Complex analysis 1

Exercises 6

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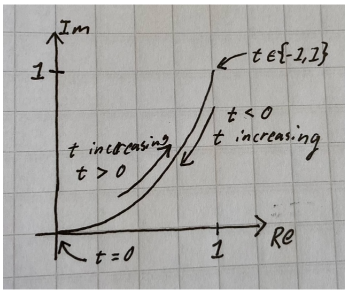
Exercise 1

Draw a rough sketch of the following paths in $\mathbb{C}$ and explain briefly how the curve traces out its trajectory:

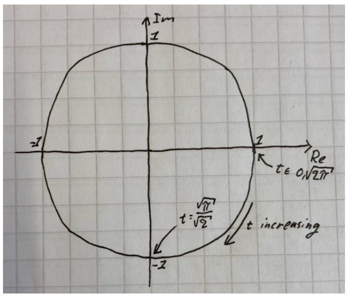
(i) $\gamma(t) = t^2 + it^4$ for $t \in [-1, 1]$,

(ii) $\gamma(t) = e^{-it^2}$ for $t \in [0, \sqrt{2\pi}]$,

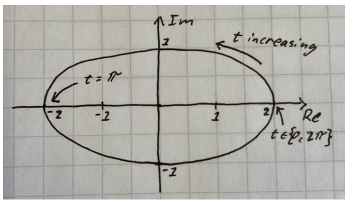
(iii) $\gamma(t) = 2\cos(t) + i\sin(t)$ for $t \in [0, 2\pi]$.



(a) path in part (i)



(b) path in part (ii)



(c) path in part (iii)

SOLUTION:

In part (i) we have a path that looks like the parabola $y = x^2$ since the imaginary part is quadratic in the real part. The one perhaps noteworthy thing I suppose is the speed at which we traverse the path, which is quite uneven because of the nonlinearities. When $t = 0$ and we're in the middle of the index set then we have traversed exactly half the path. But when $t = -1/2$ and we're only a quarter way through the index set then we're already more that a quarter through the path because of the nonlinear behaviour.

In part (ii), the path looks like the unit cirle which we traverse clockwise. Again the speed at which we traverse the path varies a lot. Exactly half way through the index set, at $t = \sqrt{\pi}/\sqrt{2}$, we have only traversed a quarter of the circle.

In part (iii), I believe we have an ellipse and we traverse anti-clockwise. In this case we again traverse the path with with varying speed. At $t = \pi$ exactly half way through the index set, we have traversed half the path. But at $t = \pi/4$ at an 8th through the index set then we have traversed more than one 8th of the path. ■

EXERCISE 2

If $\gamma(t) = te^{it}$ for $0 \leq t \leq \pi$, evaluate

(i) $\int_\gamma \bar{z}\, dz$,

(ii) $\int_\gamma |z|\, |dz|$,

(iii) $\int_\gamma |z|\, dz$.

SOLUTION:

So $\gamma(t) = te^{it}$ and $\gamma'(t) = (1 + it)e^{it}$. Then

$$\int_\gamma \bar{z}\, dz = \int_0^\pi \bar{\gamma(t)}\gamma'(t)\, dt = \int_0^\pi t + it^2\, dt = \frac{\pi^2}{2} + i\frac{\pi^3}{3}.$$

Next,

$$\int_\gamma |z|\, |dz| = \int_0^\pi |\gamma(t)||\gamma'(t)|\, dt = \int_0^\pi t\sqrt{1+t^2}\, dt = \frac{1}{3}[(1+\pi^2)^{3/2} - 1]$$

and finally

$$\begin{aligned}
\int_\gamma |z|\,dz = \int_0^\pi |\gamma(t)|\gamma'(t)\,dz &= \int_0^\pi (t+it^2)e^{it}\,dt \\
&= \underbrace{\int_0^\pi t\cos(t)\,dt}_{=-2} - \underbrace{\int_0^\pi t^2\sin(t)\,dt}_{=\pi^2-4} \\
&\quad + i\underbrace{\int_0^\pi t\sin(t)\,dt}_{=\pi} + i\underbrace{\int_0^\pi t^2\cos(t)\,dt}_{-2\pi} \\
&= 2-\pi^2-i\pi.
\end{aligned}$$

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EXERCISE 3

Evaluate $\int_\gamma z^2\,dz$ and $\int_\gamma e^z\,dz$, where $\gamma(t) = t + i\frac{t^2}{\pi}$ for $0 \le t \le \pi$.

SOLUTION:

We can use the fundamental theorem of calculus to compute

$$\int_\gamma z^2\,dz = \frac{1}{3}z^3\Big|_{z=\gamma(0)}^{z=\gamma(\pi)} = \frac{1}{3}(\pi+i\pi)^3 = \frac{2\pi^3}{3}(-1+i)$$

and

$$\int_\gamma e^z\,dz = e^z\Big|_{z=\gamma(0)}^{z=\gamma(\pi)} = e^{\pi(1+i)} - e^0 = -(e^\pi+1)$$

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EXERCISE 4

Prove proposition 4.2.10: Let $U \subset \mathbb{C}$ be open and let $\gamma : [a,b] \to U$ be a piecewise C^1-path. Assume that $f, g : U \to \mathbb{C}$ are analytic. Assume additionally that f', g' are continuous. Show that

$$\int_\gamma f(z)g'(z)\,dz = [f(\gamma(b))g(\gamma(b)) - f(\gamma(a))g(\gamma(a))] - \int_\gamma f'(z)g(z)\,dz$$

SOLUTION:

This follows straight from Theorem 4.3.4 in the lecture notes. Since f, g are analytic, the product fg is also analytic and $(fg)' = f'g + fg'$.

Clearly fg is a primitive of $(fg)'$, so from the fundamental theorem of calculus we get

$$\begin{aligned}\int_\gamma f(z)g'(z)\,dz &= \int_\gamma (fg)'(z)\,dz - \int_\gamma f'(z)g(z)\,dz \\ &= f(\gamma(b))g(\gamma(b)) - f(\gamma(a))g(\gamma(a)) - \int_\gamma f'(z)g(z)\,dz.\end{aligned}$$

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EXERCISE 5

Consider the map $f(z) = \bar{z}$, defined for $z \in \mathbb{C}$.

(a) Show that f does not have a primitive in any open set $U \subset \mathbb{C}$.

(b) In Theorem 5.1.4, if U is convex and $a \in U$, we explicitly defined

$$F(z) := \int_{[a,z]} f(\zeta)\,d\zeta.$$

and showed that F is a primitive of f. What goes wrong in the proof when $f(z) = \bar{z}$ and why isn't F above a primitive of f? Calculate F explicitly!

SOLUTION:

If f had a primitive then we would have

$$\int_\gamma f(z)\,dz = 0$$

for any closed C^1-path γ. But for any path $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$, we have

$$\int_\gamma f(z)\,dz = \int_0^{2\pi} i\bar{z}_0 re^{it} + ir^2\,dt = 2\pi r^2 i.$$

So $f(z)$ cannot have a primitive.

For the second part, the proof that the given function $F(z)$ is a primitive relies on Cauchy's theorem for triangles(Theorem 5.1.1) which assumes f to be analytic. So unless the assumptions in Theorem 5.1.1 can be weakened to not use analyticity then this is a reason why the theorem doesn't apply to $f(z) = \bar{z}$. But the analyticity assumption in Theorem 5.1.1 really is used in the proof, since the formula $f(z) = f(z_0) + f'(z_0)(z - z_0) + e(z)(z - z_0)$ is used. Finally, we check what the function $F(z)$ actually is in this case

$$F(z) = \int_{[a,z]} \bar{s}\, ds = \int_0^1 \bar{a} + t|z-a|^2\, dt = \bar{a} + \frac{1}{2}|z-a|^2.$$

This is not a primitive of any function since it's not even differentiable.

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Exercise 6

Let $U = \mathbb{C} \setminus \{0\}$, let $f : U \to \mathbb{C}$, $f(z) = 1/z$, and let Δ be a triangle containing the origin. Assume it to be known that $\int_{\partial\Delta} f(z)\, dz = 2\pi i$. Why does this not contradict Cauchys theorem(Theorem 5.1.1)? What would go wrong in the proof of Cauchy's theorem if one tries to apply the proof to this f? Finally, prove that $\int_{\partial\Delta} f(z)\, dz = 2\pi i$ for a suitable triangle Δ containing 0.

Solution:

Theorem 5.1.1 requires continuity of $f(z)$ in U and in particular boundedness, which does not hold for $f(z)$ in this exercise. The boundedness of $f(z)$ is used in the proof of Theorem 5.1.1 to ensure that the norm $\|f\|_{L^\infty(\Delta)}$ is finite. So the argument fails for $f(z) = 1/z$.

Next let's compute $\int_{\partial\Delta} f(z)\, dz$ for some triangle Δ containing the origin. I'll use the triangle with the corners $a = -1$, $b = 1 - i$ and $c = 1 + i$,

$$\partial\Delta = [a, b] \star [b, c] \star [c, a].$$

and then

$$\int_{\partial\Delta} f(z)\, dz = \int_{[a,b]} f(z)\, dz + \int_{[b,c]} f(z)\, dz + \int_{[c,a]} f(z)\, dz.$$

We can compute the integral over $[b, c]$ using the fundamental theorem of calculus since the principal logarithm is a primitive of $f(z)$ on $[b, c]$. Then we find

$$\int_{[b,c]} f(z)\, dz = i\frac{\pi}{2}.$$

I'll use a limit argument to handle the paths $[a, b]$ and $[c, a]$. For $\varepsilon \in (0, 1)$ arbitrary, define the two paths

$$\lambda_\varepsilon(t) = [a, b](t) = -(1-t) + (1-i)t \quad \text{for } t \in [0, \varepsilon],$$
$$\eta_\varepsilon(t) = [a, b](t) = -(1-t) + (1-i)t \quad \text{for } t \in [\varepsilon, 1].$$

then $[a,b] = \lambda_\varepsilon \star \eta_\varepsilon$ and

$$\int_{[a,b]} f(z)\,dz = \int_{\lambda_\varepsilon} f(z)\,dz + \int_{\eta_\varepsilon} f(z)\,dz$$

Now the principal logarithm is a primitive of $f(z)$ on η_ε, so we get

$$\begin{aligned}\int_{\eta_\varepsilon} f(z)\,dz =& \operatorname{Log}(\eta_\varepsilon(1)) - \operatorname{Log}(\eta_\varepsilon(\varepsilon)) \\ =& \operatorname{Log}(1-i) - \operatorname{Log}(-1+2\varepsilon - i\varepsilon) \\ =& \log(\sqrt{2}) - i\frac{\pi}{4} \\ & - \big[\underbrace{\log(\sqrt{(-1+2\varepsilon)^2+\varepsilon^2})}_{\to 0 \text{ as } \varepsilon\to 0} + i \underbrace{\operatorname{Arg}(-1+2\varepsilon-i\varepsilon)}_{\to -\pi \text{ as } \varepsilon\to 0}\big] \\ \to& \log(\sqrt{2}) - i\frac{\pi}{4} + i\pi.\end{aligned}$$

Note that I'm not using $\lim_\varepsilon \operatorname{Arg}(z_\varepsilon) = \operatorname{Arg}(\lim_\varepsilon z_\varepsilon)$ to evaluate the limit of $\operatorname{Arg}(z)$, since $\operatorname{Arg}(z)$ is not continuous at the negative real line so this equatliy doesn't hold. But to justify that the claimed limit holds, recall from Remark 1.3.14 in the lecture notes that $\operatorname{Arg}(z)$ can be defined in terms of $\arccos\left(\frac{\operatorname{Re}(z)}{|z|}\right)$. It's not mentioned specifically how this is done, but the following seems reasonable

$$\operatorname{Arg}(z) = \begin{cases} \arccos\left(\frac{\operatorname{Re}(z)}{|z|}\right) & \text{for } \operatorname{Im}(z) > 0, \\ 0 & \text{for } \operatorname{Re}(z) > 0 \text{ and } \operatorname{Im}(z) = 0, \\ -\arccos\left(\frac{\operatorname{Re}(z)}{|z|}\right) & \text{for } \operatorname{Im}(z) < 0. \end{cases}$$

Now I think it should be clear that $\operatorname{Arg}(-1+2\varepsilon-i\varepsilon) \to -\pi$ as $\varepsilon \to 0$ since $\frac{\operatorname{Re}(-1+2\varepsilon-i\varepsilon)}{|(-1+2\varepsilon-i\varepsilon)|} \to -1$ from below negative real line. Next, we estimate the integral over λ_ε by something that converges to 0,

$$\left|\int_{\lambda_\varepsilon} f(z)\,dz\right| \le \|f\|_{L^\infty(\lambda_\varepsilon)} \operatorname{length}(\lambda_\varepsilon) \le \|f\|_{L^\infty(\partial\Delta)} \operatorname{length}(\lambda_\varepsilon).$$

Since f is bounded on $\partial\Delta$ and $\operatorname{length}(\lambda_\varepsilon) \to 0$, we see that

$$\int_{\lambda_\varepsilon} f(z)\,dz \to 0$$

as $\varepsilon \to 0$. Now we get

$$\int_{[a,b]} f(z)\,dz = \int_{\lambda_\varepsilon} f(z)\,dz + \int_{\eta_\varepsilon} f(z)\,dz \to \log(\sqrt{2}) - i\frac{\pi}{4} + i\pi$$

In the exact same way we compute

$$\int_{[c,a]} f(z)\,dz = -\log(\sqrt{2}) - i\frac{\pi}{4} + i\pi$$

We finally have

$$\begin{aligned}\int_{\partial\Delta} f(z)\,dz &= \int_{[a,b]} f(z)\,dz + \int_{[b,c]} f(z)\,dz + \int_{[c,a]} f(z)\,dz \\ &= \log(\sqrt{2}) - i\frac{\pi}{4} + i\pi + i\frac{\pi}{2} - \log(\sqrt{2}) - i\frac{\pi}{4} + i\pi \\ &= 2\pi i.\end{aligned}$$

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