Complex analysis 1

Supplementary exercises

1. Let $z = 1 + 2i$ and $w = 3 + 4i$. Express the following points in the form $x + iy$:

$$z = 3iw, \quad 2|w|^2 + (1-i)z^2, \quad (z + 5z^{-1})^{-1}, \quad \text{and} \quad \bar{w}^{-1} - w.$$

   Find $\mathrm{Arg}(z)$ and $\mathrm{Arg}(w)$, then $\arg(z)$ and $\arg(w)$. Find the value of zw by the "geometric technique", using polar coordinate representations. Check that your result agrees with the "analytic" way of computing zw, based on the definition of the complex product.

2. Describe geometrically the sets of points z in the complex plane satisfying the following:
   (a) $\mathrm{Re}(z) > 3$
   (b) $|z - i| = |z - 1|$
   (b) $|z| = \mathrm{Re}(z) + 1$

3. Show that for any $z \in \mathbb{C}$ with $z \neq 1$ and any integer $n \geq 0$, one has

$$1 + z + z^2 + \ldots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

4. Prove the *parallelogram law*

$$|z + w|^2 + |z - w|^2 = 2|z|^2 + 2|w|^2, \qquad z, w \in \mathbb{C}.$$

5. Let $f(z) = e^z$ and $Q := [0,1] \times [0,\pi]$. What does $f(Q)$ look like?

6. Show that for any $z, w \in \mathbb{C}$ one has the formulas

$$\sin(z + w) = \sin z \cos w + \cos z \sin w,$$
$$\cos(z + w) = \cos z \cos w - \sin z \sin w.$$

7. Prove by using the definitions that $D(z, r)$ is an open set and $\overline{D}(z, r)$ is a closed set.

8. Let $z \in \mathbb{C}$ with $|z| = 1$, but $z \neq 1$. Show that the limit $\lim_{n\to\infty} z^n$ does not exist. Draw a picture!

(More exercises on the next page)

9. Prove that the sets $D(z,r)$ and $\mathbb{C}\setminus\{0\}$ are connected, and that $D(-2,1)\cup D(2,1)$ is not connected. (See Example 2.4.2.)

10. Determine the largest open set where the function is analytic, and compute the derivative of

$$z^4(1-z)^6, \qquad [(z+1)(z^3-8)]^4, \qquad (z-2\sqrt{z})^{-1}$$

11. Let $U \subset \mathbb{C}$ be open, and let $f\colon U \to \mathbb{C}$ be complex differentiable at $z \in U$. Assume $f'(z) \neq 0$. Show that there exists a radius $r > 0$ such that

$$f(w) \neq f(z), \qquad w \in D(z,r) \setminus \{z\}.$$

12. If f is n times complex differentiable in a connected open set $U \subset \mathbb{C}$ and $f^{(n)}(z) = 0$ for $z \in U$, show that f is a polynomial in z of degree at most $n-1$.

13. Let $\gamma\colon [0,1] \to \mathbb{C}$ be a path. Suppose $z_1, z_2 \in \gamma^*$ are distinct points with

$$\operatorname{card} \gamma^{-1}(\{z_j\}) = 1, \qquad j \in \{1,2\}.$$

Show that $\overleftarrow{\gamma}$ is **not** a reparametrisation of γ. Show by example that one point is not enough to guarantee the same conclusion.

14. Compute the path integrals

$$\int_{\gamma_r} \frac{-3z+2}{z^2-8z+12}\, dz,$$

where $\gamma_r(t) = re^{it}$, $r \in [0,2\pi]$, and $r > 0$ is such that the integrand is continuous on γ_r^* (so that the integral is well-defined). Which values of "r" do you have to exclude?

15. Compute the path integrals

$$\int_{\partial D(0,1)} \frac{dz}{z^n}, \qquad n \geq 2.$$

16. Compute the path integral

$$\int_\gamma ze^z\, dz,$$

where γ is the path $\gamma(t) = te^{2\pi it}$, $t \in [0,1]$. (*Hint:* integration by parts.)

17. Calculate the integral

$$\int_0^\infty \frac{dx}{x^{10}+1}.$$