, m \geq N$.

2.6. Definition. It is said that the function sequence $s_k : A \to \mathbf{C}$ converges *locally uniformly* on the set A (towards the function $s : A \to \mathbf{C}$) if $s_k \to s$ uniformly on every compact subset K of the set A.

Example. Let's consider functions $s_k(z) = z^k$, $z \in B(0,1)$. Let $K \subset B(0,1)$ be a compact set, then $K \subset B(0,r)$ for some $0 < r < 1$. Assume $\epsilon > 0$ and choose N such that $r^N < \epsilon$. Now

$$|s_k(z) - 0| = |z|^k < r^k \leq r^N < \epsilon\,,$$

for all $z \in K$, when $k \geq N$. Thus $s_k \to 0$ uniformly on the set K. Since K was an arbitrary compact subset of $B(0,1)$, then $s_k \to 0$ locally uniformly in the disk $B(0,1)$. Note! The function sequence s_k does not converge uniformly in the disk $B(0,1)$ (exercise).

2.7. Remark. A singleton $\{z\}$ is compact, so if $s_k \to s$ locally uniformly, then $s_k \to s$ pointwise. In particular, the limit function of a locally uniformly convergent sequence is unique.

2.8. Lemma. *Let $s_k : G \to \mathbf{C}$ be continuous functions, $k \in \mathbf{N}$, which converge locally uniformly in an open set G (towards some function $s : G \to \mathbf{C}$). If γ is a path in the set G, then*

$$\int_\gamma s(z)\,dz = \lim_{k\to\infty} \int_\gamma s_k(z)\,dz\,.$$

PROOF: Exercise; remember that the trace $|\gamma| \subset G$ is compact. □

Uniform convergence of analytic functions $s_k : G \to \mathbf{C}$ (locally) is so strong that the order of differentiation and taking limits can be interchanged.

2.9. Theorem. *Assume that the sequence of analytic functions $s_k : G \to \mathbf{C}$ converges locally uniformly in an open set G towards a function $s : G \to \mathbf{C}$. Then s is analytic and $s_k^{(n)} \to s^{(n)}$ locally uniformly in the set G for all $n = 0, 1, 2, \ldots$.*

Proof: Since $s_k \to s$ locally uniformly in the open set G, s is continuous in the set G. From Lemma 2.8 and Cauchy's theorem, it follows that

$$\int_{\partial R} s(z)dz = \lim_{k\to\infty} \int_{\partial R} s_k(z)dz = 0$$

for all closed rectangles R in the set G. Therefore, by Morera's theorem CA1.5.9, $s : G \to \mathbf{C}$ is analytic.

To prove the local uniform convergence of derivatives, it suffices to show that $s_k' \to s'$ locally uniformly (the claim follows from this inductively). For this purpose, it suffices to show that $s_k' \to s'$ uniformly in the disk $B(z_0, r)$, where $\overline{B} = \overline{B}(z_0, 2r) \subset G$. (Why?) Next, we apply Cauchy's estimate CA1.5.12 to the analytic function $s_k - s$ in the disk $B(z_0, 2r)$. So for all $z \in B(z_0, r)$

$$\begin{aligned} |s_k'(z) - s'(z)| &\leq \frac{\sup_{\zeta\in\overline{B}} |s(\zeta) - s_k(\zeta)| 2r}{(2r - |z - z_0|)^2} \\ &\leq \underbrace{\frac{2}{r} \sup_{\zeta\in\overline{B}} |s(\zeta) - s_k(\zeta)|}_{\text{independent of } z} \xrightarrow{k\to\infty} 0, \end{aligned}$$

since $s_k \to s$ uniformly in the disk $\overline{B} \subset G$. Since the last upper limit converges to zero independently of the point z, $s_k' \to s'$ uniformly in the disk $B(z_0, r)$. □

2.10. Remark. Theorem 2.9 does not hold if we only assume that $s_k \to s$ pointwise, even if s is known to be analytic. If the sequence s_k is locally uniformly bounded, then by Cauchy's estimate (Theorem CA1.5.12), it can be seen that pointwise convergence implies local uniform convergence (exercise).

Series of functions. Let $f_n : A \to \mathbf{C}$, $n = 1, 2, \ldots$. We say that the function series $\sum_{n=1}^{\infty} f_n$ *converges (pointwise) in the set* A, if the series

$$\sum_{n=1}^{\infty} f_n(z) \quad \text{converges for every } z \in A\,.$$

The series $\sum_{n=1}^{\infty} f_n$ converges absolutely in the set A if $\sum_{n=1}^{\infty} |f_n|$ converges in the set A.

We say that the function series $\sum\limits_{n=1}^{\infty} f_n$ *converges uniformly in the set* A if the sequence of functions

$$s_k = \sum_{n=1}^{k} f_n : A \to \mathbf{C}\,, \qquad k \in \mathbf{N}\,,$$

uniformly converges in the set A.

We say that the function series $\sum\limits_{n=1}^{\infty} f_n$ *converges locally uniformly in the set* A if the sequence of functions s_k (as above) converges locally uniformly in the set A.

2.11. Remark. If the function series $\sum\limits_{n=1}^{\infty} f_n$ converges uniformly in the set A, then the sequence of functions $(f_n)_{n\in\mathbf{N}}$ converges uniformly in the set A towards the zero function. This follows from the uniform convergence criterion, i.e., Remark 2.5.

2.12. Lemma. *Let $f_n : G \to \mathbf{C}$ be continuous, for which the series*

$$\sum_{n=1}^{\infty} f_n = f$$

converges locally uniformly in the open set G. If γ is a path in the set G, then

$$\int_\gamma f(z)\,dz = \sum_{n=1}^{\infty} \int_\gamma f_n(z)dz.$$

Proof: Let $s_k = \sum_{n=1}^{k} f_n$ for each $k \in \mathbf{N}$. By assumption, the sequence s_k of continuous functions converges locally uniformly in the set G towards the function f. Now, by the linearity of the integral and Lemma 2.8, we have

$$\int_\gamma f(z)\,dz = \lim_{k\to\infty} \int_\gamma s_k(z)\,dz = \lim_{k\to\infty} \sum_{n=1}^{k} \int_\gamma f_n(z)\,dz = \sum_{n=1}^{\infty} \int_\gamma f_n(z)\,dz\,.$$

□

2.13. Theorem. *Let $f_n : G \to \mathbf{C}$ be analytic functions, such that the series*

$$\sum_{n=1}^{\infty} f_n$$

converges locally uniformly in the open set G. Then

$$f = \sum_{n=1}^{\infty} f_n$$

is analytic in the set G, and for all $k = 1, 2, \ldots$ the series of derivatives $\sum_{n=1}^{\infty} f_n^{(k)}$ converges locally uniformly in the set G towards the derivative of f $f^{(k)}$,

$$f^{(k)}(z) = \sum_{n=1}^{\infty} f_n^{(k)}(z) \quad \text{when } z \in G\,.$$

PROOF: Follow from Theorem 2.9 by applying linearity of differentiation (exercise). □

The following test is often a useful way to verify uniform (and pointwise) convergence of a function series.

2.14. Theorem (Weierstrass M-test). *Let $f_n : A \to \mathbf{C}$ be functions, and suppose that for all $n \in \mathbf{N}$, there exists $M_n < \infty$ such that*

$$|f_n(z)| \leq M_n$$

for all $z \in A$. If the series $\sum_{n=1}^{\infty} M_n$ converges, then the function series $\sum_{n=1}^{\infty} f_n$ converges uniformly and absolutely on the set A.

PROOF: Pointwise convergence is an easy exercise. Let $\varepsilon > 0$. Since $\left(\sum_{n=1}^{k} M_n\right)_k$ is a Cauchy sequence, there exists a number $N = N(\varepsilon)$ such that

$$\sum_{n=1}^{k} M_n - \sum_{n=1}^{m} M_n < \varepsilon$$

when $k \geq m \geq N$. Let

$$s_k = \sum_{n=1}^{k} f_n : A \to \mathbf{C}\,.$$

Now, when $k > m \geq N$, for all $z \in A$

$$\begin{aligned}
|s_k(z) - s_m(z)| &= \left| \sum_{n=m+1}^{k} f_n(z) \right| \\
&\leq \sum_{n=m+1}^{k} |f_n(z)| \leq \sum_{n=m+1}^{k} M_n \\
&= \sum_{n=1}^{k} M_n - \sum_{n=1}^{m} M_n < \varepsilon\,.
\end{aligned}$$

Thus, the sequence of functions s_k satisfies uniform convergence by the Cauchy criterion (see Remark 2.5). □

2.15. Definition. If $a_n \in \mathbf{R}$, $n = 1, 2, \ldots$, then define

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} (\sup\{a_n, a_{n+1}, \ldots\}) = \inf_n (\sup\{a_n, a_{n+1}, \ldots\}) \in [-\infty, \infty]$$

and

$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty} (\inf\{a_n, a_{n+1}, \ldots\}) = \sup_n (\inf\{a_n, a_{n+1}, \ldots\}) \in [-\infty, \infty].$$

2.16 . Remark. Instead of the interpretation of complex limits familiar from Complex Analysis 1 course, which we usually use, the two limits appearing above are taken on the extended real line $\mathbf{R} \cup \{\pm\infty\}$. Since (for example)

$$b_n = \sup\{a_n, a_{n+1}, \ldots\}$$

is a decreasing sequence of numbers in the interval $(-\infty, \infty]$, it must have a limit (on the extended real line) and thus lim sup is well-defined. Also lim inf is well-defined.

2.17. Remark. If the limit $\lim_{n\to\infty} a_n$ exists on the extended real line, then

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n\,.$$

Example. Let

$$a_n = i^{2n}, \qquad \text{then} \qquad \liminf_{n\to\infty} a_n = -1 < 1 = \limsup_{n\to\infty} a_n\,.$$

2.18. Definition. Let $z_0 \in \mathbf{C}$. Series of the form

$$(*) \qquad \sum_{n=0}^{\infty} a_n(z-z_0)^n,$$

$a_0, a_1, \dots \in \mathbf{C}$, are called *power series around point z_0 (or Taylor series at point z_0)*. The numbers a_n are *coefficients* of the power series $(*)$. Let

$$\rho = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}},$$

then[2] $0 \le \rho \le \infty$ is the *radius of convergence* of series $(*)$. In this case, the disk $B(z_0,\rho)$ is called the *disk of convergence* of series $(*)$. If $\rho = \infty$, then $B(z_0,\rho) = \mathbf{C}$.

2.19. Theorem. *Let ρ be the radius of convergence of the power series $\sum\limits_{n=0}^{\infty} a_n(z-z_0)^n$. Then the following holds:*

- *The series $\sum\limits_{n=0}^{\infty} a_n(z-z_0)^n$ diverges for all z such that $|z-z_0| > \rho$.*
- *If $\rho > 0$, then the series $\sum\limits_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely and locally uniformly in the disk $B(z_0,\rho)$ and thus the function*

  $$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

  is analytic in the disk $B(z_0,\rho)$. Moreover,

  $$f^{(n)}(z_0) = n!a_n. \tag{2.1}$$

[2] We define $\frac{1}{0} := \infty$ and $\frac{1}{\infty} := 0$.

2.20. Remark. Theorem 2.19 does not discuss the convergence of the series at the boundary of the convergence disk $B(z_0, \rho)$. The series may either converge or diverge there (exercise).

Proof of Theorem 2.19: **1.** Let's consider a number $z \in \mathbf{C}$ such that $|z - z_0| = r > \rho$. Now $r^{-1} < \rho^{-1} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. This means there are infinitely many indices n, for which

$$\sqrt[n]{|a_n|} > r^{-1} \quad \text{or} \quad |a_n| > r^{-n},$$

so

$$|a_n(z - z_0)^n| = |a_n||z - z_0|^n > r^{-n}r^{-n} = 1$$

for infinitely many n. Especially,

$$|a_n(z - z_0)^n| \not\to 0\,,$$

so the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ diverges.

2. Let $\rho > 0$ and $0 < r < \rho$. It suffices to show that $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges uniformly and absolutely in the closed disk $\bar{B}_r := \overline{B}(z_0, r)$. (Why?) Let $s \in (r, \rho)$. Since $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \rho^{-1} < s^{-1}$, there exists $N \in \mathbb{N}$ such that

$$\sqrt[n]{|a_n|} < s^{-1}, \quad \text{when} \quad n \geq N\,.$$

Let $c = \max\{1, |a_0|, |a_1|s, \ldots, |a_N|s^N\}$, then

$$|a_n| \leq cs^{-n} \quad \text{for all } n.$$

In other words,

$$|a_n(z - z_0)^n| \leq c\left(\frac{r}{s}\right)^n =: M_n \quad \text{for all } z \in \bar{B}_r.$$

Since $\sum_{n=0}^{\infty} M_n$ converges, as $\frac{r}{s} < 1$, by the Weierstrass M-test (2.14), it follows that the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges uniformly and absolutely in the disk $\bar{B}_r$.

3. The claim regarding the analyticity of function f and its derivatives follows from Theorem 2.13; note that the partial sum functions $s_k = \sum_{n=0}^{k} a_n(z - z_0)^n$ are entire. □

2.2. Power series expansion

An analytic function is locally given by a power series:

2.21. Theorem (Power series expansion)**.** *Let f be analytic in an open set G, and $B = B(z_0, r) \subset G$. Then f has a power series expansion around the point z_0 and f determines the power series uniquely:*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n\,, \quad z \in B,$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!}\,.$$

Proof: *Existence of power series expansion:* Let

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

and $z \in B$. Let s be such that $|z - z_0| < s < r$. If $\zeta \in \partial B(z_0, s)$, then

$$\left|\frac{z - z_0}{\zeta - z_0}\right| = \frac{|z - z_0|}{s} < \frac{s}{s} = 1\,,$$

thus by the formula for the sum of a geometric series, we have

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{\zeta - z_0}\frac{1}{1 - \frac{z-z_0}{\zeta-z_0}} = \frac{f(\zeta)}{\zeta - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{\zeta - z_0}\right)^n = \sum_{n=0}^{\infty}\frac{f(\zeta)(z - z_0)^n}{(\zeta - z_0)^{n+1}}.$$

By the Weierstrass M-test, the last series converges (ζ-wise) locally uniformly in the open set $G \setminus \overline{B}(z_0, |z - z_0|) \supset \partial B(z_0, s)$, thus if $\gamma(t) = z_0 + se^{it}$, $0 \le t \le 2\pi$, then by the Cauchy integral formula, we get

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty}\frac{f(\zeta)(z - z_0)^n}{(\zeta - z_0)^{n+1}}d\zeta\,.$$

By Lemma 2.12,

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_\gamma \frac{f(\zeta)(z - z_0)^n}{(\zeta - z_0)^{n+1}}d\zeta = \sum_{n=0}^{\infty}\left((z - z_0)^n\ \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta\right),$$

and since by the Cauchy integral formula (1.10),

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}d\zeta=\frac{f^{(n)}(z_0)}{n!}=a_n\,,$$

we obtain

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n.$$

Uniqueness of power series: Let

$$f(z)=\sum_{n=0}^{\infty}b_n(z-z_0)^n\quad\text{ for all } z\in B\,.$$

Then the convergence radius of the series is at least as large as r, and from Theorem 2.19, it follows that

$$b_n=\frac{f^{(n)}(z_0)}{n!}=a_n\,,$$

which proves the claim. □

Example. Let's find the power series expansion of the function $f(z)=e^z$ around the origin.

Since $f^{(n)}(z)=e^z$ for all z, we have $f^{(n)}(0)=1$ for all n, hence

$$e^z=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}z^n=1+z+\frac{z^2}{2!}+\cdots=\sum_{n=0}^{\infty}\frac{z^n}{n!}\quad\text{ for all } z\in\mathbb{C}\,.$$

As an application of power series expansion, we prove the following theorem of uniqueness:

2.22. Theorem. *Let f be analytic in domain D. Then the following conditions are equivalent:*

i) $f\equiv 0$ in domain D (i.e., $f(z)=0$ for all $z\in D$).

ii) The set

$$N = \{z \in D : \ f(z) = 0\}$$

has an accumulation point in D.

iii) There exists a point $z_0 \in D$, such that

$$f^{(k)}(z_0) = 0 \quad \text{for all } k = 0, 1, 2, 3, \ldots .$$

Proof: Clearly, i) $\Rightarrow$ ii).

Now we show that ii) $\Rightarrow$ iii): Let $z_0 \in D$ be an accumulation point of the set N. Since f is continuous, we have $f(z_0) = 0$. If

$$f^{(k)}(z_0) \neq 0 \quad \text{for some } k \geq 1$$

then let

$$k_0 = \min\{k : \ f^{(k)}(z_0) \neq 0\} \in \mathbb{N}.$$

In this case, the power series of f at the point z_0 is

$$\sum_{n=k_0}^{\infty} a_n (z - z_0)^n \,, \qquad \text{where } a_{k_0} \neq 0.$$

If

$$g(z) = \sum_{n=0}^{\infty} a_{n+k_0} (z - z_0)^n \,,$$

then g is analytic in some neighborhood B of z_0 and

$$f(z) = (z - z_0)^{k_0} g(z) \quad \text{for all } z \in B.$$

Since g is continuous and $g(z_0) = a_{k_0} \neq 0$, there exists a punctured neighborhood U^* of z_0 where

$$g(z) \neq 0 \quad \text{for all } z \in U^*.$$

In particular,

$$0 \neq (z - z_0)^{k_0} g(z) = f(z) \text{ for all } z \in U^* \,,$$

which contradicts the assumption that z_0 is an accumulation point of the set N. Therefore, iii) follows.

Finally, we prove that iii) $\Rightarrow$ i): Let

$$U = \{z \in D : f^{(k)}(z) = 0 \quad \text{for all } k = 0, 1, 2, 3, \ldots\}.$$

Since all derivatives of the analytic function f are continuous (even analytic), the set U is closed in D. Since $z_0 \in U$, it suffices, by the connectedness of D, to show that U is also open: Let $w \in U$ and let

$$f(z) = \sum_{n=0}^{\infty} a_n (z - w)^n$$

be the power series expansion of f, which converges in the disk $B(w, r) \subset D$. By Theorem 2.21,

$$a_n = \frac{f^{(n)}(w)}{n!} = 0 \qquad \text{for all } n = 0, 1, 2, 3, \dots ,$$

so $f(z) = 0$ for all $z \in B(w, r)$, and thus $B(w, r) \subset U$, which means that U is an open set. □

A function f is called *discrete* if for any $w \in \mathbf{C}$ the set $f^{-1}(\{w\})$ a discrete set, i.e., it has no accumulation point in the domain of f. According to Theorem 2.22, an analytic function in domain D is discrete unless it is a constant map.

By Theorem 2.22, the following definition makes sense.

2.23. Definition. Let f be analytic in an open set G and $z_0 \in G$. If $f(z_0) = 0$ and $f \not\equiv 0$ in every neighborhood of z_0, then the number

$$k_0 = \min\{k \in \mathbb{N} : f^{(k)}(z_0) \neq 0\}$$

is called the *order of vanishing at* z_0.

2.24. Theorem. *Assume that f is analytic in an open set G, $z_0 \in G$, and $k_0 = 1, 2, 3, \dots$. Then the following conditions are equivalent:*

i) The point z_0 is a zero of f of order k_0.

ii) There exists an analytic function $g : G \to \mathbb{C}$ such that $g(z_0) \neq 0$ and

$$f(z) = (z - z_0)^{k_0} g(z) \qquad \text{for all } z \in G\,.$$

PROOF: Let z_0 be a zero of f of order k_0, and let

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

be the power series expansion of f in the disk $B(z_0, r) \subset G$. Then $a_n = 0$ for all $n < k_0$, thus

$$\begin{aligned}\frac{f(z)}{(z-z_0)^{k_0}} &= (z-z_0)^{-k_0} \sum_{n=k_0}^{\infty} a_n (z-z_0)^n \\ &= \sum_{n=0}^{\infty} a_{n+k_0} (z-z_0)^n \,, \qquad \text{when } z \in B^*(z_0, r)\,,\end{aligned}$$

where the right-hand side defines an analytic function in the disk $B(z_0, r)$, and its value at z_0 is

$$a_{k_0} = \frac{f^{(k_0)}(z_0)}{k_0!} \neq 0\,.$$

Thus, the function

$$g(z) = \begin{cases} \dfrac{f(z)}{(z-z_0)^{k_0}}\,, & \text{when } z \neq z_0 \\ a_{k_0}\,, & \text{when } z = z_0\,, \end{cases}$$

is the desired function.

Conversely, suppose g is an analytic function such that $g(z_0) \neq 0$ and

$$f(z) = (z-z_0)^{k_0} g(z) \qquad \text{for all } z \in G\,.$$

Then $f \not\equiv 0$ in any neighborhood of the point z_0, and if $k \leq k_0$, then

$$f^{(k)}(z) = \sum_{n=0}^{k} \binom{k}{n} \frac{k_0!}{(k_0 - k + n)!} (z-z_0)^{k_0+n-k} g^{(n)}(z)\,,$$

where

$$\binom{k}{n} = \frac{k!}{n!(k-n)!}\,.$$

Thus,

$$f^{(k)}(z_0) = \begin{cases} 0\,, & \text{when } k < k_0 \\ k_0! g(z_0) \neq 0\,, & \text{when } k = k_0\,, \end{cases}$$

so z_0 is a zero of f of order k_0. □

2.3. Laurent series

For Laurent series, we need double series that are infinite in both directions:

2.25. Definition. *The double series* $\sum_{n=-\infty}^{\infty} z_n$ *converges* if

$$\text{both} \quad \sum_{n=0}^{\infty} z_n \quad \text{and} \quad \sum_{n=1}^{\infty} z_{-n} \quad \text{converge.}$$

Then

$$\sum_{n=-\infty}^{\infty} z_n = \sum_{n=0}^{\infty} z_n + \sum_{n=1}^{\infty} z_{-n}.$$

The series $\sum_{n=-\infty}^{\infty} z_n$ *converges absolutely* if both $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=1}^{\infty} z_{-n}$ converge absolutely.

We say that the series $\sum_{n=-\infty}^{\infty} a_n 0^n$ *diverges* if $a_n \neq 0$ for some $n < 0$.

The *convergence, absolute convergence, and (local) uniform convergence in a set* A of double function series are defined analogously to the corresponding concepts of function series (see p. 16).

2.26. Definition. A *Laurent series at point* $z_0 \in \mathbf{C}$ is of the form

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad , \quad a_n \in \mathbf{C}$$

as a double function series.

The number

$$\rho_O = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}}$$

is the *outer convergence radius* of the Laurent series $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ and

$$\rho_I = \limsup_{n\to\infty} \sqrt[n]{|a_{-n}|}$$

its *inner convergence radius*. When $\rho_I < \rho_O$, we say that

$$D = \{z \in \mathbf{C} : \rho_I < |z - z_0| < \rho_O\}$$

is the *convergence ring* of the series. If $\rho_I = 0$ and $\rho_O = \infty$, then $D = \mathbf{C} \setminus \{0\}$.

2.27. Remark. The Taylor series, i.e., the ordinary power series, is a Laurent series when $a_n = 0$ for all $n < 0$. Its inner convergence radius is 0.

2.28. Theorem. *Let ρ_O and ρ_I be the outer and inner convergence radii of the Laurent series $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$. The series diverges if*

$$|z-z_0| > \rho_O \quad \text{or} \quad |z-z_0| < \rho_I .$$

Additionally, the Laurent series has the following convergence properties:

- *If $\rho_O > 0$, then $\sum_{n=0}^{\infty} a_n(z-z_0)$ converges absolutely and locally uniformly in the disk $B_O = B(z_0, \rho_O)$ and thus defines an analytic function f_O in the disk B_O.*
- *If $\rho_I < \infty$, then $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converges absolutely and locally uniformly in the open set*

  $$\complement B_I = \{z : |z-z_0| > \rho_I\} ,$$

  so the function

  $$f_I(z) = \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$$

  is analytic in the set $\complement B_I$.
- *If $\rho_I < \rho_O$, then the Laurent series under consideration converges absolutely and locally uniformly in the annulus*

  $$D = \{z : \rho_I < |z-z_0| < \rho_O\}$$

  and thus

  $$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0) = f_I(z) + f_O(z)$$

  is analytic in the set D. Moreover, for all $n \in \mathbf{Z}$

  $$a_n = \frac{1}{2\pi i} \int_{\{|z-z_0|=r\}} \frac{f(z)}{(z-z_0)^{n+1}} dz ,$$

  as long as $\rho_I < r < \rho_O$ and $\{z : |z-z_0| = r\}$ is oriented counterclockwise.

Proof: Since ρ_O is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, it follows from it and the statements concerning the function f_O in Theorem 2.19.

Then consider the series

$$\sum_{n=1}^{\infty} a_{-n}\zeta^n,$$

which is a power series with convergence radius of $1/\rho_I$. The series diverges if $|\zeta| > \frac{1}{\rho_I}$, and if $\rho_I < \infty$, it converges absolutely and locally uniformly in the disk $B(0, 1/\rho_I)$. Denote

$$\zeta = \frac{1}{z-z_0},$$

then from the local uniform convergence of the power series follows (exercise) that the series

$$\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$$

converges absolutely and locally uniformly in the set $\{z : |z-z_0| > \rho_I\}$ and diverges when $|z-z_0| < \rho_I$. The function f_I becomes analytic in particular in the set $\complement B_I$.

From these, it follows that the double series $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ diverges, if

$$\text{either} \quad |z-z_0| > \rho_O \quad \text{or} \quad |z-z_0| < \rho_I\,.$$

If $\rho_I < \rho_O$ i.e., $\rho_O > 0$ and $\rho_I < \infty$, then from the above considerations it follows that $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ converges absolutely and locally uniformly in the open ring

$$D = B_O \cap \complement B_I$$

and there

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = f_O(z) + f_I(z)$$

is analytic.

Let $\rho_I < r < \rho_O$ and $k \in \mathbf{Z}$. Denote

$$\frac{f(z)}{(z-z_0)^{k+1}} = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^{n-k-1} \quad , \quad z \in D.$$

Let $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$. According to Lemma 2.12, the order of integration and summation can be exchanged, because the series converges locally uniformly in the set $D \supset |\gamma|$, and we obtain

$$\begin{aligned}
\frac{1}{2\pi i} \int\limits_{\gamma} \frac{f(z)}{(z - z_0)^{k+1}} dz &= \frac{1}{2\pi i} \int\limits_{\gamma} \sum_{n=-\infty}^{\infty} a_n (z - z_0)^{n-k-1} dz \\
&= \sum_{n=-\infty}^{\infty} \frac{a_n}{2\pi i} \int\limits_{\gamma} (z - z_0)^{n-k-1} dz \\
&= a_k \, ,
\end{aligned}$$

since the function $(z - z_0)^{n-k-1}$ has a primitive, except when $n = k$. □

Conversely, for an analytic function in the annulus, the Laurent series expansion is:

2.29. Theorem (Laurent series expansion of an analytic function). *Assume that* $0 \le a < b \le \infty$*. Let* f *be analytic in the annulus*

$$D = \{z \in \mathbf{C} : a < |z - z_0| < b\}.$$

Then f *can be represented by a Laurent series in the set* D*,*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \, , \quad z \in D.$$

The representation is unique: for all $n \in \mathbf{Z}$*,*

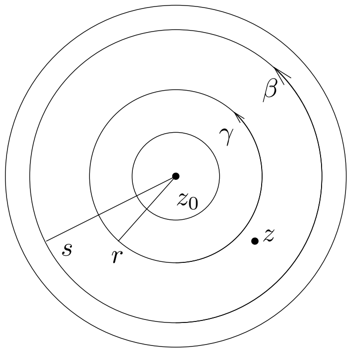
$$a_n = \frac{1}{2\pi i} \int\limits_{\{|z - z_0| = r\}} \frac{f(z)\, dz}{(z - z_0)^{n+1}} \, ,$$

provided that $a < r < b$ *and the circle is* $\{|z - z_0| = r\}$ *oriented counterclockwise.*

Proof: Let $r_0 \in (a, b)$. Let

$$a_n = \frac{1}{2\pi i} \int\limits_{\{|z - z_0| = r_0\}} \frac{f(z)\, dz}{(z - z_0)^{n+1}}.$$

Note that, by Cauchy's theorem, the choice of radius r_0 does not matter because the paths of two differently radiused contours are homologous in D.



Let $z \in D$. Choose r, s such that $a < r < |z - z_0| < s < b$. Let

$$\gamma(t) = z_0 + re^{it}, \quad 0 \le t \le 2\pi$$
$$\beta(t) = z_0 + se^{it}, \quad 0 \le t \le 2\pi.$$

Then the cycle $\sigma = (\beta, \overleftarrow{\gamma})$ is null-homologous in D and $n(\sigma, z) = 1$, so from Cauchy's integral formula 1.9 follows that

$$(*) \qquad f(z) = \frac{1}{2\pi i} \int_\sigma \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_\beta \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Now proceed as in the proof of the power series representation (see Theorem 2.21). If $\zeta \in |\beta|$, then

$$\left| \frac{z - z_0}{\zeta - z_0} \right| = \frac{|z - z_0|}{s} < 1,$$

thus

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_{n=0}^{\infty} \frac{f(\zeta)(z - z_0)^n}{(\zeta - z_0)^{n+1}}.$$

By the Weierstrass M-test, convergence (with respect to ζ) is locally uniform in $D \setminus \overline{B}(z_0, |z - z_0|) \supset |\beta|$, thus Lemma 2.12 implies that

$$(**) \qquad \frac{1}{2\pi i} \int_\beta \frac{f(\zeta)}{\zeta - z} d\zeta = \sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{2\pi i} \int_\beta \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right)}_{= a_n} (z - z_0)^n$$
$$= \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Similarly, $\left|\frac{\zeta-z_0}{z-z_0}\right| = \frac{r}{|z-z_0|} < 1$ for all $\zeta \in |\gamma|$, and

$$\frac{f(\zeta)}{\zeta - z} = -\frac{f(\zeta)}{z-z_0}\frac{1}{1-\frac{\zeta-z_0}{z-z_0}} = -\sum_{n=0}^{\infty}\frac{f(\zeta)(\zeta-z_0)^n}{(z-z_0)^{n+1}} = -\sum_{n=1}^{\infty}\frac{f(\zeta)(\zeta-z_0)^{n-1}}{(z-z_0)^n}.$$

By virtue of the Weierstrass M-test, convergence is locally uniform with respect to ζ in the set $D \cap B(z_0, |z-z_0|) \supset |\gamma|$, so

$$(***) \qquad \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta = -\sum_{n=1}^{\infty}\underbrace{\left(\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-z_0)^{-n+1}}d\zeta\right)}_{=a_{-n}}(z-z_0)^{-n}$$
$$= -\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}.$$

Now the double series $\sum\limits_{n=-\infty}^{\infty} a_n(z-z_0)^n$ converges in the ring D, and from equations $(*), (**)$, and $(***)$, it follows that

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n - \left(-\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}\right)$$
$$= \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

for all $z \in D$.

Uniqueness follows from Theorem 2.28 as in the case of power series (exercise).

□

Example. Let $f(z) = e^{1/z}$, then f is analytic in the set $\mathbb{C} \setminus \{0\}$. Since

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!} \qquad \text{for all } z \in \mathbb{C},$$

we obtain

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty}\frac{\left(\frac{1}{z}\right)^n}{n!} = \sum_{n=0}^{\infty}\frac{z^{-n}}{n!} = \sum_{n=-\infty}^{0}\frac{z^n}{(-n)!}.$$

This is the Laurent series of f, which is unique.

3. Singular points and residue theorem

3.1. Singular points

A function f has an *isolated singular point* (or simply: *singular point*) $z_0 \in \mathbf{C}$ if there exists such $r > 0$ that f is analytic in the punctured disk

$$B^* = B^*(z_0, r) = B(z_0, r) \setminus \{z_0\} = \{z : 0 < |z - z_0| < r\}.$$

We say that f is *analytic (in an open set) G except for isolated singular points*, if there is a set of isolated singular points $E \subset G$ such that the set E has no accumulation points in the set G and f is analytic in the set $G \setminus E$.

Let f have an isolated singular point z_0 and let $r > 0$ be chosen such that f is analytic in the punctured disk $B^* = B(z_0, r) \setminus \{z_0\}$. According to Theorem 2.29, the function f has a unique Laurent series, that is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

for all $z \in B^*$.

The singular points are classified as follows:

- z_0 is a *removable singularity* of the function f if $a_n = 0$ for all $n < 0$.
- z_0 is a *pole* of the function f if there exists $k < 0$ such that $a_k \neq 0$, but $a_k \neq 0$ for only finitely many $k < 0$.
- z_0 is an *essential singularity* of the function f if $a_k \neq 0$ for infinitely many $k < 0$.

Example. The origin (i.e., the point 0) is

a removable singularity of the function $\quad f(z) = \dfrac{\sin z}{z}$,

a pole of the function $\quad g(z) = \dfrac{1}{z}$, and

an essential singularity of the function $\quad h(z) = e^{\frac{1}{z}}$.

3.1. Remark (Important!). The point z_0 is a removable singularity of the function f if and only if

$$\tilde{f}(z) = \begin{cases} f(z)\,, & \text{when } z \neq z_0 \\ a_0\,, & \text{when } z = z_0\,, \end{cases}$$

is well-defined and analytic in a neighborhood of point z_0; here, the a_0 is the 0^{th} coefficient of the Laurent series of f.

3.2. Remark. Let z_0 be an isolated singular point of the function f. We say that

$$S(z) = \sum_{n=-\infty}^{-1} a_n (z - z_0)^n$$

is the *singular part* (or *principal part*) of the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Then S is analytic in the set $\mathbf{C} \setminus \{z_0\}$ (exercise).

Additionally, z_0 is a singularity of the function $f - S$. However, it is removable because

$$f(z) - S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

in some punctured neighborhood of the point z_0.

3.3. Remark. The coefficient a_{-1} appearing in the Laurent series of f plays a special role. Suppose z_0 is a singular point of the function f and let

$$a_{-1} =: \operatorname{Res}(f, z_0)$$

denote the *residue* of the function f at point z_0. If $r > 0$ is sufficiently small, then by Theorem 2.29, we have

$$\operatorname{Res}(f, z_0) = \frac{1}{2\pi i} \int_{\{|z - z_0| = r\}} f(z)\, dz\,,$$

where the contour is oriented counterclockwise.

3.2. Removable singularities

Next, we characterize removable singularities. The following result is quite useful in many situations.

3.4. Theorem (Riemann's removable singularity theorem)**.** *Let f be a function with an isolated singular point z_0. Then the following are equivalent:*

i) *z_0 is a removable singularity.*

ii) *There exists $r > 0$ such that f is bounded in the punctured disk*

$$B^* = \{z \in \mathbf{C} : 0 < |z - z_0| < r\}.$$

iii) *We have:*

$$\lim_{z \to z_0} (z - z_0) f(z) = 0\,.$$

PROOF: i) $\Rightarrow$ ii). Exercise.

ii) $\Rightarrow$ iii). Let $M, r > 0$ such that

$$|f(z)| \leq M\,, \qquad \text{when } z \in B^* = \{z \in \mathbf{C} : 0 < |z - z_0| < r\}\,.$$

Now

$$|z - z_0||f(z)| \leq M|z - z_0| \xrightarrow{z \to z_0} 0\,.$$

iii) $\Rightarrow$ i). Define

$$g(z) = \begin{cases} (z - z_0) f(z)\,, & \text{when } z \neq z_0 \\ 0\,, & \text{when } z = z_0. \end{cases}$$

Since g is continuous in the disk $B = B(z_0, r)$ for small $r > 0$ and analytic in the punctured disk $B^*(z_0, r)$, it follows from Theorem CA1.5.10 that g is analytic in the entire disk B. Thus, when $z \neq z_0$,

$$\begin{aligned} f(z) &= \frac{1}{z - z_0} g(z) = \frac{1}{z - z_0} \sum_{n=0}^{\infty} b_n (z - z_0)^n \\ &\overset{b_0 = g(z_0) = 0}{=} \frac{1}{z - z_0} \sum_{n=1}^{\infty} b_n (z - z_0)^n = \sum_{n=0}^{\infty} b_{n+1} (z - z_0)^n, \end{aligned}$$

which is a Laurent series with singular part equal to 0. Hence z_0 is removable. □

3.3. Poles

3.5. Definition. Let z_0 be a pole of the function f and

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n .$$

Then the number

$$k = -\min\{n < 0 : a_n \neq 0\}$$

is called the *order of the pole at* z_0.

3.6. Theorem. *Let $k \in \mathbf{N}$ and f be analytic in the punctured disk $B^* = B^*(z_0, r)$. Then z_0 is a pole of the function f of order k if and only if there exists an analytic function g in the disk $B(z_0, r)$ such that $g(z_0) \neq 0$ and*

$$f(z) = \frac{g(z)}{(z - z_0)^k} \qquad \textit{for all } z \in B^*.$$

PROOF: First, let's prove the necessity part. For all $z \in B^*$

$$f(z) = \sum_{n=-k}^{\infty} a_n (z - z_0)^n, \quad a_{-k} \neq 0.$$

Define

$$g(z) := \sum_{n=0}^{\infty} a_{n-k} (z - z_0)^n.$$

Then g is analytic in the disk $B(z_0, r)$, as the series converges in the disk $B(z_0, r)$. Moreover, $g(z_0) = a_{-k} \neq 0$. Now

$$\begin{aligned} f(z) &= \sum_{n=-k}^{\infty} a_n (z - z_0)^n = (z - z_0)^{-k} \sum_{n=0}^{\infty} a_{n-k} (z - z_0)^n \\ &= (z - z_0)^{-k} g(z). \end{aligned}$$

Next, let's prove the sufficiency part. When

$$g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n , \quad b_0 \neq 0 ,$$

and

$$f(z) = (z - z_0)^{-k} g(z)\,,$$

then

$$f(z) = (z - z_0)^{-k} \sum_{n=0}^{\infty} b_n (z - z_0)^n = \sum_{n=-k}^{\infty} b_{n+k} (z - z_0)^n,$$

which is the Laurent series of the function f with $-k$th coefficient being $b_0 \neq 0$. □

3.7. Remark (Important!). Let z_0 be a pole of the function f of order k and g be analytic in a neighborhood of point z_0 such that for all $z \in B(z_0, r) \setminus \{z_0\}$

$$f(z) = (z - z_0)^{-k} g(z).$$

Then

$$\begin{aligned} f(z) &= (z - z_0)^{-k} \sum_{n=0}^{\infty} b_n (z - z_0)^n \\ &= \sum_{n=-k}^{\infty} b_{n+k} (z - z_0)^n\,, \end{aligned}$$

which is the Laurent series of the function f. Here, the coefficient of the term $(z - z_0)^{-1}$ is

$$b_{-1+k} = b_{k-1} = \frac{g^{(k-1)}(z_0)}{(k-1)!}$$

that is

$$\operatorname{Res}(f, z_0) = \frac{g^{(k-1)}(z_0)}{(k-1)!}.$$

Since f is not defined at point z_0, it's best to write this in the form: If z_0 is a pole of order k of the function f, then

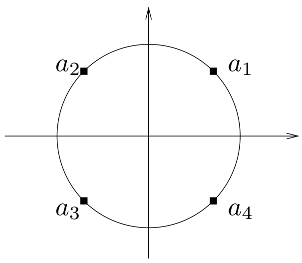
$$\operatorname{Res}(f, z_0) = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}} \big((z - z_0)^k f(z) \big). \tag{3.1}$$

In particular, if z_0 is a first-order pole, we have

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Example. Let
$$f(z) = \frac{1}{e^z - 1}$$
Then
$$f(z) = \frac{1}{z + \frac{z^2}{2!} + \dots} = \frac{1}{z} \frac{1}{1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots} = \frac{1}{z} g(z),$$
where g is analytic and $g(0) \neq 0$. Thus the function f has a first-order pole at point 0 and
$$\operatorname{Res}(f, 0) = \lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{z}{e^z - 1} = \frac{1}{\frac{d}{dz}(e^z)|_{z=0}} = 1.$$

3.8. Example. Let
$$f(z) = \frac{z^2}{1 + z^4},$$
then the function f has poles at
$$a_1 = e^{i\frac{\pi}{4}}, \quad a_2 = e^{i\frac{3\pi}{4}}, \quad a_3 = e^{i\frac{5\pi}{4}} \quad \text{and} \quad a_4 = e^{i\frac{7\pi}{4}}.$$



Clearly, the poles a_n are simple (exercise). Thus
$$\begin{aligned} \operatorname{Res}(f, a_1) &= \lim_{z \to a_1} (z - a_1) f(z) = \lim_{z \to a_1} \frac{z^2}{(z - a_2)(z - a_3)(z - a_4)} \\ &= \frac{a_1^2}{(a_1 - a_2)(a_1 - a_3)(a_1 - a_4)} = \sqrt{2} \frac{2i}{2i \cdot 2(1+i) \cdot 2i} \\ &= \frac{\sqrt{2}}{4(i+1)} = \frac{1-i}{4\sqrt{2}} = \frac{1}{4} e^{-\frac{i\pi}{4}}. \end{aligned}$$
Similarly
$$\operatorname{Res}(f, a_2) = \frac{-1-i}{4\sqrt{2}} = \frac{1}{4} e^{-\frac{3i\pi}{4}}.$$
The residues at the poles a_3 and a_4 can be determined similarly.

3.9. Theorem. *Let z_0 be an isolated singularity of the function f. Then*

i) point z_0 is a pole of the function f if and only if $\lim_{z\to z_0} |f(z)| = \infty$.

ii) point z_0 is a pole of order k if and only if k is the positive integer for which

$$\lim_{z\to z_0} |z - z_0|^k |f(z)| \in]0, \infty[.$$

PROOF: Let z_0 be a pole of the function f of order k. Then there exists an analytic function g in a neighborhood of point z_0 such that $g(z_0) \neq 0$ and

$$f(z) = \frac{g(z)}{(z - z_0)^k}\,.$$

Thus

$$\lim_{z\to z_0} |f(z)| = \lim_{z\to z_0} \left| \frac{g(z)}{(z - z_0)^k} \right| = \infty,$$

because $g(z_0) \neq 0$. Also,

$$|z - z_0|^l |f(z)| = |z - z_0|^{l-k} |g(z)| \to \begin{cases} \infty\,, & \text{if } l < k \\ |g(z_0)| \in (0, \infty)\,, & \text{if } l = k \\ 0\,, & \text{if } l > k. \end{cases}$$

This proves the "only if" parts of both claims.

Conversely, suppose

$$\lim_{z\to z_0} |f(z)| = \infty \quad \text{and} \quad h = \frac{1}{f}\,.$$

Now h is bounded ($|h| \leq 1$) in a neighborhood of point z_0, so z_0 is a removable singularity for the function h and

$$h(z_0) = \lim_{z\to z_0} h(z) = 0\,.$$

Since $h \not\equiv 0$, then if

$$\sum_{n=0}^{\infty} b_n (z - z_0)^n$$

is the power series expansion of the function h, then $b_0 = 0$, but

$$k := \min\{n \in \mathbf{N} : b_n \neq 0\} \in \{1, 2, 3, \dots\}.$$

Thus the function

$$g(z) = \frac{1}{\sum_{n=0}^{\infty} b_{n+k}(z-z_0)^n}$$

is analytic in a neighborhood of point z_0 and

$$g(z_0) = \frac{1}{b_k} \neq 0\,.$$

Thus

$$f(z) = \frac{1}{h(z)} = \frac{1}{\sum_{n=k}^{\infty} b_n(z-z_0)^n} = \frac{1}{(z-z_0)^k \sum_{n=0}^{\infty} b_{n+k}(z-z_0)^n} = \frac{g(z)}{(z-z_0)^k}.$$

According to Theorem 3.6, z_0 is a pole of the function f with order k, which proves the "if" part of the first claim.

Finally, if it is known that

$$\lim_{z\to z_0} |z-z_0|^k |f(z)| = a \in (0,\infty)$$

for some $k \in \mathbf{N}$, then

$$|f(z)| \to \infty\ ,\ \text{when } z \to z_0,$$

and the above proof shows that z_0 is a pole of order k (note that $1/|b_k| = a$). □

3.4. Essential singularities

The behavior of an analytic function around an essential singularity is wild. One has **Picard's great theorem**: *If z_0 is an essential singularity of an analytic function f, then $\mathbf{C} \setminus f(B^*(z_0,r))$ has at most one point, for any $r > 0$.*

However, we will not prove this deep theorem, but instead a weaker (and easier) result.

3.10. Theorem (Casorati-Weierstrass)**.** *If z_0 is an isolated essential singularity of an analytic function f in the punctured disk $B^*(z_0,r)$, then the image set $f(B^*(z_0,r))$ is dense in $\mathbf{C}$, i.e.,*

$$\overline{f(B^*(z_0,r))} = \mathbf{C}\,.$$

PROOF: Suppose the image set is not dense, then in its complement there is a disk

$$B(w,\varepsilon) \cap f(B^*(z_0,r)) = \emptyset\,.$$

Then the function

$$g(z) = \frac{1}{f(z) - w}$$

would be analytic in the punctured disk $B^*(z_0,r)$. Moreover, g is bounded, because

$$\begin{aligned} |g(z)| &= \frac{1}{|f(z) - w|} \\ &\leq \frac{1}{\varepsilon}\,, \end{aligned}$$

so by Riemann's removable singularity theorem, Theorem 3.4, z_0 is a removable singularity for g. Thus, since z_0 has a limit for g, z_0 is either a removable singularity or a pole of $1/g$. Hence, z_0 is a singularity for the function

$$f(z) = w + \frac{1}{g(z)}$$

which is either removable or a pole, which contradicts the assumption that z_0 was an essential singularity of f. □

An entire function f is either a polynomial or it has an essential singularity at infinity:

3.11. Theorem. *Let $f : \mathbf{C} \to \mathbf{C}$ be an entire function. Then the function*

$$g(z) = f(\frac{1}{z})$$

has an essential singularity at the origin or f is a polynomial.

PROOF: If f is not a polynomial, then f has a power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n\,, \quad z \in C,$$

where $a_n \neq 0$ for infinitely many n. Thus, the Laurent series of the function g is

$$g(z) = \sum_{n=-\infty}^{0} b_n z^n\,, \quad b_n = a_{-n}\,,\ z \in \mathbf{C} \setminus \{0\}\,,$$

so since $b_n = a_{-n} \neq 0$ for infinitely many $n < 0$, 0 is its essential singularity. □

Example. Let f be analytic in the set $B^* = B^*(z_0, r)$. Suppose that f is not identically zero in B^* and that there exist points $z_n \subset B^*$ such that $\lim_{n\to\infty} z_n = z_0$ and $f(z_n) = 0$ for all $n \in \mathbf{N}$. (As a concrete example, consider the function

$$f(z) = \cos\left(\frac{1}{z-1}\right),$$

which has an isolated singularity at $z_0 = 1$.)

Then:

- z_0 cannot be a removable singularity of the function f, since otherwise f would be identically zero in the set B^*, see Theorem 2.22.
- Since $\lim_{n\to\infty}|f(z_n)| = 0 < \infty$ and $\lim_{n\to\infty} z_n = z_0$, then z_0 cannot be a pole of the function f, see Theorem 3.9.

Therefore, the point z_0 must be an essential singularity (no other options are possible, QED).

3.5. Residue theorem

The residue theorem provides an efficient tool for computing complex integrals.

3.12. Theorem (Residue theorem)**.** *Let σ be a null-homologous cycle in an open set G, and let f be analytic in G except for an isolated set of singularities E, where $E \cap |\sigma| = \emptyset$. Then*

$$\int_\sigma f(z)\,dz = 2\pi i \sum_{a\in E} n(\sigma, a)\,\mathrm{Res}(f, a).$$

3.13. Remark. In the proof of Cauchy's theorem, it was shown that if σ is a null-homologous cycle in G, then

$$K_\sigma = |\sigma| \cup \{z \in \mathbf{C} \setminus |\sigma| : n(\sigma, z) \neq 0\}$$

is a compact subset of G. Let $D \subset\subset G$ be an open set such that $K_\sigma \subset D$. Then σ is null-homologous in D. Since the set E has no accumulation points in G, the set $E \cap D = \{z_1, \ldots, z_p\}$ is finite. Thus, the sum in the statement is well-defined:

$$\sum_{a\in E} n(\sigma, a)\,\mathrm{Res}(f, a) = \sum_{j=1}^{p} n(\sigma, z_j)\,\mathrm{Res}(f, z_j).$$

PROOF: Let $z_1, \dots, z_p$ be as in the remark, and let S_k be the singular part of the function f at the point z_k, $k = 1, \dots, p$. Then S_k is analytic in $\mathbf{C} \setminus \{z_k\}$ by Theorem 2.28, and thus the function $f - S_k$ has a removable singularity at z_k. Therefore,

$$g = f - S_1 - S_2 - \cdots - S_p$$

is analytic in D except for removable singularities at $z_1, \dots, z_p$. Since these are removable, we can assume that g is analytic in the whole of D. Now, from Cauchy's theorem it follows that

$$0 = \int_\sigma g(z)dz = \int_\sigma f(z)\,dz - \sum_{k=1}^{p} \int_\sigma S_k(z)dz$$

or

$$(*) \qquad \int_\sigma f(z)\,dz = \sum_{k=1}^{p} \int_\sigma S_k(z)dz.$$

Now, if

$$S(z) = \sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$$

is the singular part of the function f at an arbitrary point $z_0 \in E$, then the series S converges locally uniformly in $\mathbf{C} \setminus \{z_0\}$. In particular, it converges uniformly along the cycle σ. Thus,

$$\int_\sigma S(z)dz = \int_\sigma \left(\sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n} \right) dz = \sum_{n=1}^{\infty} a_{-n} \int_\sigma \frac{dz}{(z - z_0)^n},$$

where we used Lemma 2.12. Since the function $(z - z_0)^{-n}$ has a primitive when $n > 1$ (or by the Cauchy integral formula for derivatives),

$$\int_\sigma (z - z_0)^{-n} dz = 0\,, \qquad \text{when } n > 1\,,$$

so we obtain

$$\int_\sigma S(z)dz = a_{-1} \int_\sigma \frac{dz}{z - z_0} = 2\pi i \operatorname{Res}(f, z_0) n(\sigma, z_0).$$

Now, from equation $(*)$ it follows that

$$\int_\sigma f(z)\,dz = \sum_{k=1}^{p} \int_\sigma S_k(z)dz = 2\pi i \sum_{k=1}^{p} n(\sigma, z_k) \operatorname{Res}(f, z_k).$$

□

Example. Let us show that

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}dx = \frac{\pi}{\sqrt{2}}.$$

First, we note that the integral converges, since

$$\frac{x^2}{1+x^4} \leq \frac{1}{x^2} .$$

Let

$$f(z) = \frac{z^2}{1+z^4} ,$$

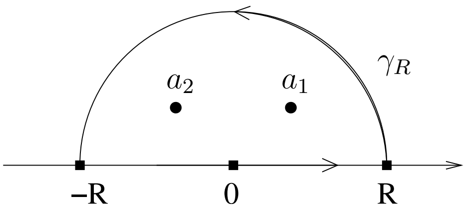
so that the function f has poles at

$$a_1 = e^{i\frac{\pi}{4}} , \quad a_2 = e^{i\frac{3\pi}{4}} , \quad a_3 = e^{i\frac{5\pi}{4}} \quad \text{and} \quad a_4 = e^{i\frac{7\pi}{4}} .$$

Following the calculation in Example 3.8, we have

$$\text{Res}(f, a_1) = \frac{1}{4}e^{-\frac{i\pi}{4}} , \qquad \text{Res}(f, a_2) = \frac{1}{4}e^{-\frac{3i\pi}{4}} .$$

Let $R > 1$ and γ_R be the semicircle $B(0, R) \cap \{z : Im(z) > 0\}$ traversed counterclockwise.



The residue theorem implies that

$$(*) \qquad \frac{1}{2\pi i} \int_{\gamma_R} f(z)\, dz = \text{Res}(f, a_1) + \text{Res}(f, a_2) = -\frac{i}{2\sqrt{2}}.$$

Since

$$\frac{1}{2\pi i} \int_{\gamma_R} f(z)\, dz = \frac{1}{2\pi i} \int_{-R}^{R} \frac{x^2}{1+x^4}dx + \frac{1}{2\pi} \int_{0}^{\pi} \frac{R^3e^{3it}}{1+R^4e^{4it}}dt,$$

we have

$$\int_{-R}^{R} \frac{x^2}{1+x^4} dx \overset{(*)}{=} \frac{\pi}{\sqrt{2}} - iR^3 \int_0^{\pi} \frac{e^{3it}}{1+R^4 e^{4it}} dt.$$

Since $t \in [0, \pi]$, we have $|1 - 1 + R^4 e^{4it}| = R^4$, so by the triangle inequality,

$$\left|1 + R^4 e^{4it}\right| \geq R^4 - 1,$$

when $R > 1$. Thus,

$$\left| R^3 \int_0^{\pi} \frac{e^{3it}}{1+R^4 e^{4it}} dt \right| \leq \frac{R^3}{R^4 - 1} \int_0^{\pi} \underbrace{|e^{3it}|}_{=1} dt = \frac{\pi R^3}{R^4 - 1} \to 0,$$

as $R \to \infty$. Therefore,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx &= \lim_{R \to \infty} \int_{-R}^{R} \frac{x^2}{x^4+1} dx \\ &= \frac{\pi}{\sqrt{2}} - \underbrace{\lim_{R \to \infty} iR^3 \int_0^{\pi} \frac{e^{3it}}{1+R^4 e^{4it}} dt}_{=0} = \frac{\pi}{\sqrt{2}}. \end{aligned}$$

3.14. Remark. When evaluating integrals as above by taking the limit $R \to \infty$, it must be ensured that the integrands converge absolutely! Compare this remark to the one regarding double series.

4. Mapping properties of analytic functions

4.1. Analytic functions are angle-preserving

Let $U \subset \mathbf{R}^2$ be open and let $f : U \to \mathbf{R}^2$ be a C^1 map (not necessarily analytic). In this section we will prove a geometric characterization of analytic functions: f is analytic if and only if the map f preserves (infinitesimal) angles between curves.

We will identify vectors in $\mathbf{R}^2$ with points in $\mathbf{C}$ and write

$$z = (x, y) = x + iy.$$

Let $z_0 \in U$ and let $\gamma : (-\varepsilon, \varepsilon) \to U$ be a C^1 curve with $\gamma(0) = z_0$. Writing $\gamma(t) = (x(t), y(t))$, the tangent vector of γ is given by

$$\dot\gamma(t) = (\dot x(t), \dot y(t)) = \dot x(t) + i\dot y(t).$$

If $f = u + iv$ is a C^1 map $U \to \mathbf{C}$ (not necessarily analytic), then $(f \circ \gamma)(t)$ is a C^1 curve through $f(z_0)$. Its tangent vector is given by the following formula.

4.1. Lemma. *One has*

$$(f \circ \gamma)^{\cdot}(t) = \partial f(\gamma(t))\dot\gamma(t) + \overline{\partial} f(\gamma(t))\overline{\dot\gamma(t)}$$

where the right hand side involves multiplication of complex numbers, and ∂f and $\overline{\partial} f$ are the Wirtinger derivatives

$$\partial f(z) = \frac{1}{2}(\partial_x f(z) - i\partial_y f(z)), \qquad \overline{\partial} f(z) = \frac{1}{2}(\partial_x f(z) + i\partial_y f(z)).$$

Proof: Write $f = u + iv$ and $\gamma(t) = (x(t), y(t))$. Then the chain rule gives

$$\begin{aligned}(f \circ \gamma)^{\cdot}(t) &= \partial_t\left[u(x(t), y(t))) + i\partial_t(v(x(t), y(t))\right] \\ &= u_x(\gamma(t))\dot x(t) + u_y(\gamma(t))\dot y(t) + i\left[v_x(\gamma(t))\dot x(t) + v_y(\gamma(t))\dot y(t)\right] \\ &= (u_x + iv_x)\dot x(t) + (u_y + iv_y)\dot y(t) \\ &= f_x \cdot x(t) + f_y \cdot y(t) \\ &= \frac{1}{2}(f_x - if_y)(\dot x(t) + i\dot y(t)) + \frac{1}{2}(f_x + if_y)(\dot x(t) - i\dot y(t)).\end{aligned}$$

□

4.2. Definition. Let $f : U \to \mathbf{C}$ be a C^1 map. We say that f is *angle-preserving*, or *conformal*, at $z_0 \in U$ if there are $r(z_0) > 0$ and $\theta(z_0) \in (-\pi, \pi]$ such that for any C^1 curve $\gamma(t)$ in U with $\gamma(0) = z_0$, one has

$$(f \circ \gamma)^{\cdot}(0) = r(z_0)e^{i\theta(z_0)}\dot{\gamma}(0).$$

Thus f is angle-preserving at z_0 if the tangent vector of the curve $(f \circ \gamma)(t)$ at $t = 0$ is obtained by scaling and rotating the tangent vector $\dot{\gamma}(0)$ by a factor $r(z_0)$ and angle $\theta(z_0)$ that are independent of the curve γ. We require $r(z_0) > 0$ to exclude the degenerate case where the tangent vectors $(f \circ \gamma)^{\cdot}(0)$ would always be zero.

4.3. Remark. With an appropriate definition of angle, one can prove (exercise) that f is angle-preserving at z_0 if and only if the angle between the tangent vectors of $(f \circ \gamma_1)(t)$ and $(f \circ \gamma_2)(t)$ at $t = 0$ is equal to the angle between $\dot{\gamma}_1(0)$ and $\dot{\gamma}_2(0)$ whenever $\gamma_1(t)$ and $\gamma_2(t)$ are C^1 curves through z_0. Thus an angle-preserving map preserves the angles between tangent vectors. However, it may change the lengths of tangent vectors (this happens when $r(z_0) \neq 1$).

4.4. Theorem (Analytic = conformal). *Let $f : U \to \mathbf{C}$ be a C^1 map and $z_0 \in U$. Then f is angle-preserving at z_0 if and only f has a complex derivative at z_0 and $f'(z_0) \neq 0$. If γ is a C^1 curve with $\gamma(0) = z_0$, the tangent vector of $(f \circ \gamma)(t)$ satisfies*

$$(f \circ \gamma)^{\cdot}(0) = f'(z_0)\dot{\gamma}(0).$$

Proof: Writing $f = u + iv$, one has

$$\overline{\partial} f = \frac{1}{2}(\partial_x f + i\partial_y f) = \frac{1}{2}(u_x + iv_x + i(u_y + iv_y)) = \frac{1}{2}(u_x - v_y + i(u_y + v_x)).$$

Thus the Cauchy-Riemann equations $u_x = v_y, u_y = -v_x$ are valid at z_0 if and only if

$$\overline{\partial} f(z_0) = 0.$$

Since the complex derivative is given by $f'(z_0) = u_x(z_0) + iv_x(z_0)$, we also have $\partial f(z_0) = f'(z_0)$ whenever the complex derivative exists.

"$\Longleftarrow$" Suppose that f has a complex derivative at z_0 and $f'(z_0) \neq 0$. Then the Cauchy-Riemann equations are valid at z_0, so $\overline{\partial} f(z_0) = 0$. From Lemma 4.1 we obtain that

$$(f \circ \gamma)^{\cdot}(0) = \partial f(z_0)\dot{\gamma}(0)$$

whenever γ is a C^1 curve with $\gamma(0) = z_0$. Writing $f'(z_0) = \partial f(z_0) = r(z_0)e^{i\theta(z_0)}$, we see that f is angle-preserving at z_0. One has $r(z_0) > 0$ since $f'(z_0) \neq 0$.

"$\Longrightarrow$" Suppose that f is angle-preserving at z_0. Then by definition there is $w_0 \in \mathbf{C}$, $w_0 \neq 0$ with

$$(f \circ \gamma)^{\cdot}(0) = w_0 \dot{\gamma}(0)$$

for any C^1 curve $\gamma(t)$ through z_0. On the other hand, Lemma 4.1 gives that

$$(f \circ \gamma)^{\cdot}(0) = \partial f(z_0)\dot{\gamma}(0) + \overline{\partial} f(z_0)\overline{\dot{\gamma}(0)}.$$

Fix any $w \in \mathbf{C}$ and consider the curve $\gamma(t) = z_0 + tw$. Then $\dot{\gamma}(0) = w$, and the previous two formulas imply

$$w_0 w = \partial f(z_0) w + \overline{\partial} f(z_0)\bar{w}.$$

This can be rewritten as

$$(\partial f(z_0) - w_0)w + \overline{\partial} f(z_0)\bar{w} = 0.$$

This is true for any $w \in \mathbf{C}$. Choosing $w = 1$ and $w = i$ yields

$$\partial f(z_0) = w_0, \qquad \overline{\partial} f(z_0) = 0.$$

Since $\overline{\partial} f(z_0) = 0$, the Cauchy-Riemann equations are valid at z_0 and therefore f has a complex derivative at z_0. One also has $f'(z_0) = \partial f(z_0) = w_0 \neq 0$. □

Theorem 4.4 shows that indeed analytic functions can be characterized by the property that they preserve (infinitesimal) angles. We will next begin to use the facts proved in the previous chapters in order to study conformal mappings.

4.2. Argument principle and Hurwitz's theorem

4.5. Theorem. *(Argument principle). Let f be an analytic function in the domain D, whose zeros are $a_1, a_2, \ldots, a_n \in D$. Let $k_j \in \mathbf{N}$ be the multiplicity of the zero a_j and σ be a null-homologous cycle in D. If $a_j \notin |\sigma|$ for all j, then*

$$\frac{1}{2\pi i} \int_{\sigma} \frac{f'(z)}{f(z)}\, dz = \sum_{j=1}^{n} k_j\, n(\sigma, a_j)\,.$$

Proof: Let $j \in \{1, 2, \ldots, n\}$. By Theorem 2.24, there exists an analytic function g in the domain D and $r > 0$ such that $g(a_j) \neq 0$ and $f(z) = (z - a_j)^{k_j} g(z)$ for all $z \in B(a_j, r)$. Moreover, we can assume that $g(z) \neq 0$ for all $z \in B(a_j, r)$, so

$$\frac{f'(z)}{f(z)} = \frac{k_j (z - a_j)^{k_j - 1} g(z)}{(z - z_0)^{k_j} g(z)} + \frac{(z - a_j)^{k_j} g'(z)}{(z - a_j)^{k_j} g(z)} = \frac{k_j}{z - a_j} + \frac{g'(z)}{g(z)}$$

when $z \in B^*(a_j, r)$. Since the function g'/g is analytic in a neighborhood of point a_j, then $\operatorname{Res}(f'/f, a_j) = k_j$.

By the residue theorem,

$$\frac{1}{2\pi i} \int\limits_\sigma \frac{f'(z)}{f(z)}\, dz = \sum_{j=1}^{n} n(\sigma, a_j) \operatorname{Res}(f'/f, a_j) = \sum_{j=1}^{n} k_j\, n(\sigma, a_j)$$

as desired. □

By applying Theorem 4.5 to the function $f(z) - w_0$, we obtain:

4.6 . Theorem. *Let f be an analytic function in the domain D, and let $a_1, a_2, \ldots, a_n \in D$ be the points where $f(z) = w_0$. Let $k_j \in \mathbf{N}$ be the multiplicity of the zero a_j of w_0 (i.e., the multiplicity of the zero of the function $f - w_0$), and let σ be a nullhomologous cycle in D. If $a_j \notin |\sigma|$ for all $j = 1, 2, \ldots, n$, then*

$$\frac{1}{2\pi i} \int\limits_\sigma \frac{f'(z)}{f(z) - w_0}\, dz = \sum_{j=1}^{n} k_j\, n(\sigma, a_j)\,.$$

4.7. Example. If $\gamma(t) = 2e^{i2\pi t}$, $t \in [0, 1]$, then

$$\int\limits_\gamma \frac{2z + 1}{z^2 + z + 1}\, dz = 4\pi i\,,$$

since both zeros of the denominator $\frac{1}{2}(-1 \pm i\sqrt{3})$ are on the circumference of the unit circle.

4.8. Example. Let γ be a null-homologous closed path in the domain D, and $f : D \to \mathbf{C}$ be analytic. Let $w_0 \in \mathbf{C} \setminus f(|\gamma|)$ and $a_1, a_2, \dots, a_n$ be the points $z \in D$ such that $f(z) = w_0$. If $k_j \in \mathbf{N}$ is the multiplicity of the zero w_0 at a_j and $\sigma = f \circ \gamma$, then σ is a closed path and

$$n(\sigma, w_0) = \frac{1}{2\pi i} \int_\sigma \frac{1}{w - w_0}\, dw = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - w_0}\, dz = \sum_{j=1}^{n} k_j\, n(\gamma, a_j)\,,$$

where the middle equality can be easily seen with a change of variables (exercise).

4.9. Theorem. *(Hurwitz's theorem). If the analytic functions f_j, $j \in \mathbf{N}$, converge locally uniformly to a function $f : D \to \mathbf{C}$ in the domain D, and $f_j(z) \neq 0$ for all $j \in \mathbf{N}$ and $z \in D$, then either $f \equiv 0$ or $f(z) \neq 0$ for all $z \in D$.*

Proof: Assume that f is not identically zero in the domain D, but there exists a zero of f of multiplicity k_0 at $z_0 \in D$. By Theorem 2.22, there exists $r > 0$ such that $B(z_0, 2r) \subset D$ and f does not attain the value zero in the set $B(z_0, 2r) \setminus \{z_0\}$. By the argument principle (applied in the disk $B(z_0, 2r)$), we have

$$\int_{\{|z-z_0|=r\}} \frac{f'(z)}{f(z)}\, dz = 2\pi i k_0 \neq 0\,.$$

On the other hand, since f_j does not vanish in the domain D, then

$$\int_{\{|z-z_0|=r\}} \frac{f_j'(z)}{f_j(z)}\, dz = 0$$

for all j. Since the sequences f_j and f_j' converge locally uniformly in D towards the respective limit functions f and f', and since the function f does not vanish in the compact set $\{|z - z_0| = r\}$, then f_j'/f_j converges uniformly to the limit function f'/f in the set $\{|z - z_0| = r\}$ (exercise). Thus,

$$0 = \lim_{j\to\infty} \int_{\{|z-z_0|=r\}} \frac{f_j'(z)}{f_j(z)}\, dz = \int_{\{|z-z_0|=r\}} \frac{f'(z)}{f(z)}\, dz = 2\pi i k_0 \neq 0\,,$$

which is a contradiction. □

From Hurwitz's theorem, we can derive the following result, which describes the uniform limit of analytic injective functions.

4.10. Theorem. *Suppose f_j, $j \in \mathbf{N}$ are injective analytic functions in a domain D. If the functions f_j converge to the function f locally uniformly in the set D, then either f is constant or it is injective in D.*

Proof left as an exercise.

4.3. Open mapping theorem for analytic functions

From Theorem 2.22, it follows that a non-constant analytic function in the domain D is discrete, i.e., the preimage of a point in D is always a discrete set, i.e., a set without accumulation points. In this section, we analyze further the local behavior of analytic functions and prove, among other things, that a non-constant analytic function is an open mapping, i.e., it maps open sets to open sets.

According to the following theorem, an analytic function behaves around its zero similarly as z^n around the origin, where n is the multiplicity of the zero.

4.11. Theorem. *Let $f : B(z_0, R) \to \mathbf{C}$ be analytic and $w_0 = f(z_0)$. If z_0 is an n-fold zero of the function $f(z) - w_0$, then there exist $\varepsilon > 0$ and $0 < \delta < R$ such that when $0 < |w - w_0| < \varepsilon$, the equation*

$$f(z) = w$$

has exactly n simple solutions for z in the disk $B(z_0, \delta)$.

Proof: Since the multiplicity of z_0 is finite, f is not constant. Since the zeros of an analytic function are isolated (Theorem 2.24), there exists $0 < \delta < \frac{1}{2}R$ such that the equation

$$f(z) = w_0$$

has no solutions in the punctured disk $B^*(z_0, 2\delta)$ and furthermore

$$f'(z) \neq 0 \quad \text{for all } z \in B^*(z_0, 2\delta)\,.$$

Let

$$\gamma(t) = z_0 + \delta e^{2\pi i t}\,, \quad t \in [0, 1]$$

and $\sigma = f \circ \gamma$. Since $w_0 \notin |\sigma|$, there exists $\varepsilon > 0$ such that

$$B(w_0, \varepsilon) \cap |\sigma| = \emptyset.$$

Thus, $B(w_0, \varepsilon)$ belongs to some component of $\mathbf{C} \setminus |\sigma|$, so by Lemma CA1.5.4, for all $w \in B(w_0, \varepsilon)$,

$$n(\sigma, w_0) = n(\sigma, w)\,.$$

Let $w \in B^*(w_0, \varepsilon)$ and let a_j be the k_j-fold zeros of the function $f(z) - w$. According to Example 4.8,

$$n = n\,n(\gamma, z_0) = n(\sigma, w_0) = n(\sigma, w) = \sum_{j=1}^{p} k_j n(\gamma, a_j)\,.$$

Since $n(\gamma, a_j)$ is either $= 0$ or $= 1$ and since $f'(z) \neq 0$ for all $z \in B^*(z_0, \delta)$, there are exactly n simple solutions of the equation $f(z) = w$ in $a_j \in B^*(z_0, \delta)$. □

4.12. Remark. Theorem 4.11 is known as the "branched covering principle", which refers to the fact that an analytic function covers the image locally n times.

By analyzing the proof, we see that any $\delta > 0$ such that

$$f(z) \neq w_0 \quad \text{and} \quad f'(z) \neq 0 \quad \text{when } 0 < |z - z_0| \leq \delta\,.$$

is sufficient. Then, we only need to require $\varepsilon > 0$ such that

$$\varepsilon \leq |w_0 - f(z)| \quad \text{whenever } |z - z_0| = \delta\,.$$

A non-constant analytic function f is an *open mapping*, i.e., the image of an open set G under f is always open:

4.13. Theorem (Open mapping theorem for analytic functions)**.** *Let f be a non-constant analytic function in the domain D. Then f is an open mapping.*

Proof: Let $G \subset D$ be open and $w_0 \in f(G)$. Choose $z_0 \in G$ such that $f(z_0) = w_0$. Since f is non-constant, Theorem 4.11 can be applied and we find $\delta > 0$ and $\varepsilon > 0$ such that $B(z_0, \delta) \subset G$ and for every $w \in B(w_0, \varepsilon)$ there is (at least one) pre-image $z \in B(z_0, \delta)$. Thus,

$$B(w_0, \varepsilon) \subset f\big(B(z_0, \delta)\big) \subset f(G)\,,$$

so $f(G)$ is open. □

4.14. Corollary. *Let $f : G \to \mathbf{C}$ be analytic and $z_0 \in G$. If $f'(z_0) \neq 0$, then there exists $r > 0$ such that*

$$f|_{B(z_0,r)} : B(z_0, r) \to f\big(B(z_0, r)\big)$$

is a homeomorphism.

Proof: Let $w_0 = f(z_0)$. Since $f'(z_0) \neq 0$, Theorem 4.11 can be applied at the point z_0 in some neighborhood $B(z_0, R) \subset G$ with $n = 1$. Thus, we find $\varepsilon > 0$ and $\delta > 0$ such that $B = B(z_0, \delta) \subset G$ and for every $w \in B(w_0, \varepsilon)$ there is exactly one pre-image $z \in B$ (we can assume that there is exactly one pre-image for w_0 as well in the disk B; additional proof left as an exercise). By the continuity of the function f, there exists $0 < r < \delta$ such that $f(B(z_0, r)) \subset B(w_0, \varepsilon)$. We show that $f|_{B(z_0,r)} : B(z_0, r) \to f\big(B(z_0, r)\big)$ is a bijection; note that this restriction is surjective by definition.

For injectivity, let $z \in B(z_0, r)$. Assume that $\zeta \in B(z_0, r)$ such that $f(\zeta) = f(z) =: w \in B(w_0, \varepsilon)$. By the previous reasoning, $z = \zeta$, otherwise point w would have (at least) two pre-images in the set $B(z_0, r) \subset B(z_0, \delta)$. Thus, $f|_{B(z_0,r)} : B(z_0, r) \to f\big(B(z_0, r)\big)$ is a bijection; the continuity of the inverse map follows from the openness of the mapping (Theorem 4.13). □

4.15. Corollary. *Let $f : G \to \mathbf{C}$ be an injective analytic function. Then*

$$f'(z) \neq 0 \qquad \text{for all } z \in G\,.$$

Proof: Let $z_0 \in G$ and $f(z_0) = w_0$. The point z_0 is a simple root of the equation $f(z) = f(z_0)$, otherwise we would find a point $w \in f(G)$ (close to w_0), which would have multiple pre-image points in G (by Theorem 4.11), contradicting the fact that f is an injection. Hence by the definition of multiplicity, $f'(z_0) \neq 0$. □

4.4. Introduction to conformal mappings

Many physical phenomena, such as fluid flow, heat conduction, etc., are mathematically modeled using Laplace's equation

$$u_{xx} + u_{yy} = 0\,, \quad z = (x, y) \in \mathbf{C}\,.$$

It is much easier to handle this equation if the domain of interest is either the upper half-plane H or the unit disk B rather than a general domain G. Using the Cauchy-Riemann equations, it can be seen that a "conformal change of variables" preserves the Laplacian, i.e., if $f : D \to G$ is a conformal map, cf. Definition 4.16, and u satisfies the Laplace equation in G, then $u \circ f$ satisfies the Laplace equation in D. For this reason, it is important to find conformal mappings between different domains.

4.16. Definition. A function $f : G \to \mathbf{C}$ is a *conformal map (univalent, schlicht)* in the domain G, if it is an analytic injection in the domain G.

4.17. Remark. Often, the phrase "f is conformal (at point z_0)" is used, meaning only that f is locally injective (i.e., its derivative does not vanish). In our language, a conformal map is a global injection. By Lemma 4.14, if $f'(z_0) \neq 0$, then there exists a neighborhood U of z_0 where $f|_U : U \to f(U)$ is a conformal map.

In the classical sense, conformality means that angles are preserved at an infinitesimal scale: If γ and σ are regular paths passing through the point z_0 and if $f'(z_0) \neq 0$, then the angle between the curves γ and σ (tangent lines) = the angle between the image curves $f \circ \gamma$ and $f \circ \sigma$ (tangent lines). This is easy to believe in light of Remark CA1.2.3: both curves γ and σ rotate by the angle $\arg(f'(z_0))$ under the mapping f at the point z_0, and thus their angle remains preserved.

Example. The map $f(z) = z^2$ is conformal outside the origin, but it is not locally injective at the origin.

On the other hand, the derivative of the exponential function never vanishes, thus it defines a locally conformal map.

From the above remarks, we obtain:

4.18. Theorem. *Let $f : G \to \mathbf{C}$ be a conformal map. Then*

(i) f defines a homeomorphism $f : G \to f(G)$.

(ii) $f'(z) \neq 0$ for all $z \in G$.

(iii) The inverse function of f, $f^{-1} : f(G) \to G$, is also a conformal map, and

$$(f^{-1})'(f(z)) = \frac{1}{f'(z)}\,.$$

Example. The conformal mapping between the domains $G = \{z \in \mathbf{C} : |z| > 1\}$ and $D = B^*(0,1)$ is given by $f(z) = \frac{1}{z}$.

The following fundamental theorem states that every simply connected domain (which is not the entire complex plane) is conformally equivalent to the unit disk.

4.19. Theorem (Riemann mapping theorem)**.** *Let $D \neq \mathbf{C}$ be a simply connected domain. Then there exists an analytic bijection $f : D \to B(0,1)$.*

The proof of the Riemann mapping theorem 4.19 is given in the following two subsections. Using it, we can easily prove Lemma 4.21, which actually characterizes the complex plane regions that are conformally equivalent to the disk; we only need the following auxiliary result.

4.20. Lemma. *Let D be a simply connected domain and $f : D \to \mathbf{C}$ a conformal map. Then $f(D)$ is simply connected.*

PROOF: Since f is not constant, $f(D)$ is a domain. Let $w_0 \in \mathbf{C} \setminus f(D)$ and let σ be a closed curve in $f(D)$. It suffices to show that $n(\sigma, w_0) = 0$.

Now $\gamma = f^{-1} \circ \sigma$ is a closed curve in D, where f^{-1} is the inverse map of f from $f(D)$ to D, which is also a conformal map. Moreover, $\sigma = f \circ \gamma$. So

$$n(\sigma, w_0) = n(f \circ \gamma, w_0) = \frac{1}{2\pi i} \int_{f\circ\gamma} \frac{dw}{w - w_0} = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - w_0}\, dz\,.$$

Since γ is a closed curve in the simply connected domain D and $z \mapsto f'(z)/(f(z)-w_0)$ is analytic in D, the last integral is zero by Cauchy's theorem. □

4.21. Theorem. *Let G be an open set. Then the following conditions are equivalent:*

(A) $G \neq \mathbf{C}$ and G is simply connected;

(B) there exists an analytic bijection $f : G \to B(0,1)$.

PROOF: The implication (A) $\Rightarrow$ (B) follows from the Riemann mapping theorem. Let G be an open set and let $f : G \to B(0,1)$ be an analytic bijection; by Liouville's theorem, $G \neq \mathbf{C}$. Furthermore, since the inverse function $f^{-1} : B(0,1) \to G$ is also an analytic bijection, Lemma 4.20 shows that $G = f^{-1}[B(0,1)]$ is a simply connected domain. □

The Riemann mapping theorem has many proofs, and the proof presented below originates from Fejér and Riesz (1922). It is based on the ideas of Dirichlet and Riemann, solving a suitably posed extremum problem. For this, we first briefly discuss the theory of normal families.

4.5. Normal families and Montel's theorem

Recall the following definition:

- A sequence $f_j : D \to \mathbf{C}$, where $D \subset \mathbf{C}$ is a domain, converges locally uniformly in the set D if and only if the functions f_j converge uniformly on every compact set $K \subset D$.

In this section, we provide an extremely useful criterion (Arzelà–Ascoli theorem), which guarantees the existence of a locally uniformly convergent subsequence.

4.22. Definition. Let D be a domain and $\mathcal{F} \subset C(D, \mathbf{C}) = \{f : D \to \mathbf{C} \text{ continuous}\}$ a family of continuous complex-valued functions. We say that the family $\mathcal{F}$ is *normal on the domain* D, if every sequence of functions from the family $\mathcal{F}$ contains a subsequence that converges locally uniformly in the set D.

Note that the limit function of the (subsequence) does not necessarily belong to the family $\mathcal{F}$. However, the limit function is always continuous on D. We skip the proof of the following lemma, which is typically presented in courses on topology or functional analysis.

4.23. Theorem (Arzelà–Ascoli)**.** *Let $D \subset \mathbf{C}$ be a domain. Then a family $\mathcal{F}$ is normal in D if and only if conditions (1) and (2) hold:*

(1) $\mathcal{F}$ is equicontinuous at every point, i.e., for every $w \in D$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - f(w)| < \varepsilon \qquad \text{whenever } z \in B(w, \delta) \text{ and } f \in \mathcal{F}$$

(2) for every $w \in D$, the set $A(w) = \overline{\{f(w) : f \in \mathcal{F}\}}$ is compact.

4.24. Remark. Let $\mathcal{F} \subset C(D, \mathbf{C})$ be a normal family of analytic functions in the domain D. Every sequence in the family $\mathcal{F}$ contains a subsequence that converges uniformly in compact subsets of D. In particular, the limit function of such subsequence is analytic by Lemma 2.9.

We apply the Arzelà–Ascoli theorem to answer the following question: when is a family $\mathcal{F}$ of analytic functions in an open set $D \subset \mathbf{C}$ normal?

4.25. Theorem (Montel's theorem)**.** *Suppose D is a domain. A family $\mathcal{F}$ of analytic functions in the set D is normal if and only if $\mathcal{F}$ is locally bounded, i.e., for every compact subset $K \subset D$, there exists $M = M(K) < \infty$ such that*

$$|f(z)| \le M \qquad \text{whenever } z \in K \text{ and } f \in \mathcal{F}\,.$$

PROOF: Let $\mathcal{F}$ be a locally bounded family of analytic functions in the domain D. First, we show that $\mathcal{F}$ is equicontinuous. Let $w \in D$ and $\varepsilon > 0$. To construct $\delta > 0$, choose $r > 0$ such that $K = \overline{B(w,r)} \subset D$. Consider a function $f \in \mathcal{F}$, then by assumption we have

$$\sup_{|z-w|\le r} |f(z)| \le M = M(K) < \infty\,.$$

Using Cauchy's integral formula (Theorem 1.10), we have

$$\begin{aligned}\sup_{|z-w|\le r/2} |f'(z)| &= \sup_{|z-w|\le r/2} \left| \frac{1}{2\pi i} \int_{\partial B(w,r)} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \right| \\ &\le \frac{M 2\pi r}{2\pi (r/2)^2} = \frac{4M}{r} =: M'\,.\end{aligned}$$

Choose $\delta = \min\{\varepsilon/(2M'), r/2\} > 0$, then

$$\begin{aligned}\sup_{|z-w|\le\delta} |f(z) - f(w)| &= \sup_{|z-w|\le\delta} \left| \int_{[w,z]} f'(\zeta)\, d\zeta \right| \\ &\le \sup_{|z-w|\le\delta} |z-w| M' \le \delta M' < \varepsilon\,.\end{aligned}$$

Since δ does not depend on the function f, the family $\mathcal{F}$ is equicontinuous.

Furthermore, since $\{w\} \subset D$ is compact, by assumption $|f(w)| \le M(\{w\})$ whenever $w \in D$ and $f \in \mathcal{F}$. In particular,

$$A(w) = \overline{\{f(w) \,:\, f \in \mathcal{F}\}}$$

is compact for every $w \in D$, since this set is always bounded.

The normality of the family $\mathcal{F}$ now follows from the Arzelà–Ascoli Theorem 4.23. The necessity of Montel's condition is left as an exercise. $\square$

4.6. Proof of the Riemann mapping theorem

We prove the Riemann mapping theorem (Theorem 4.19): if $D \neq \mathbf{C}$ is a simply connected domain, then there exists an analytic bijection $f : D \to B(0,1)$. We proceed with the proof. Fix some $z_0 \in D$ and define the family

$$\mathcal{F} = \{f : D \to B(0,1) \,:\, f \text{ is conformal and } f(z_0) = 0\}.$$

The proof consists of three steps, roughly outlined as follows:

1. $\mathcal{F} \neq \emptyset$;
2. There exists $f \in \mathcal{F}$ such that $|f'(z_0)| \geq |g'(z_0)|$ for all $g \in \mathcal{F}$;
3. If $f \in \mathcal{F}$ satisfies the condition in step 2, then f has the desired properties.

Step 1. The proof of the first step $\mathcal{F} \neq \emptyset$ is based on the assumption $D \neq \mathbf{C}$ and the simply connectedness of D; details are left as an exercise.

Step 2. Let $s = \sup\{|g'(z_0)| \,:\, g \in \mathcal{F}\}$. Choose a sequence f_j indexed by $j \in \mathbf{N}$ from the family $\mathcal{F}$ such that

$$\lim_{j\to\infty} |f_j'(z_0)| = s\,.$$

Since $\mathcal{F} \neq \emptyset$ and the members of this family are conformal mappings that are analytic injections in D, by Corollary 4.15 we have $s > 0$.

By definition, the family $\mathcal{F}$ is locally bounded (since for every compact set $K \subset D$, we have $|g(z)| \leq 1$ for all $z \in K$ and $g \in \mathcal{F}$). According to Montel's theorem 4.25, the family $\mathcal{F}$ is normal in D. Therefore, there exists a subsequence $(f_{j_k})_{k\in\mathbf{N}}$ of the sequence (f_j) and a function $f : D \to \mathbf{C}$ such that

$$f_{j_k} \xrightarrow{k\to\infty} f \text{ locally uniformly in } D\,.$$

By Theorem 2.9, the limit function f is analytic in D. By the same theorem, $f_{j_k}' \to f'$ locally uniformly as $k \to \infty$. In particular,

$$0 < s = \lim_{k\to\infty} |f_{j_k}'(z_0)| = |f'(z)|\,,$$

so f is not constant in D. Since the functions f_{j_k} are analytic injections in D, by Lemma 4.10, f is an injection from D to $\mathbf{C}$. Furthermore, note that $f(D) \subset \overline{B(0,1)}$ and f is an open mapping by Lemma 4.13. Thus, $f(D) \subset B(0,1)$. Since $f_{j_k}(z_0) = 0$ for all $k \in \mathbf{N}$, we also have $f(z_0) = 0$.

From the above reasoning, it follows that $f \in \mathcal{F}$ and f satisfies the condition $|f'(z_0)| \geq |g'(z_0)|$ for all functions $g \in \mathcal{F}$.

Step 3. We claim that f is an analytic bijection $D \to B(0,1)$. After step 2, it suffices to show that

$$f : D \to B(0,1) \text{ is surjective.}$$

This is proved by contradiction: Suppose there exists $w_0 \neq 0$ such that

$$w_0 \in B(0,1) \setminus f(D)\,.$$

Using this assumption, we will construct a map $\rho \in \mathcal{F}$ such that $|\rho'(z_0)| > |f'(z_0)|$. This is a contradiction with the condition $|f'(z_0)| \geq |g'(z_0)|$ for all $g \in \mathcal{F}$.

We need auxiliary mappings (details omitted). If $a \in B(0,1)$, then define

$$\phi_a(z) = \frac{z-a}{1-\bar{a}z} \qquad \text{for } z \in B(0,1)\,.$$

The map ϕ_a is an analytic bijection from $B(0,1)$ to $B(0,1)$, and

$$\phi_a'(z) = \frac{1-|a|^2}{(1-\bar{a}z)^2} \qquad \text{whenever } z \in B(0,1)\,. \tag{4.1}$$

The composite mapping $\tau = \phi_{w_0} \circ f$ is a well-defined conformal mapping $D \to B(0,1)$. By Lemma 4.20, $\tau[D] \subset B(0,1)$ is simply connected and $0 \notin \tau[D]$. By Lemma 1.15, there exists a branch of the logarithm $\log : \tau[D] \to \mathbf{C}$ in the region $\tau[D]$, which is an analytic injection satisfying $e^{\log(z)} = z$ for all $z \in \tau[D]$. Define

$$S(w) = e^{\frac{1}{2}\log(w)}\,, \qquad \text{for } w \in \tau(D)\,.$$

Then $S(w)^2 = w$ for $w \in \tau(D)$, so S is a branch of the square root in $\tau(D)$. We will show that S is a conformal mapping $\tau(D) \to B(0,1)$. For all $w \in \tau(D)$, we have

$$|S(w)|^2 = |w| < 1, \qquad w \in \tau(D).$$

Thus $S(\tau(D)) \subset B(0,1)$. Moreover, if $S(w) = S(w')$ for some $w, w' \in \tau(D)$, then

$$w = S(w)^2 = S(w')^2 = w'.$$

This proves the injectivity of the mapping S.

The map $S \circ \phi_{w_0} \circ f$ is then conformal and injective $D \to B(0,1)$, but it maps z_0 to $S(\phi_{w_0}(f(z_0))) = S(-w_0)$. We "normalize" this function by defining

$$\rho = \phi_{S(-w_0)} \circ S \circ \phi_{w_0} \circ f.$$

Then ρ is a conformal map $D \to B(0,1)$ with $\rho(z_0) = 0$. In particular, $\rho \in \mathcal{F}$. To prove the contradiction, it suffices to show that

$$|\rho'(z_0)| > |f'(z_0)|\,. \tag{4.2}$$

We still need to prove inequality (4.2). Observe that

$$\rho = G \circ f$$

where $G = \phi_{S(-w_0)} \circ S \circ \phi_{w_0} : f(D) \to B(0,1)$ satisfies $G(0) = 0$. Then the chain rule gives

$$\rho'(z_0) = G'(0) f'(z_0).$$

We will show that $G'(0) > 1$. To estimate $G'(0)$ we use the chain rule and the differentiation formula (4.1):

$$\begin{aligned} G'(0) &= \phi'_{S(-w_0)}(S(-w_0)) S'(-w_0) \phi'_{w_0}(0) \\ &= \frac{1 - |S(-w_0)|^2}{(1 - |S(-w_0)|^2)^2} S'(-w_0)(1 - |w_0|^2) \end{aligned}$$

Since $S(w)^2 = w$, we have $2S(w)S'(w) = 1$ and $|S(w)|^2 = |w|$. Thus

$$G'(0) = \frac{1 - |w_0|}{(1 - |w_0|)^2} \frac{1}{2\sqrt{|w_0|}} (1 - |w_0|^2) = \frac{1 + |w_0|}{2\sqrt{|w_0|}}.$$

Finally, since

$$\frac{1 + x^2}{2x} = 1 + \frac{(\sqrt{x} - \frac{1}{\sqrt{x}})^2}{2} > 1 \qquad \text{for all } x \in (0,1)\,,$$

one has $G'(0) > 1$. Thus $|\rho'(z_0)| = |G'(0)|\,|f'(z_0)| > |f'(z_0)|$. The estimate (4.2) follows.

This completes the proof of the Riemann mapping theorem. □

4.26. Remark. The argument in Step 3 probably seems miraculous and it is not clear where it came from. Here is a short explanation. The map G was conformal $f(D) \to B(0,1)$, and therefore it has a conformal inverse $H = G^{-1} : B(0,1) \to f(D)$. Since $f(D) \subset B(0,1)$ the map H is conformal from $B(0,1)$ to itself. By the Schwarz lemma (Theorem 5.17), there are only two possibilities for such a map H:

- either H is a rotation and $|H'(0)| = 1$;
- or one has $|H(z)| < |z|$ and $|H'(0)| < 1$.

Since G involves a square root S, the map H cannot be a rotation and one must have $|H'(0)| < 1$. Therefore necessarily $|G'(0)| = 1/|H'(0)| > 1$.

A deeper explanation for this phenomenon is that the Möbius transformations ϕ_{w_0} and $\phi_{S(-w_0)}$ preserve areas with respect to the hyperbolic metric in $B(0,1)$. The square root function S must instead increase the area, because the square function $w \mapsto w^2$ shrinks the area. This last fact follows from a geometric interpretation of the Schwarz lemma in terms of the hyperbolic metric: any conformal map $H : B(0,1) \to B(0,1)$ is either a Möbius transformation that preserves hyperbolic distances, or it is a contraction that strictly decreases hyperbolic distances. This is another explanation for the fact that $|G'(0)| > 1$ in Step 3.

5. Extended complex plane and Möbius transformations

5.1. Extended complex plane

Often it is convenient to add the point at infinity $\infty \notin \mathbf{C}$ to the complex plane $\mathbf{C}$. The resulting set

$$\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$$

is called the *extended complex plane*. Algebraic operations in $\hat{\mathbf{C}}$ are defined similar to the extended real number set:

$$\begin{cases} \infty \pm z = z \pm \infty = \infty\,, & \text{when } z \in \mathbf{C} \\ \infty \cdot z = z \cdot \infty = \infty\,, & \text{when } z \in \hat{\mathbf{C}} \setminus \{0\} \\ \frac{z}{\infty} = 0\,, & \text{when } z \in \mathbf{C} \\ \frac{z}{0} = \infty\,, & \text{when } z \in \hat{\mathbf{C}} \setminus \{0\}\,. \end{cases}$$

Note that operations

$$\infty - \infty\,, \quad \frac{\infty}{\infty}\,, \quad \frac{0}{0} \quad \text{and} \quad 0 \cdot \infty$$

are not defined at all (and thus should not be used).

Traditionally, the extended complex plane is geometrically visualized as a three-dimensional sphere, called the *Riemann sphere*, as follows: Let

$$S = \{(u, v, w) \in \mathbf{R}^3 : u^2 + v^2 + w^2 = 1\}$$

be the unit sphere in $\mathbf{R}^3$, and let's identify the complex plane $\mathbf{C}$ with the plane $\{(x, y, 0) : x, y \in \mathbf{R}\}$. Consider the straight lines L passing through the "north pole" $N = (0, 0, 1)$ of S. If L is not in the tangent plane of S, then it intersects the spherical shell of S at exactly one point $P = (u, v, w) \neq N$, and the plane $\mathbf{C}$ at exactly one point

$$\pi(P) = \pi(u, v, w) = (\frac{u}{1-w}, \frac{v}{1-w}, 0)\,.$$

This yields a bijection $\pi : S \setminus \{N\} \to \mathbf{C}$, called the *stereographic projection*. Setting $\pi(N) = \infty$ establishes correspondence between the extended complex plane $\hat{\mathbf{C}}$ and the sphere S. The inverse map π^{-1} can be easily determined:

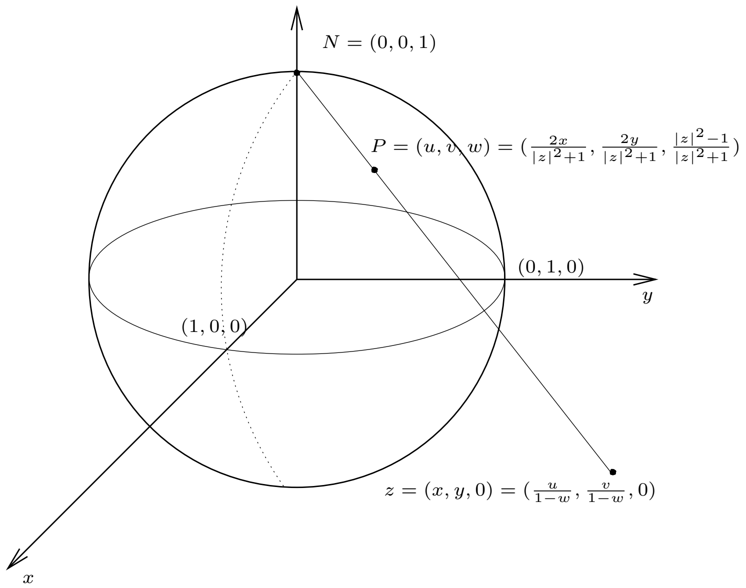
$$\pi^{-1}(z, 0) = (\frac{2Re(z)}{|z|^2 + 1}, \frac{2Im(z)}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1})\,.$$

As an exercise, it is fairly easy to verify that in the stereographic projection, arcs passing through the north pole correspond to straight lines in the complex plane, and circles in the complex plane correspond to circles on the spherical shell. For this reason, circles and lines (extended to include the point at infinity ∞) in $\mathbf{C}$ are called *generalized circles* of the extended complex plane $\hat{\mathbf{C}}$.

Half-planes and disks in $\mathbf{C}$ correspond to spherical caps on the Riemann sphere. Additionally, the stereographic projection is 'conformal', meaning it preserves angles.

Note that the point at infinity ∞ on the Riemann sphere $S = \hat{\mathbf{C}}$ holds the same position as any other point, so by rotating the point at infinity to another location, one can get a good understanding of the local properties of the extended complex plane. For our purposes, it suffices to define the ∞-centered disk as

$$B(\infty, r) = \infty \cup \{z \in \mathbf{C} : |z| > \frac{1}{r}\}\,.$$



Riemann sphere and stereographic projection.

5.2. Möbius transformations

5.1. Definition. A map $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$

$$f(z) = \frac{az+b}{cz+d}\,, \quad \text{where } a,b,c,d \in \mathbf{C}\,, \quad ad - bc \neq 0\,,$$

is called a *Möbius transformation.* [3] Here we use the conventions:

$$\text{If } c = 0\,, \qquad \text{then } \begin{cases} f(z) = \dfrac{az+b}{d}\,, & \text{when } z \in \mathbf{C} \\ f(\infty) = \infty. \end{cases}$$

$$\text{If } c \neq 0\,, \qquad \text{then } \begin{cases} f(z) = \dfrac{az+b}{cz+d}\,, & \text{when } z \in \mathbf{C} \setminus \{-\frac{d}{c}\}, \\ f(-\frac{d}{c}) = \infty \\ f(\infty) = \frac{a}{c}\,. \end{cases}$$

One can easily prove:

5.2. Theorem. *A Möbius transformation $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ is a homeomorphism and a conformal map on $\mathbf{C} \setminus \{-\frac{d}{c}\}$ (thus on $\mathbf{C}$ if $c = 0$).*

□

5.3. Theorem. *Möbius transformations*

$$\mathcal{G} = \{f : \hat{\mathbf{C}} \to \hat{\mathbf{C}} : f \text{ is a Möbius transformation}\}$$

form a group, where the operation is composition of maps.

PROOF: Straightforward calculation (exercise). The inverse map of a Möbius transformation

$$f(z) = \frac{az+b}{cz+d}$$

is

$$f^{-1}(z) = \frac{dz-b}{-cz+a}.$$

□

Note that the group of Möbius transformations is not commutative, i.e., there exist Möbius transformations f, g for which $f \circ g \neq g \circ f$ (exercise).

[3] A.F. Möbius, 1790–1868. Often Möbius transformations are referred to as linear fractional transformations.

5.4. Remark. Möbius transformations

$$f(z) = \frac{az+b}{cz+d}$$

and complex (2×2) matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det(A) = ad - bc \neq 0\,,$$

correspond to each other such that all matrices

$$\lambda A = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}, \quad \det(\lambda A) = \lambda^2 \det(A) = \lambda^2(ad - bc) \neq 0\,,$$

correspond to the same Möbius transformation

$$f(z) = \frac{az+b}{cz+d}\,.$$

The one-to-one correspondence can be achieved, for example, by normalizing (2×2) matrices such that their determinant is $= 1$. Note that matrix multiplication corresponds to composition of Möbius transformations (exercise).

5.5. Remark (Elementary Möbius transformations)**.** A Möbius transformation

$$f(z) = z + w\,, \quad w \in \mathbf{C}$$

is called a *translation*. It translates the points of the complex plane $\mathbf{C}$ by the amount w and keeps the point at infinity fixed $f(\infty) = \infty$.

Another significant type of elementary Möbius transformations are the mappings

$$g(z) = \lambda z\,, \qquad \text{where } \lambda \in \mathbf{C}\,, \lambda \neq 0\,.$$

If $|\lambda| = 1$, it is a *rotation* of the complex plane: g rotates the points of the complex plane by the angle $\arg(\lambda)$ and keeps the point at infinity fixed $g(\infty) = \infty$.

If $\lambda \in \mathbf{R}$ and $\lambda > 0$, then the mapping g is a *dilation*: *expansion* if $\lambda \geq 1$ and *contraction* if $0 < \lambda \leq 1$.

The general case $\lambda \in \mathbf{C}$, $\lambda \neq 0$, is obtained by combining rotation and dilation, as

$$\lambda = |\lambda| \frac{\lambda}{|\lambda|}\,.$$

The point at infinity remains fixed, $g(\infty) = \infty$.

The third type of elementary Möbius transformations is *inversion*:

$$h(z) = \frac{1}{z}\,.$$

Inversion swaps the point at infinity and the origin: $h(\infty) = 0$ and $h(0) = \infty$. Inversion can be obtained by reflecting the point z with respect to the circumference $\partial B(0,1)$ of the unit disk and then with respect to the real axis (taking the conjugate), or vice versa:

$$h(z) = \frac{1}{z} = \overline{\frac{z}{|z|^2}} = \frac{\bar{z}}{|\bar{z}|^2}\,,$$

from which it follows that

$$|h(z)| = \frac{1}{|z|} \qquad \text{and} \qquad \arg(h(z)) = \arg(\bar{z}) = -\arg(z)\,.$$

All Möbius transformations can be obtained by combining translation, dilation, rotation, and inversion.

5.6. Theorem. *Every Möbius transformation can be obtained by combining translation $z \mapsto z + w$, mapping $z \mapsto \lambda z$, and inversion $z \mapsto \frac{1}{z}$.*

Proof: When $ad - bc \neq 0$, then

$$\frac{az+b}{cz+d} = \begin{cases} \dfrac{bc-ad}{c^2(z+\frac{d}{c})} + \dfrac{a}{c}\,, & \text{if } c \neq 0, \\ \dfrac{a}{d}z + \dfrac{b}{d}\,, & \text{if } c = 0, \end{cases}$$

which are of the required type. □

5.7. Theorem. *Möbius transformations map generalized circles to generalized circles.*

Proof: Straight lines in the complex plane are of the form

$$(5.1) \qquad Bz + \bar{B}\bar{z} + c = 0\,, \qquad \text{where } B \in \mathbf{C}\,, \quad c \in \mathbf{R}\,,$$

which can be seen by substituting the equation of a line $ax + by + c = 0$ in $\mathbf{R}^2$ into the equation $z = x + iy$, where $2B = a - ib$. Furthermore, the equation of a circle

$$|z - z_0| = r$$

can be written as

$$z\bar{z} + \bar{B}z + B\bar{z} + c = 0\,, \qquad \text{where } B = -z_0 \in \mathbf{C}\,, \quad c = |B|^2 - r^2 \in \mathbf{R}\,. \tag{5.2}$$

Clearly, lines are preserved as lines and circles as circles under the mappings $z \mapsto z + w$ and $z \mapsto \lambda z$. If z lies on the line (5.1), then its image point $w = 1/z$ under the inversion $z \mapsto 1/z$ satisfies the equation

$$cw\bar{w} + B\bar{w} + \bar{B}w = 0\,,$$

which, for $c = 0$, corresponds to a line passing through the origin. If $c \neq 0$, then according to equation (5.2), this is a circle with radius $|B|/|c|$ and center $-B/c$.

Similarly, if a point z lies on the circle (5.2), then its image point $w = 1/z$ under the inversion satisfies the equation

$$cw\bar{w} + Bw + \bar{B}\bar{w} + 1 = 0\,,$$

which, for $c = 0$ (meaning the circumference passes through the origin), corresponds to the equation of a line. For other values of c, this is the equation of a circle. □

Since Möbius transformations are homeomorphisms, we obtain:

5.8. Corollary. *A Möbius transformation maps every open set that is either*

- *a disk* $B(z_0, r)$,
- *the exterior of a disk* $\hat{\mathbf{C}} \setminus \bar{B}(z_0, r)$, *or*
- *a half-plane determined by a line in* $\mathbf{C}$,

to a set that is of one of the above types.

□

A point $z_0 \in \hat{\mathbf{C}}$ for which $f(z_0) = z_0$ is called a *fixed point* of the transformation f.

5.9. Theorem. *A Möbius transformation $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ has either 1 or 2 fixed points unless f is the identity map $f(z) = z$ for all $z \in \hat{\mathbf{C}}$.*

PROOF: Exercise. Determine the number of solutions of the second degree equation

$$\frac{az+b}{cz+d} = z$$

(be careful with the case $z = \infty$). Another way is to use Theorem 5.10 below. □

In the definition of a Möbius transformation, the images of three points can be specified, but not more.

5.10. Theorem. *Let $z_1, z_2, z_3 \in \hat{\mathbf{C}}$ be three distinct points. If $w_1, w_2, w_3 \in \hat{\mathbf{C}}$ are three distinct points, then there exists exactly one Möbius transformation $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$, for which*

$$f(z_1) = w_1\,, \quad f(z_2) = w_2 \quad \text{and} \quad f(z_3) = w_3\,.$$

PROOF: We can assume (why?) that

$$w_1 = 1\,, \quad w_2 = 0 \quad \text{and } w_3 = \infty\,.$$

Existence: The sought transformation is (exercise: prove this!)

$$f(z) = \frac{(z-z_2)(z_1-z_3)}{(z-z_3)(z_1-z_2)}$$

Note: if z_1, z_2, or z_3 is ∞, then (respectively)

$$f(z) = \frac{z-z_2}{z-z_3}\,, \quad \frac{z_1-z_3}{z-z_3} \quad \text{or} \quad \frac{z-z_2}{z_1-z_2}.$$

Uniqueness: Let g be another such Möbius transformation. Then

$$f \circ g^{-1}(z) = \frac{az+b}{cz+d}$$

is a Möbius transformation, for which

$$\begin{cases} f \circ g^{-1}(0) = 0\,, & \text{implying } b = 0 \\ f \circ g^{-1}(\infty) = \infty\,, & \text{implying } c = 0, \\ f \circ g^{-1}(1) = 1\,, & \text{implying } \frac{a}{d} = 1\,. \end{cases}$$

Thus,

$$f \circ g^{-1}(z) = z \quad \text{for all } z \in \hat{\mathbf{C}}\,,$$

so $f = g$ and the claim is proved. □

The formula that defines the mapping f is called the cross ratio:

5.11. Definition. Let $z_1, z_2, z_3, z_4 \in \hat{\mathbf{C}}$ be four distinct points. The quantity

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)}$$

is called the *cross ratio* of the points z_1, z_2, z_3, z_4.[4]

The cross ratio is invariant under Möbius transformations.

5.12. Theorem. *Let $z_1, z_2, z_3, z_4 \in \hat{\mathbf{C}}$ be four distinct points and $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ a Möbius transformation. Then*

$$[z_1, z_2, z_3, z_4] = [f(z_1), f(z_2), f(z_3), f(z_4)]\,.$$

PROOF: Let

$$g(z) = [z_1, z_2, z_3, z]$$

be the unique Möbius transformation (Theorem 5.10), which maps the points z_1, z_2, and z_3 (respectively) to points 1, 0, and ∞. Then $g \circ f^{-1}$ maps the points $f(z_1)$, $f(z_2)$, and $f(z_3)$ (respectively) to points 1, 0, and ∞, so

$$[z_1, z_2, z_3, z_4] = g(z_4) = g \circ f^{-1}(f(z_4)) = [f(z_1), f(z_2), f(z_3), f(z_4)]\,,$$

where the last equality follows from the uniqueness part of Theorem 5.10. □

[4] Warning: do not confuse this notation with a fraction. The cross ratio can be written in many different orders, so always check from your source what the current definition is.

5.13. Remark. With Theorem 5.12, we can quickly find the (unique) Möbius transformation f that maps the given three distinct points $z_1, z_2, z_3 \in \hat{\mathbf{C}}$ to the given three distinct points $w_1, w_2, w_3 \in \hat{\mathbf{C}}$: solve for $f(z)$ from the cross ratio

$$[w_1, w_2, w_3, f(z)] = [z_1, z_2, z_3, z]\,.$$

Example. Let's construct a Möbius transformation f such that $0 \mapsto 0$, $1 \mapsto 1$, and $2 \mapsto i$. Calculate

$$[1, 0, i, f(z)] = [1, 0, 2, z] \quad \text{or} \quad \frac{(1-i)(-f(z))}{(i - f(z))} = \frac{(1-2)(-z)}{2-z}\,,$$

yielding

$$f(z) = \frac{iz}{z(2-i) - 2 + 2i}\,.$$

Note that in the light of Theorems 5.7 and 5.8, f maps the real axis to the circle determined by the points 0, 1, and i. Furthermore, since $f(-i) = -1/3$, f maps (bijectively) to the upper half-plane the disk bounded by the aforementioned circle

$$B(\frac{1}{2} + \frac{1}{2}i, \frac{\sqrt{2}}{2}).$$

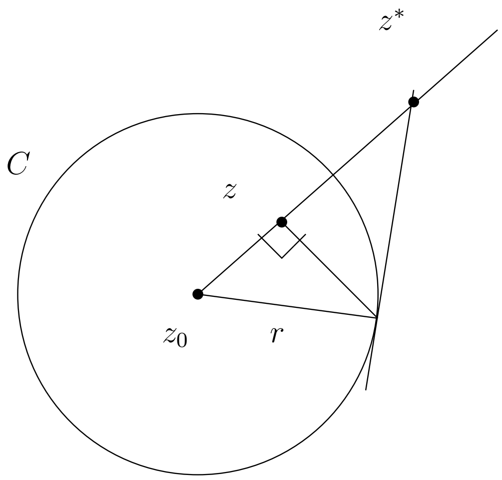
Let

$$C = \{z \in \mathbf{C} : |z - z_0| = r\}$$

be a circle centered at z_0 with radius r. If $z \in \hat{\mathbf{C}}$, then its *reflection point with respect to the circle* C is

$$z^* = \begin{cases} z_0 + \dfrac{r^2}{\bar{z} - \bar{z}_0}\,, & \text{if } z \neq z_0, \infty \\ \infty\,, & \text{if } z = z_0 \\ z_0\,, & \text{if } z = \infty\,. \end{cases}$$

Sometimes, the points z and z^* are said to be *symmetric* with respect to the circle C.



5.14. Theorem. *Points z and w are symmetric with respect to the circle C if and only if*

$$[w, z_1, z_2, z_3] = \overline{[z, z_1, z_2, z_3]} \quad \textit{for all } z_1, z_2, z_3 \in C\,.$$

PROOF: First, it is observed that

$$z_j^* = z_0 + \frac{r^2}{\bar{z}_j - \bar{z}_0} = z_0 + \frac{(z_j - z_0)(\bar{z}_j - \bar{z}_0)}{\bar{z}_j - \bar{z}_0} = z_j\,, \qquad \text{when } z_j \in C\,,$$

implying

$$z^* - z_j = \frac{r^2(\bar{z}_j - \bar{z})}{(\bar{z}_j - \bar{z}_0)(\bar{z} - \bar{z}_0}\,,$$

and therefore

$$[z^*, z_1, z_2, z_3] = \overline{[z, z_1, z_2, z_3]}\,.$$

The proof of the converse direction is an exercise. Note that any three distinct points $z_1, z_2, z_3 \in C$ uniquely determine the reflection point. □

5.15. Corollary. *Let f be a Möbius transformation and C a generalized circle. Then, the image points $f(z)$ and $f(z^*)$ of symmetric points z and z^* with respect to circle C are symmetric with respect to the generalized circle $f(C)$.*

□

Example. Let's construct a Möbius transformation f that maps the unit disk $B(0,1)$ to itself, $f(0) = 0$, and $f(1) = i$. Note that since the origin remains fixed, its symmetric point ∞ also remains fixed. Hence,

$$[i, 0, \infty, f(z)] = [1, 0, \infty, z] ,$$

implying

$$\frac{f(z)}{i} = \frac{z}{1} \qquad \text{or} \qquad f(z) = iz .$$

5.3. Möbius transformations as conformal mappings

We demonstrate that conformal mappings of the disk to itself are restrictions of Möbius transformations to the disk.

We start with the following result.

5.16. Theorem. *Let $z_0 \in B(0,1)$, $\lambda \in \mathbf{C}$, $|\lambda| = 1$. Then the Möbius transformation*

$$f(z) = \lambda \frac{z - z_0}{1 - z\bar{z}_0}$$

defines a (conformal) bijection $f : B(0,1) \to B(0,1)$ with $f(z_0) = 0$.

Proof: Note that f is a Möbius transformation since

$$\lambda(1 - z_0\bar{z}_0) = \lambda(1 - |z_0|^2) \neq 0 .$$

Therefore, since $f(z_0) = 0$, it suffices to check that

$$f(\partial B(0,1)) \subset \partial B(0,1) .$$

Indeed, when $|z| = 1$, we have

$$|f(z)| = |\lambda| \frac{|z - z_0|}{|1 - z\bar{z}_0|} = \frac{|z - z_0|}{|z||\frac{\bar{z}}{|z|^2} - \bar{z}_0|} = \frac{|z - z_0|}{|\bar{z} - \bar{z}_0|} = \frac{|z - z_0|}{|z - z_0|} = 1 .$$

□

We still need the Schwarz lemma:

5.17. Theorem. *If $f : B(0,1) \to B(0,1)$ is analytic with $f(0) = 0$, then*

$$|f'(0)| \le 1 \quad \text{and} \quad |f(z)| \le |z| \quad \text{for all } z \in B(0,1).$$

Moreover, if there exists $z_0 \in B(0,1)$, $z_0 \ne 0$, such that $|f(z_0)| = |z_0|$, or if $|f'(0)| = 1$, then there exists $\lambda \in \mathbf{C}$, $|\lambda| = 1$, such that

$$f(z) = \lambda z \quad \text{for all } z \in B(0,1).$$

Proof: Proven in exercises: immediately follows when applying the maximum modulus principle CA1.5.16 to the function

$$g(z) = \begin{cases} \dfrac{f(z)}{z}\,, & \text{when } z \ne 0, \\ f'(0)\,, & \text{when } z = 0\,, \end{cases}$$

which is analytic in the entire disk $B(0,1)$. □

5.18. Theorem. *Let $f : B(0,1) \to B(0,1)$ be a conformal bijection and $z_0 = f^{-1}(0)$. Then there exists $\lambda \in \mathbf{C}$, $|\lambda| = 1$, such that*

$$f(z) = \lambda \frac{z - z_0}{1 - z\bar{z}_0} \quad \text{for all } z \in B(0,1)\,.$$

Proof: Let

$$g(z) = \frac{z - z_0}{1 - z\bar{z}_0}\,,$$

then by Theorem 5.16, g defines a conformal bijection of the disk to itself with $g(z_0) = 0$. Hence, $h = f \circ g^{-1}$ satisfies the assumptions of Schwarz lemma 5.17, so

$$|h(z)| \le |z| \quad \text{for all } z \in B(0,1)\,.$$

Similarly, h^{-1} satisfies the assumptions of Schwarz lemma 5.17, so

$$|h^{-1}(w)| \le |w| \quad \text{for all } w \in B(0,1)\,.$$

In particular, when $z = h^{-1}(w)$, we have $h(z) = w$ and

$$|h(z)| \le |z| = |h^{-1}(w)| \le |h(z)|$$

thus

$$|h(z)| = |z| \quad \text{for all } z \in B(0,1)\,.$$

Therefore, by a special case of Schwarz lemma, we have

$$h(z) = \lambda z\,,$$

and consequently

$$f(z) = h(g(z)) = \lambda g(z) = \lambda \frac{z - z_0}{1 - z\bar{z}_0}\,,$$

as claimed. □

5.19. Corollary. *If B is a disk, then every conformal mapping $f : B \to B$ is a restriction of a Möbius transformation to the disk B.*

□

Conformal mappings of the entire plane are affine mappings:

5.20. Theorem. *Let $f : \mathbf{C} \to \mathbf{C}$ be an entire injective function. Then there exist $a, b \in \mathbf{C}$ with $a \neq 0$ such that*

$$f(z) = az + b \quad \text{for all } z \in \mathbf{C}\,.$$

In particular, f is a Möbius transformation and a conformal bijection.

Proof: First, we show that f is a polynomial. If f is not a polynomial, then the point at infinity is its essential singularity (Theorem 3.11), i.e., the origin is an essential singularity of the function

$$z \mapsto f(\frac{1}{z})$$

By the Casorati-Weierstrass theorem 3.10, the set

$$f(\mathbf{C} \setminus B(0,1))$$

is dense in $\mathbf{C}$, so by the open mapping theorem (Theorem 4.13), $f(B(0,1))$ intersects it,

$$f(\mathbf{C} \setminus B(0,1)) \cap f(B(0,1)) \neq \emptyset\,.$$

Thus, there exist points $z_1 \in B(0,1)$ and $z_2 \notin B(0,1)$ such that contrary to the injectivity of f,

$$f(z_1) = f(z_2)\,.$$

Hence, f is a polynomial.

By the fundamental theorem of algebra (Theorem CA1.5.15) and Theorem 4.11, an injective polynomial is of degree 1, so the claim follows. □