Complex analysis 1

Exercises 7

David Johansson

March 1, 2024

Exercise 1

Let γ and β be closed, piecewise C^1-paths in $\mathbb{C}$ with the same initial point. Show that the winding numbers satisfies

$$\begin{aligned} n_{\overleftarrow{\gamma}}(z) &= -n_\gamma(z), && z \in \mathbb{C} \setminus \gamma^*, \\ n_{\gamma\star\beta}(z) &= n_\gamma(z) + n_\beta(z), && z \in \mathbb{C} \setminus (\gamma^* \cup \beta^*) \end{aligned}$$

Solution:

The function $\gamma^* \ni y \mapsto \frac{1}{y-z}$ is continuous since $z \notin \gamma^*$. Then we get from Proposition 4.2.15 that

$$n_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{1}{y-z}\,dy = -\frac{1}{2\pi i}\int_{\overleftarrow{\gamma}} \frac{1}{y-z}\,dy = -n_{\overleftarrow{\gamma}}(z).$$

Similarly, $(\gamma \star \beta)^* \ni y \mapsto \frac{1}{y-z}$ is continuous when $z \notin (\gamma \star \beta)^*$, so we again get from Proposition 4.2.15 that

$$\begin{aligned} n_{\gamma\star\beta}(z) &= \frac{1}{2\pi i}\int_{\gamma\star\beta} \frac{1}{y-z}\,dy \\ &= \frac{1}{2\pi i}\int_\gamma \frac{1}{y-z}\,dy + \frac{1}{2\pi i}\int_\beta \frac{1}{y-z}\,dy \\ &= n_\gamma(z) + n_\beta(z). \end{aligned}$$

∎

Exercise 2

Let η be the circle path $\eta(t) = e^{it}$, $t \in [0, 2\pi]$. Compute the path integral

$$\int_\eta \sqrt{9 - z^2}\, dz,$$

where $\sqrt{\cdot}$ is the principal square root.

Solution:

Since $B(0,3) \ni z \mapsto z\sqrt{9 - z^2} =: f(z)$ is analytic and $\eta^* \in B(0,3)$ then

$$\int_\eta \sqrt{9 - z^2}\, dz = \int_\eta \frac{f(z)}{z}\, dz = 2\pi i \underbrace{f(0)}_{=0} n_\eta(0) = 0$$

by Theorem 5.2.13. ■

Exercise 3

With η as above, compute

$$\int_\eta \frac{1}{z^2 - \frac{5}{2}z + 1}\, dz.$$

Solution:

First note that $z^2 - \frac{5}{2}z + 1 = (z-2)(z - \frac{1}{2})$ and

$$\frac{1}{z^2 - \frac{5}{2}z + 1} = \frac{2}{3}\left(\frac{1}{z-2} - \frac{1}{z - \frac{1}{2}}\right).$$

Then

$$\int_\eta \frac{1}{z^2 - \frac{5}{2}z + 1}\, dz = 2\pi i \frac{2}{3}\left(\underbrace{\frac{1}{2\pi i}\int_\eta \frac{1}{z-2}\, dz}_{=0} - \underbrace{\frac{1}{2\pi i}\int_\eta \frac{1}{z - \frac{1}{2}}\, dz}_{=1}\right)$$

$$= -\frac{4\pi i}{3}$$

where the first integral is 0 since the integrand is analytic inside γ and the second integral is evaluated using Theorem 5.2.13. ■

Exercise 4

Compute

$$\int_0^{2\pi} \frac{1}{5 - 4\cos\theta}\, d\theta.$$

Solution:

Again using the same path $\eta(\theta) = e^{i\theta}$, $\theta \in [0, 2\pi]$ as before, we have

$$\cos(\theta) = \frac{\eta(\theta) + \frac{1}{\eta(\theta)}}{2}$$

and

$$1 = \frac{1}{i\eta(\theta)}\eta'(\theta).$$

Inserting this in the integral, we get

$$\begin{aligned}\int_0^{2\pi} \frac{1}{5 - 4\cos(\theta)}\, d\theta &= \int_0^{2\pi} \frac{1}{5 - 2(\eta(\theta) + \frac{1}{\eta(\theta)})} \frac{1}{i\eta(\theta)}\eta'(\theta)\, dz \\ &= \int_\eta \frac{1}{5 - 2(z + \frac{1}{z})} \frac{1}{iz}\, dz \\ &= -\frac{1}{2i} \int_\eta \frac{1}{z^2 - \frac{5}{2}z + 1}\, dz = \frac{2\pi}{3}\end{aligned}$$

where the last equality comes from Exercise 3. ■

Exercise 5

Let $\gamma_r(t) = re^{it}$, $t \in [0, \pi]$. Show that

$$\lim_{r\to 0} \int_{\gamma_r} \frac{e^{iz}}{z}\, dz = i\pi \quad \text{and} \quad \lim_{R\to\infty} \int_{\gamma_R} \frac{e^{iz}}{z}\, dz = 0.$$

Solution:

Let's start with the limit $r \to 0$. I start by Taylor expanding e^{iz} at 0 using Proposition 3.1.6:

$$e^{iz} = 1 + iz + \varepsilon(z)z$$

where $\varepsilon(z) \to 0$ when $z \to 0$. Then

$$\int_{\gamma_r} \frac{e^{iz}}{z}\, dz = \int_{\gamma_r} \frac{1}{z} + i + \varepsilon(z)\, dz \tag{1}$$

Using Corollary 4.2.12 we have

$$\left| \int_{\gamma_r} i + \varepsilon(z)\, dz \right| \leq (1 + \|\varepsilon\|_{L^\infty(\gamma_r^*)}) \operatorname{length}(\gamma_r).$$

Note that $\operatorname{length}(\gamma_r) = \pi r \to 0$. So if only $\|\varepsilon\|_{L^\infty(\gamma_r^*)}$ is bounded in r then this converges to 0. Actually, $\varepsilon(z)$ is analytic and therefore bounded on compact sets, by Proposition 2.3.9. To see that it is analytic, write

$$\varepsilon(z) = \frac{e^{iz}}{z} - \frac{1}{z} - i.$$

Since the right-hand side is analytic outside of $z = 0$, the same is true for $\varepsilon(z)$. Then $\varepsilon(z)$ is also continuous outside of $z = 0$. But it is also continuous at $z = 0$ since we know that $\varepsilon(z) \to 0$ as $z \to 0$. By Corollary 5.3.9 we get that $\varepsilon(z)$ is analytic in some neighborhood of 0. Let U be a neighborhood of the origin and C a constant such that $|\varepsilon(z)| \leq C$ for all $z \in U$. Since $\gamma_r^* \subset U$ for all small enough $r > 0$, this shows that

$$\|\varepsilon\|_{L^\infty(\gamma_r^*)} \leq C$$

and we conclude that

$$\lim_{r \to 0} \int_{\gamma_r} i + \varepsilon(z)\, dz = 0. \tag{2}$$

Next we have

$$\begin{aligned} \int_{\gamma_r} \frac{1}{z}\, dz &= \int_0^\pi \frac{1}{re^{it}} ire^{it}\, dt \\ &= \int_0^\pi i\, dt \\ &= i\pi. \end{aligned} \tag{3}$$

Combining (1),(2),(3) we get

$$\lim_{r \to 0} \int_{\gamma_r} \frac{e^{iz}}{z}\, dz = i\pi.$$

Now let's prove that

$$\lim_{R \to \infty} \int_{\gamma_R} \frac{e^{iz}}{z}\, dz = 0.$$

Let's start by just estimating the integral from above:

$$\begin{aligned}
\left|\int_{\gamma_R} \frac{e^{iz}}{z}\,dz\right| &\leq \int_{\gamma_R} \frac{|e^{iz}|}{|z|}\,|dz| \\
&= \int_{\gamma_R} \frac{|e^{i\,\mathrm{Re}(z)-\mathrm{Im}(z)}|}{|z|}\,dz \\
&= \int_{\gamma_R} \frac{e^{-\mathrm{Im}(z)}}{|z|}\,dz \\
&= \int_0^{\pi} \frac{e^{-R\sin(t)}}{|Re^{it}|}|iRe^{it}|\,dz \\
&= \int_0^{\pi} e^{-R\sin(t)}\,dz.
\end{aligned} \tag{4}$$

One way to handle this is to use the dominated convergence theorem. Since I don't know if students are expected to know about dominated convergence in this course, I'll present an alternative solution. But the price of not using powerful tools like dominated convergence is that it gets more involved. Let's start by writing

$$\int_0^{\pi} e^{-R\sin(t)}\,dt = \int_0^{\frac{1}{2}} e^{-R\sin(t)}\,dt + \int_{\frac{1}{2}}^{\pi-\frac{1}{2}} e^{-R\sin(t)}\,dt + \int_{\pi-\frac{1}{2}}^{\pi} e^{-R\sin(t)}\,dt.$$

Using $\sin(t) = \sin(\pi - t)$ and a change of variables, we find

$$\int_0^{\frac{1}{2}} e^{-R\sin(t)}\,dt = \int_{\pi-\frac{1}{2}}^{\pi} e^{-R\sin(t)}\,dt,$$

and we get

$$\int_0^{\pi} e^{-R\sin(t)}\,dt = 2\int_0^{\frac{1}{2}} e^{-R\sin(t)}\,dt + \int_{\frac{1}{2}}^{\pi-\frac{1}{2}} e^{-R\sin(t)}\,dt \tag{5}$$

It's well-known that $-t \leq \sin(t) \leq t$ for $t \geq 0$, but if we additionally have $t \leq 1$ then this can actually be improved to $\frac{5}{6}t \leq \sin(t) \leq t$. To see this, we can use another basic but perhaps less well-known inequality(see e.g. [1, Theorem 8.4.8]) for $\sin(t)$, namely

$$t - \frac{1}{6}t^3 \leq \sin(t)$$

for $t \geq 0$. When $0 \leq t \leq 1$ then $-t \leq -t^3$ and it follows that

$$\frac{5}{6}t \leq \sin(t)$$

for $0 \leq t \leq 1$. From this we get

$$e^{-R\sin(t)} \leq e^{-R\frac{5}{6}t}$$

for $0 \leq t \leq 1$. Also, $e^{-R\sin(t)}$ attains its maximum over $[\frac{1}{2}, \pi - \frac{1}{2}]$ at the boundary points, so

$$e^{-R\sin(t)} \leq e^{-R\sin(\frac{1}{2})}$$

when $\frac{1}{2} \leq t \leq \pi - \frac{1}{2}$. To see this, we can look at its derivative, $-R\cos(t)e^{-R\sin(t)}$, which is negative when $t < \pi/2$ and positive when $t > \pi/2$. Now we have

$$\int_0^{\frac{1}{2}} e^{-R\sin(t)}\, dt \leq \int_0^{\frac{1}{2}} e^{-R\frac{5}{6}t}\, dt = \frac{6}{5R}\left(1 - e^{-\frac{5R}{12}}\right) \to 0. \tag{6}$$

Note also that $e^{\sin(\frac{1}{2})} > 1$, so we get

$$\int_{\frac{1}{2}}^{\pi-\frac{1}{2}} e^{-R\sin(t)}\, dt \leq \int_{\frac{1}{2}}^{\pi-\frac{1}{2}} (e^{-R\sin(\frac{1}{2})})\, dt = \left(\frac{1}{e^{\sin(\frac{1}{2})}}\right)^R (\pi - 1) \to 0. \tag{7}$$

From (5),(6),(7) we get

$$\lim_{R\to\infty} \int_0^R e^{-R\sin(t)}\, dt = 0.$$

and it follows that

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{e^{iz}}{z}\, dz = 0.$$

■

Exercise 6

Show that

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\sin(t)}{t}\, dt = \pi.$$

Solution:

I want relate the integral to a complex path integral. Specifically, I want to integrate on the paths $[-R, -\frac{1}{R}]$ and $[\frac{1}{R}, R]$, so I first need to show that we can ignore the path $[-\frac{1}{R}, \frac{1}{R}]$. Using symmetry around the origin and $\sin(t) \leq t$ for $t \geq 0$ we have

$$\left| \int_{-\frac{1}{R}}^{\frac{1}{R}} \frac{\sin(t)}{t} \, dt \right| \leq \int_{-\frac{1}{R}}^{\frac{1}{R}} \frac{|\sin(t)|}{|t|} \, dt \leq 2 \int_0^{\frac{1}{R}} \, dt = \frac{2}{R} \to 0.$$

On the remaining part of the interval $[-R, R]$, I want to use complex path integrals. Note that for $z \in \mathbb{R}$, $\sin(z) = \operatorname{Im}(e^{iz})$. Consider the two paths $[-R, -\frac{1}{R}](t) = -R(1-t) - \frac{t}{R}$ and $[\frac{1}{R}, R](t) = \frac{1}{R}(1-t) + Rt$. Then we get, by using variable substitution,

$$\begin{aligned}
\int_{\frac{1}{R}}^{R} \frac{\sin(t)}{t} \, dt &= \int_0^1 \frac{\sin(\frac{1}{R}(1-t) + Rt)}{\frac{1}{R}(1-t) + Rt} (R - \frac{1}{R}) \, dt \\
&= \operatorname{Im}\Big(\int_{[\frac{1}{R}, R]} \frac{e^{iz}}{z} \, dz \Big)
\end{aligned}$$

and similarly,

$$\int_{-R}^{-\frac{1}{R}} \frac{\sin(t)}{t} \, dt = \operatorname{Im}\Big(\int_{[-R, -\frac{1}{R}]} \frac{e^{iz}}{z} \, dz \Big).$$

Using the paths γ_R, $\gamma_{\frac{1}{R}}$ and limits from from Exercise 5, we get

$$\begin{aligned}
\int_{[-R,-\frac{1}{R}] \cup [\frac{1}{R}, R]} \frac{e^{iz}}{z} \, dz &= \underbrace{\int_{[-R,-\frac{1}{R}] \star \gamma_{\frac{1}{R}} \star [\frac{1}{R}, R] \star \gamma_R} \frac{e^{iz}}{z} \, dz}_{=0} \\
&\quad - \underbrace{\int_{\gamma_{\frac{1}{R}}} \frac{e^{iz}}{z} \, dz}_{\to i\pi} - \underbrace{\int_{\gamma_R} \frac{e^{iz}}{z} \, dz}_{\to 0} \\
&\to i\pi.
\end{aligned}$$

The integral over the closed path is 0 since the function is analytic analytic away from $z = 0$. Combining all the above we get

$$\begin{aligned}
\lim_{R \to \infty} \int_{-R}^{R} \frac{\sin(t)}{t} \, dt &= \lim_{R \to \infty} \int_{-\frac{1}{R}}^{\frac{1}{R}} \frac{\sin(t)}{t} \, dt \\
&\quad + \lim_{R \to \infty} \operatorname{Im}\Big(\int_{[-R,-\frac{1}{R}]} \frac{e^{iz}}{z} \, dz + \int_{[\frac{1}{R}, R]} \frac{e^{iz}}{z} \, dz \Big) \\
&= \pi.
\end{aligned}$$

■

References

[1] Robert G Bartle and Donald R Sherbert. *Introduction to real analysis.* Wiley New York, 2011.